Anyone know these guys?

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We observe that some of our diptych varieties have a beautiful description in terms of key 5-folds $V(k) \subset \mathbb{A}^{k+5}$ that are almost homogeneous spaces. By poetic licence, we could call them quantum rational normal curves. We write them out here, in the hope that someone can inform us that they occur elsewhere in math or sci-fi.

From the point of view of equations, $V(k)$ are serial unprojections or non-general crazy-Pfaffians. In geometry, they are almost homogeneous spaces for a central extension group $G$ of $\text{GL}(2)$ – in other words, closed orbits for a highest weight vector (in a slightly nonfamiliar representation of a nonsimple group).

The variety $V(k)$ in equations

We define 5-folds $V(k) \subset \mathbb{A}^{k+5} \langle x_{0..k}, a, b, c, z \rangle$ for each $k \geq 3$. First set up $2 \times k$ and $k \times (k-2)$ matrixes

\[
M = \begin{pmatrix}
    x_0 & \ldots & x_{i-1} & \ldots & x_{k-1} \\
    x_1 & \ldots & x_i & \ldots & x_k
\end{pmatrix}
\quad \text{and} \quad
N = \begin{pmatrix}
    a & b & a \\
    c & b & a \\
    \vdots & \vdots & \vdots \\
    c & b & a \\
    c & b & c
\end{pmatrix}
\]

Our variety $V(k)$ is defined by two sets of equations (see below)

\[
\begin{align*}
    (I) \quad & MN = 0 \quad \text{and} \quad (II) \quad \bigwedge^2 M = z \cdot \bigwedge^{k-2} N. \quad (1)
\end{align*}
\]
(I) is a recurrence relation

\[ ax_{i-1} + bx_i + cx_{i+1} = 0 \quad \text{for} \quad i = 1, \ldots, k - 1. \]

(II) is a \((k-2) \times k\) adaptation of Cramer’s rule giving the Plücker coordinates of the space of solutions of (I) up to a scalar factor \(z\).

The ordering of minors in (II) is best understood in terms of the guiding cases

\[ x_{i-1}x_{i+1} - x_i^2 = a^{i-1}e^{k-i-1}z \quad \text{and} \quad x_{i-1}x_{i+2} - x_ix_{i+1} = a^{i-1}bc^{k-i-2}z. \quad (2) \]

Note that the maximal \((k-2) \times (k-2)\) minors of \(N\) include \(a^{k-2}\) (delete the last two row) and \(c^{k-2}\) (delete the first two). More generally, deleting two adjacent rows \(i-1, i\) gives \(a^{i-1}e^{k-i-1}\) as a minor (only the diagonal contributes), whereas deleting two rows \(i-1, i+1\) gives the minor \(a^{i-1}bc^{k-i-2}\).

Thus our second set of equations is

\[ x_{i-1}x_{j+1} - x_ix_j = zN(i-1, j). \]

Relations for \(x_ix_j - x_kx_l\) for all \(i + j = k + l\) can be obtained as combinations of these; for example

\[
x_{i-1}x_{j+2} - x_{i+1}x_j = x_{i-1}x_{j+2} - x_ix_{j+1} + x_ix_{j+1} - x_{i+1}x_j
\]

\[ = zN(i-1, j+1) + zN(i, j). \]

**Theorem 1** For \(k \geq 3\), (I) and (II) define a reduced irreducible Gorenstein 5-fold

\[ V(k) \subset \mathbb{A}^{k+5} \langle x_0, \ldots, x_k, a, b, c, z \rangle. \]

Also for \(k = 2\), with (II) involving the \(0 \times 0\) minors interpreted as the single equation \(1 \cdot z = x_0x_2 - x_1^2\).

**Lemma 2**

(i) \(z\) is a regular element for \(V(k)\).

(ii) The section \(z = 0\) of \(V(k)\) is the quotient of the hypersurface

\[ \tilde{W} : (g := au^2 + buv + cv^2 = 0) \subset \mathbb{A}^5 \langle a, b, c, u, v \rangle \]

by the \(\mu_k\) action \(\frac{1}{k}(0, 0, 0, 1, 1)\). It is Gorenstein because

\[ \frac{da \wedge db \wedge dc \wedge du \wedge dv}{g} \in \omega_{\mathbb{A}^5}(\tilde{W}). \]

is \(\mu_k\) invariant.

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(iii) Also \( z, a, c \) is a regular sequence, and the section \( z = a = c = 0 \) of \( V(k) \) is the tent consisting of \( \frac{1}{k}(1,1) \) with coordinates \( x_0, \ldots, x_k \) and two copies of \( \mathbb{A}^2 \) with coordinates \( x_0, b \) and \( x_k, b \).

**Proof** First, if \( c \neq 0 \) then \( a, b, c, x_0, x_1 \) are free parameters, and the recurrence relation (I) gives \( x_2, \ldots, x_k \) as rational function of these. One checks that the first equation in (II) gives \( z = -\frac{ax_0^2 + bx_1x_0 + cx_1^2}{x_0} \) and the remainder follow. Similarly if \( a \neq 0 \).

If \( a = c = 0 \) and \( b \neq 0 \) then one checks that \( x_0, x_k, b \) are free parameters, \( x_i = 0 \) for \( i = 1, \ldots, k - 1 \) and \( z = \frac{b_0x_4}{x_0} \). Finally, if \( a = b = c = 0 \) then \( x_0, \ldots, x_k \) and \( z \) obviously parametrise \( \frac{1}{k}(1,1) \times \mathbb{A}^1 \).

Therefore, no component of \( V(k) \) is contained in \( z = 0 \), which proves (i).

After we set \( z = 0 \), the equations (II) become \( \bigwedge^2 \mathcal{M} = 0 \), and define the cyclic quotient singularity \( \frac{k}{k}(1,1) \) (the cone over the rational normal curve). Introducing \( u, v \) as the roots of \( x_0, \ldots, x_k \), with \( x_i = u^{k-i}v^i \), boils the equations \( MN = 0 \) down to the single equation \( g := au^2 + buv + cv^2 = 0 \). This proves (ii). (iii) is easy.

The equations as Pfaffians

The equations of \( V(k) \) fit together as \( 4 \times 4 \) crazy Pfaffians of a skew matrix. For this, edit \( M \) and \( N \) to get two new matrixes,

\[
M' = \begin{pmatrix}
  x_0 & \cdots & x_{i-1} & x_i & \cdots & x_{k-2} \\
  x_1 & \cdots & x_i & x_{i+1} & \cdots & x_{k-1} \\
  x_2 & \cdots & x_{i+1} & x_{i+2} & \cdots & x_k
\end{pmatrix}
\]

which is \( 3 \times (k-1) \) and \( N' \), the \((k-1) \times (k-3)\) matrix with the same display as \( N \) (that is, delete the first (or last) row and column of \( N \)). Equations (I) can be rewritten \((a, b, c)M' = 0 \).

Now the equations (1) can be written as the Pfaffians of the \((k+2) \times (k+2)\) skew matrix

\[
\begin{pmatrix}
  cz & -bz \\
  az & M'
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
  c & -b \\
  a & M'
\end{pmatrix}
\]
the factor \( z \) in the first \( 3 \times 3 \) block floats over to the final \( 3 \times 3 \) block, allowing us to cancel \( z \) in \( \text{Pf}_{12.3(i-4)} \) for the recurrence relation \( ax_{i-1} + bx_i + cx_{i+1} = 0 \). The remaining Pfaffians give (II).

**Remark 3** This is a mild form of crazy Pfaffian (by analogy with Riemen-schneider’s quasi-determinantal): there is a multiplier \( z \) between the \((3, 3)\) and \((4, 4)\) entries, and when evaluating crazy Pfaffians you include \( z \) as a factor whenever you cross it.

Written out in more detail, the big matrix is

\[
\begin{pmatrix}
cz & -bz & x_0 & \ldots & x_{i-1} & x_i & \ldots & x_{k-2} \\
az & x_1 & \ldots & x_i & x_{i+1} & \ldots & x_{k-1} \\
x_2 & \ldots & x_{i+1} & x_{i+2} & \ldots & x_k \\
c^{k-3} & \ldots & \ldots & \ldots & \ldots & \ldots \\
c^{k-1}a^{i-2} & -bc^{k-2}a^{i-2} & \ldots & \ldots \\
c^{k-2}a^{i-1} & \ldots & \ldots & \ldots \\
a^{k-3}
\end{pmatrix}
\]

with bottom right \((k - 1) \times (k - 1)\) block equal the \((k - 3)\)rd wedge of \( N' \).

**Sanity check**

Our family starts with \( k \geq 3 \); the case \( k = 2 \) would give the hypersurface \( ax_0 + bx_1 + cx_2 = 0 \), with \( z := x_0x_2 - x_1^2 \). The first regular case is \( k = 3 \), which gives the \( 5 \times 5 \) skew determinantal

\[
\begin{pmatrix}
c & -b & x_0 & x_1 \\
a & x_1 & x_2 \\
x_2 & x_3 \\
z
\end{pmatrix}
\]
a regular section of the affine Grassmannian $aGr(2,5)$. The case $k = 4$ is

$$
\begin{pmatrix}
c & -b & x_0 & x_1 & x_2 \\
ap & x_1 & x_2 & x_3 \\
x_2 & x_3 & x_4 \\
zc & -zb & za
\end{pmatrix},
$$

the standard extra symmetric $6 \times 6$ determinantal of [Dicks] and [Reid1].

The first really new case is $k = 5$, with equations

$$
\begin{pmatrix}
c & -b & x_0 & x_1 & x_2 & x_3 \\
ap & x_1 & x_2 & x_3 & x_4 \\
x_2 & x_3 & x_4 & x_5 \\
zc^2 & -zb & z(b^2 - ac) & zac & -zab & za^2
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
zc & -zb & x_0 & x_1 & x_2 & x_3 \\
za & x_1 & x_2 & x_3 & x_4 \\
x_2 & x_3 & x_4 & x_5 \\
c^2 & -bc & b^2 - ac & ac & -ab & a^2
\end{pmatrix}
$$

We first arrived at this matrix by guesswork, determining the superdiagonal entries $c^2, ac, a^2$ and those immediately above $-bc, -ac$ by eliminating variables to smaller cases; the entry $b^2 - ac$ is then fixed so that the bottom $4 \times 4$ Pfaffian vanishes identically.

**Alternative Proof of Theorem 1** A by-now standard application of serial unprojection [PR] and [Reid2]. We can start with any of the codimension 2 complete intersections

$$
\left( x_{i-1}x_{i+1} = x_i^2 + a^{i-1}c^{k-i-1}z \right) \subset \mathbb{A}^7 (x_{i-1}, x_i, x_{i+1}, a, b, c, z)
$$

and add the remaining variables one at a time by unprojection.

**The variety $V(k)$ by apolarity**

We can treat $V(k)$ as an almost homogeneous space under $GL(2) \times \mathbb{G}$. For this, view $x_0, \ldots, x_k$ as coefficients of a binary form and $a, b, c$ as coefficients
of a binary quadratic form in dual variables, so that the equations $MN = 0$ or $(a, b, c)M' = 0$ are the apolarity relations.

More formally, write $U$ for the given representation of $GL(2)$ and write

$$q = au'^2 + 2bu'v' + cv'^2 \in S^2 U^\vee$$

and

$$f = x_0u^k + kx_1u^{k-1}v + \cdots + x_kv^k \in S^k U.$$

One includes a binomial coefficient $\binom{k}{i}$ as multiplier in the coefficient of $u_i v^{k-i}$, a standard move in this game.

The second polar of $f$ is the polynomial

$$\Phi(u, v, u', v') = \frac{1}{k(k-1)} \left( \frac{\partial^2 f}{\partial u^2} u'^2 + 2 \frac{\partial^2 f}{\partial u \partial v} u'v' + \frac{\partial^2 f}{\partial v^2} v'^2 \right)$$

$$= \sum_{i=0}^{k-2} \binom{k-2}{i} x_i u^{k-i-2} v^i u'^2$$

$$+ 2 \sum_{i=1}^{k-1} \binom{k-2}{i-1} x_i u^{k-i-1} v^{i-1} u'v' + \sum_{i=2}^{k} \binom{k-2}{i-2} x_i u^{k-i} v^{i-2} v'^2$$

$$= \sum_{i=0}^{k-2} \binom{k-2}{i} x_i u^{k-2-i} v^i u'^2$$

$$+ 2 \sum_{i=0}^{k-2} \binom{k-2}{i} x_{i+1} u^{k-2-i} v^i u'v' + \sum_{i=0}^{k-2} \binom{k-2}{i} x_{i+2} u^{k-2-i} v^{i+1} v'^2$$

Substituting $u'^2 \mapsto a$, $u'v' \mapsto \frac{1}{2}b$, and $v'^2 \mapsto c$ in this and equating to zero gives our recurrence relation $(a, b, c)M = 0$.

Moreover, the second set of equations follow from the first by substitution, provided (say) that $c \neq 0$ and we fix the value of $x_0x_2 - x_1^2$; for example, in

$$x_ix_{i+2} - x_{i+1}^2$$

substituting $x_{i+2} = -\frac{a}{c}x_i - \frac{b}{c}x_{i+1}$ gives

$$x_i(-\frac{a}{c}x_i - \frac{b}{c}x_{i+1}) - x_i^2 = -\frac{a}{c}x_i^2 - \frac{b}{c}(x_i + x_{i+1})x_{i+1},$$
and we can substitute $-\frac{a}{c}x_{i-1}$ for the bracketed expression, to deduce that

$$x_i x_{i+2} - x_{i+1}^2 = \frac{a}{c}(x_{i-1} x_{i+1} - x_i^2).$$

etc.

A normal form for a quadric under $GL(2)$ is $uv$, so that a typical solution to the equations is

$$(a, b, c) = (0, 1, 0), \quad (x_{0..k}) = (1, 0, \ldots, 1).$$

This is a “highest weight vector”, and $V(k)$ is its closed orbit.

**Application to diptych varieties**

The diptych varieties for $d, e$ with $de = 4$ are unprojections of pullbacks of $V(k)$.

**Case [2, 2]**

The diptych variety has variables the $x_{0..k}, y_{0..2}$ of Figure 1, together with

\[ A/B, 2/1, \ldots, k/(-1) \]

Figure 1: Case [2, 2]

$A, B, L, M$. The two bottom equations are

$$x_1 y_0 = A^{k-1} B^k + x_0^2 L \quad \text{and} \quad x_0 y_1 = A B x_1 + y_0 M$$
The pentagram $y_1, y_0, x_0, x_1, x_2$ adjoins $x_2$, then the long rally of flat pentagrams $y_1, x_{i-1}, x_i, x_{i+1}, x_{i+2}$ adjoin $x_3, \ldots, x_k$, with matrixes

$$
\begin{pmatrix}
y_1 & x_1 & M & x_2 \\
y_0 & AB & x_0L & x_0 \\
x_0 & A^{k-2}B^{k-1} & x_1
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
y_1 & x_{i+1} & LM & x_{i+2} \\
y_0 & AB & x_{i-1} & x_i \\
x_i & (AB)^{k-i-2}(LM)^{i-1}BM & x_{i+1}
\end{pmatrix}
$$

and Pfaffian equations

$$
y_1 x_i = ABx_{i+1} + LMx_{i-1}, \quad x_{i-1}x_{i+1} = x_i^2 + (AB)^{k-i-1}(LM)^{i-1}BM,$$

and

$$x_{i-1}x_{i+2} = x_ix_{i+1} + (AB)^{k-i-2}(LM)^{i-1}BM y_1.$$

These are the equations of $V(k)$ after the substitution

$$(a, b, c, z) \mapsto (LM, y_1, AB, BM).$$

Thus to make our diptych variety, pull back $V(k) \subset A^{k+5}$ by that substitution, then adjoin $y_0, y_2$ as unprojection variables.  

Case $[4, 1]$ with even $l = 2k$

Omit the odd numbered $x_i$, giving Figure 2. The diptych variety has variables

```
A
3
2
2
: 
(-1)
```

```
\begin{array}{c}
\text{B} \\
\text{1} \\
\text{2} \\
\text{k} \\
\text{2} \\
\text{0}
\end{array}
```

Figure 2: Case $[4, 1]$ with even $l = 2k$

$x_{0..k}, y_{0..4}, A, B, L, M$ with the two bottom equations

$$x_1y_0 = A^{k-1}B^{2k-1}y_1 + x_0^3L \quad \text{and} \quad x_0y_1 = A^kB^{2k+1} + y_0M$$

\[1\] We still have to deal with the unprojection, here and below.
We adjoin $y_2$, then $x_2, \ldots, x_k$ by a game of pentagrams centred on a long rally of flat pentagrams, with $y_2$ against $x_{i-1}, x_i, x_{i+1}, x_{i+2}$ and Pfaffian equations

$$y_2 x_i = AB^2 x_{i+1} + LM^2 x_{i-1}, \quad x_{i-1} x_{i+1} = x_i^2 + (AB^2)^{k-i-1}(LM^2)^{i-1} BM$$

and

$$x_{i-1} x_{i+2} = x_i x_{i+1} + (AB^2)^{k-i-2}(LM^2)^{i-1} BM y_2$$

These are the equations of $V(k)$ after the substitution

$$(a, b, c, z) \mapsto (LM^2, y_2, AB^2, BM).$$

**Case $[1, 4]$ with even $l = 2k$**

Omit the even numbered $x_i$, giving Figure 3. The diptych variety has variables $x_0, \ldots, x_k, y_0, \ldots, y_2, A, B, L, M$ with the two bottom equations

$$x_1 y_0 = A^{2k-1} B^k + x_0 L \quad \text{and} \quad x_0 y_1 = x_1^2 A^2 B + y_0^2 M$$

As before, adjoining $x_2, \ldots, x_k$ features a long rally of flat pentagrams, with $y_1$ against $x_{i-1}, x_i, x_{i+1}, x_{i+2}$ and Pfaffian equations

$$y_1 x_i = A^2 B x_{i+1} + L^2 M x_{i-1}, \quad x_{i-1} x_{i+1} = x_i^2 + (A^2 B)^{k-i-1}(L^2 M)^{i-1} AL$$

and

$$x_{i-1} x_{i+2} = x_i x_{i+1} + (A^2 B)^{k-i-2}(L^2 M)^{i-1} BM y_2$$

These are the equations of $V(k)$ after the substitution

$$(a, b, c, z) \mapsto (L^2 M, y_1, A^2 B, BM).$$

![Figure 3: Case [4, 1] with even $l = 2k$](image)
Case \([1, 4]\) with odd \(l = 2k + 1\)

This is \([1, 4]\) read from the top, but \([4, 1]\) read from the bottom, so is a mix of the two preceding cases. Omit the odd numbered \(x_i\), giving Figure 4. The diptych variety has variables \(x_0...k, y_0...3, A, B, L, M\) with the two bottom equations

\[
x_1y_0 = y_1A^{2k-3}B^{k-1} + x_0^3L \quad \text{and} \quad x_0y_1 = A^{2k-1}B^k + y_0M
\]

Adjoin \(y_2\) then \(x_2\) by

\[
\begin{pmatrix}
y_1 & A^2B & M & y_2 \\
y_0 & A^{2k-3}B^{k-1} & x_0^2L & x_1 \\
x_0 & y_1 & \end{pmatrix}
\]

then

\[
\begin{pmatrix}
y_2 & x_1 & M & x_2 \\
y_1 & A^2B & x_0LM & x_0A^{2k-5}B^{k-2} \\
x_0 & y_2A^{2k-5}B^{k-2} & x_1 & \end{pmatrix}
\]

After this, adjoining \(x_3, \ldots, x_{k-1}\) is the usual long rally of flat pentagrams, with \(y_2\) against \(x_{i-1}, x_i, x_{i+1}, x_{i+2}\) and

\[
\begin{pmatrix}
y_2 & x_{i+1} & LM^2 & x_{i+2} \\
x_{i-1} & A^2B & x_i & \end{pmatrix}
\]

\[
\begin{pmatrix}
x_i & (A^2B)^{k-i-3}(LM^2)^{i-1}ABMy_2 & x_{i+1} & \end{pmatrix}
\]
and the Pfaffian equations

\[ y_2 x_i = A^2 B x_{i+1} + LM^2 x_{i-1}, \quad x_{i-1} x_{i+1} = x_i^2 + (A^2 B)^{k-i-2} (LM^2)^{i-1} ABM y_2 \]

and

\[ x_{i-1} x_{i+2} = x_i x_{i+1} + (A^2 B)^{k-i-3} (LM^2)^{i-1} ABM y_2^2 \]

These are the equations of \( V(k-1) \) after the substitution

\[ (a, b, c, z) \mapsto (LM^2, y_2, A^2 B, BM). \]