Fun in codimension 4

Miles Reid

Abstract

I discuss some graded ring constructions of algebraic varieties, mostly motivated by work on algebraic surfaces by Horikawa and his followers. My aim is to see the geometric constructions insofar as possible as pullbacks of key varieties. The ideal would be to lift case divisions such as Horikawa's I, II_a and II_b out of the geometry of surfaces and into general theory of key varieties, codimension 4 ideals, Tom and Jerry unprojections and so on.

1 The Horikawa quintics revisited

Horikawa's famous paper [H1] studies canonical surfaces with $p_g = 4$, $K^2 = 5$, with the case division

Type I $|K|$ is free and embeds to a quintic $X_5 \subset \mathbb{P}^3$;

Type II $|K|$ has a transversal base point P, and, after blowing it up, φ_K defines a double cover to a quadric $Q \subset \mathbb{P}^3$, which may be of rank 4 (Type II_a) or 3 (Type II_b).

For details, see [H1], [G], [R2].

1.1 The curve and the choice of rendition

In Type II, the curve section $C \in |K_X|$ is a genus 6 hyperelliptic curve with $P \in C$ a Weierstrass point, polarised by a half-canonical divisor $A = 5P = P + 2g^1_2$. In coordinates t_1, t_2 on $\mathbb{P}^1$, with $P = (0,1)$ and $P_2, \dots, P_{14}$ given by $f_{13}(t_1, t_2) = 0$, the ring $R(C, A) = R(C, P)^{[5]}$ is generated by

$$\begin{array}{ll} \text{in degree 1} & x_1 = ut_1^2,\ \ x_2 = ut_1t_2,\ \ x_3 = ut_2^2, \\ \text{in degree 2} & y_2 = t_2^5, \\ \text{in degree 3} & z_1 = vt_1,\ \ z_2 = vt_2, \end{array} \tag{1}$$

where $u^2 = t_1$ and $v^2 = f_{13}(t_1, t_2)$, and related by

$$\bigwedge^2 N = 0 \quad \text{where} \quad N = \begin{pmatrix} x_1 & x_2 & x_3^2 & z_1 \\ x_2 & x_3 & y_2 & z_2 \end{pmatrix}, \quad \text{and} \quad \begin{aligned} z_1^2 &= [t_1^2 f_{13}], \\ z_1 z_2 &= [t_1 t_2 f_{13}], \\ z_2^2 &= [t_2^2 f_{13}], \end{aligned} \tag{2}$$

where the square brackets mean I *render* forms in t_1, t_2 of degree divisible by 5 (in this case, 15) as weighted forms in x_1, x_2, x_3, y_2 (in this case, of degree 6). The main point is that different possible renditions give rise to different deformation families of $R(C, A)$ and components of the moduli space of X, growing out of the apparently harmless substitution $x_1x_3 \mapsto x_2^2$. The identity $x_1x_3 = x_2^2$ in $R(C, P)$ becomes a relation in $R(C, A)$, but, once we deform the ring, will only be a congruence modulo deformation parameters.

Every monomial in $t_1^i t_2^j$ of degree 13 has $i \geq 3$ or $j \geq 8$, so that we can write f_{13} in the form

$$f_{13}(t_1, t_2) = A_{10}t_1^3 - B_5 t_2^8. \tag{3}$$

Fix once and for all some rendition α_4, b_2 of A_{10}, B_5; for example, do

$$x_1x_3 \mapsto x_2^2, \quad x_1y_2 \mapsto x_2x_3^2, \quad x_2y_2 \mapsto x_3^3 \tag{4}$$

repeatedly to remove all occurrences of x_1x_3, x_1y_2, x_2y_2. Then $(t_1^2, t_1t_2, t_2^2)f_{13}$ in (2) render as:

$$\text{(A)} \quad \begin{matrix} ax_1 - bx_3^4 \\ ax_2 - bx_3^2y_2 \\ ax_3 - by_2^2 \end{matrix} \quad \text{with } a = \alpha x_1\text{; or} \quad \text{(B)} \quad \begin{matrix} \alpha x_1^2 - bx_3^4 \\ \alpha x_1x_2 - bx_3^2y_2 \\ \alpha x_2^2 - by_2^2 \end{matrix} \tag{5}$$

the only difference being $\alpha x_1x_3 \mapsto \alpha x_2^2$ in the last line. Case A corresponds to Horikawa's Types II_b and I, whereas Case B corresponds to Type II_a,

1.2 Case A

Case A in (5) allows me to roll factors $x_1 \mapsto x_2 \mapsto x_3$ without putting in terms that are explicitly quadratic in the rows of N; it depends on the coincidence $n_{12} = n_{21} = x_2$ in N, so forbids deforming the quadric $x_1x_3 - x_2^2$ to rank > 3. The variable x_1 appears linearly in 4 equations multiplying x_3, y_2, z_2, a, and not in the others, so we can eliminate it, and treat the ring as a Kustin–Miller unprojection from the Pfaffians of

$$M = \begin{pmatrix} 0 & x_2 & x_3^2 & z_1 \\ & x_3 & y_2 & z_2 \\ & & z_2 & -by_2 \\ & & & -a \end{pmatrix} \quad \text{of weights} \quad \begin{matrix} 0 & 1 & 2 & 3 \\ & 1 & 2 & 3 \\ & & 3 & 4 \\ & & & 5 \end{matrix} \tag{6}$$

with unprojection ideal the codimension 4 c.i. $I = (x_3, y_2, z_2, a)$. The weight 0 of the entry $m_{12} = 0$ is noteworthy. Apart from $m_{13} = x_2$ and $m_{15} = z_1$, every entry of M is in I, so we can treat it in the bigger families of Tom_1 or Jerry_{24} unprojections.

$\mathbf{Jer}_{24}$ The format requires $m_{12} \in I$, so keeps the 0. The deformation of the entry $m_{14} = x_3^2$ in the ideal $I = (x_3, y_2, z_2, A)$ can be nullified by coordinate changes. The entry $m_{35} = -by_2$ is a free entry, so treat it as a token B. After this, the pivot $m_{24} = y_2$ can be projected out, and the deformation family calculated as a parallel unprojection. It is a variant on rolling factors:

$$\bigwedge^2 \begin{pmatrix} x_1 & x_2 & x_3^2 & z_1 \\ x_2 & x_3 & y_2 & z_2 \end{pmatrix} = 0 \quad \text{and} \quad \begin{aligned} z_1^2 &= ax_1 - x_2x_3B, \\ z_1z_2 &= ax_2 - x_3^2B, \\ z_2^2 &= ax_3 - y_2B. \end{aligned} \tag{7}$$

Conclusion: This depends on *both* the coincidences $x_2 = x_2$ and $x_3 \mid n_{13}$; for Horikawa surfaces, it only gives deformations inside Type II_b.

$\mathbf{Tom}_1$ The original ring and its general Tom_1 deformation are the Pfaffians of the following extrasymmetric 6×6 matrixes:

$$M_0 = \begin{pmatrix} 0 & x_3^2 & x_1 & x_2 & z_1 \\ & y_2 & x_2 & x_3 & z_2 \\ & & z_1 & z_2 & a \\ & & & 0 & bx_3^2 \\ & & & & by_2 \end{pmatrix} \quad \mapsto \quad M_\lambda = \begin{pmatrix} \lambda & y_1 & x_1 & x_2 & z_1 \\ & y_2 & x_2 & x_3 & z_2 \\ & & z_1 & z_2 & a \\ & & & \lambda b & by_1 \\ & & & & by_2 \end{pmatrix}, \tag{8}$$

where λ, y_1, b, a are indeterminates of weight $0, 2, 2, 5$. Write

$$\mathcal{CV} \subset \mathbb{A}^9_{\langle x_{1\dots3}, y_1, y_2, b, z_1, z_2, a\rangle} \times \mathbb{A}^1_\lambda \tag{9}$$

for the key variety defined by the 4×4 Pfaffians of M_λ, the affine cone over the weighted projective variety $\mathcal{V} \subset \mathbb{P}(1^3, 2^3, 3^2, 5) \times \mathbb{A}^1$.

The fibre $\mathcal{C}V_{\lambda\neq0}$ is just a copy of $\mathbb{A}^5_{\langle x_{1\dots3},y_1,y_2\rangle}$ cunningly set up to degenerate as $\lambda\to0$ to the codimension 4 variety $\mathcal{C}V_0$ given by the Pfaffians of M_0.

Proposition 1.1 *Assume first that λ is invertible; then $\mathcal{C}V_\lambda$ is the graph over $\mathbb{A}^5_{\langle x_{1\dots3},y_1,y_2\rangle}$ of the functions b,z_1,z_2,a defined by four of the Pfaffians* $\mathrm{Pf}_{12.ij}$*:*

$$\begin{aligned}&-\lambda z_1=x_1y_2-x_2y_1,\quad -\lambda z_2=x_2y_2-x_3y_1,\\ &-\lambda a=y_2z_1-y_1z_2\quad\text{and}\quad \lambda^2b=x_1x_3-x_2^2;\end{aligned}\tag{10}$$

after this, the remaining Pfaffian equations hold as identities. Therefore the fibre $\mathcal{C}V_{\lambda\neq0}\cong\mathbb{A}^5$ and $V_{\lambda\neq0}\cong\mathbb{P}^4(1^3,2^2)$.

When $\lambda=0$, the variety $\mathcal{C}V_0$ is given by

$$\bigwedge^2\begin{pmatrix}x_1&x_2&y_1&z_1\\x_2&x_3&y_2&z_2\end{pmatrix}=0\quad\text{and}\quad\begin{aligned}z_1^2&=ax_1-by_1^2\\z_1z_2&=ax_2-by_1y_2\\z_2^2&=ax_3-by_2^2\end{aligned}\tag{11}$$

It is an affine Gorenstein codimension 4 variety that can be viewed in an obvious way as an anticanonical divisor in a scroll.

The affine cone $\mathcal{C}V_0$ also has a parametric form that displays it as a simple birational transformation away from the quotient of $\mathbb{A}^5_{\langle b,s_1,s_2,u,v\rangle}$ by the μ_2 action $\frac{1}{2}(0,1,1,1,1)$. Indeed, the quantities

$$\begin{aligned}x_1=s_1^2,\quad x_2=s_1s_2,\quad x_3=s_2^2&\\ y_1=s_1u,\quad y_2=s_2u,&\qquad\text{and}\quad a=v^2+bu^2,\\ z_1=s_1v,\quad z_2=s_2v,&\end{aligned}\tag{12}$$

satisfy the relations (11) identically. In straight projective space $x_1,\dots,z_2$ would be coordinates on the scroll $\mathbb{P}_{\mathbb{P}^1}(2,1,1)$, the blowup of $\mathbb{P}^1\subset\mathbb{P}^3$ or the projection of $v_2(\mathbb{P}^3)$ from a conic.

1.3 Case B and the obstruction

In Case B of (5), the roll $x_1^2\mapsto x_1x_2\mapsto x_2^2$ is just quadratic in the rows. The 9 equations are in rolling factors format:

$$\bigwedge^2\begin{pmatrix}x_1&x_2'&y_1&z_1\\x_2&x_3&y_2&z_2\end{pmatrix}=0\quad\text{and}\quad\begin{aligned}z_1^2&=\alpha x_1^2-by_1^2,\\z_1z_2&=\alpha x_1x_2-by_1y_2,\\z_2^2&=\alpha x_2^2-by_2^2,\end{aligned}\tag{13}$$

with $x_2' = x_2$ and $y_1 = x_3^2$. This format allows the quadric to deform to rank 4 (when x_2' becomes an independent variable). The key variety in Case B is an anticanonical divisor in a weighted form of $\mathbb{P}^1 \times \mathbb{P}^3$; I do not know how to interpret it as a Kustin–Miller unprojection.

The two deformation families of Case A and Case B are incompatible already at the first infinitesimal level: you can deform to the extrasymmetric format (8) with $\lambda \neq 0$, or you can deform the rank 3 quadric $x_1x_3 - x_2^2$ to rank 4, but you can't do both. Compare [R2], Section 5, which calculates Horikawa's obstruction to deformation as $\lambda(x_2 - x_2') = 0$.

1.4 Degeneration of quintic curves, embedded

The key variety $\mathcal{V}$ of (9) describe the degenerations of quintics hypersurfaces, starting from the degeneration of a plane quintic curve to the nonsingular hyperelliptic curve $C_0 \subset \mathbb{P}(1,1,1,2,3,3)$ as described in Griffin [G]: for this, consider the complete intersection

$$(b = B_2, y_1 = Y_{1,2}, a = A_5) \subset \mathcal{V} \subset \mathbb{P}(1^3, 2^3, 3^2, 5) \times \mathbb{A}^1, \qquad (14)$$

where $B_2(x_{1..3}, y_2)$, $Y_{1,2}(x_{1..3}, y_2)$, $A_5(x_{1..3}, y_2)$ are general forms in x_i, y_2 of the stated weights. For $\lambda \neq 0$, the fibre V_λ is $\mathbb{P}(1,1,1,2,2)$, and the two quadratic equations in (14) eliminate y_1, y_2 (because B_2 contains y_2), so C_λ is a general plane quintic $C_5 \subset \mathbb{P}^2_{\langle x_{1\dots3}\rangle}$. For $\lambda = 0$, the equations of V_0 with the specialisation of y_1, b, a are essentially the hyperelliptic equations we started from in (1).

1.5 Generalities on regular pullbacks

The above *embedded* treatment inside V_λ works for plane quintic curves, but not for higher dimensional quintic hypersurfaces, because $h^0(V_\lambda, \mathcal{O}(1)) = 3$. The more general notion is *regular pullback* from a key variety; I explain this briefly for completeness.

By definition, a *key variety* is an affine variety $W \subset \mathbb{A}^N$ that I wish to treat as a key variety (in other words, it is a psychological state); as usual, write $k[W] = k[x_{1\dots n}]/I_W$ for its affine ideal and coordinate ring. W might be, say, the affine cone $a\,\mathrm{Grass}(2,5) \subset \bigwedge^2 \mathbb{C}^5$ over the Plücker embedding of $\mathrm{Grass}(2,5)$ of [CR], with equations the 4×4 Pfaffians of a generic 5×5 skew matrix, or the extrasymmetric variety $\mathcal{CV} \subset \mathbb{A}^9 \times \mathbb{A}^1_\lambda$ of (9), or the origin $0 \in \mathbb{A}^n$ defined by the regular sequence $x_{1\dots n}$.

Given an ambient ring R (either regular local or polynomial and graded in positive degrees), and a morphism $\varphi\colon \operatorname{Spec} R \to \mathbb{A}^N$ to the ambient space of W, take the pullback or scheme theoretic inverse image $\varphi^{-1}W \subset \operatorname{Spec} R$, and require it to be a regular pullback in the sense of Proposition 1.2. The morphism φ specifies values $\varphi^*(x_i) = X_i \in R$; the pullback is then defined by the ideal $\varphi^* I_W \subset R$, in other words, by substituting elements $X_i \in R$ for x_i in the equations of W. It is the same thing as the intersection with the graph of φ

$$\Gamma_\varphi \cap (\operatorname{Spec} R \times W) \subset \operatorname{Spec} R \times \mathbb{A}^N; \tag{15}$$

the graph Γ_φ is of course the complete intersection cut out by the equations $X_i = x_i$ for $i = 1, \dots, n$.

Proposition 1.2 *Equivalent conditions:*

(i) $X_i - x_i$ *for* $i = 1, \dots, n$ *form a regular sequence for* $\operatorname{Spec} R \times W$.

(ii) The resolution complex of W *remains exact on pulling back to* $\operatorname{Spec} R$.

Assume also that W *is Cohen–Macaulay; then (i) and (ii) are equivalent to*

(iii) $\varphi^{-1}W$ *has the expected dimension, that is,* $\operatorname{codim} \varphi^{-1}W = \operatorname{codim} W$.

In my case (9), I substitute specific values $X_1, \dots, A \in R$ for the variables $x_1, \dots, a$ of $\mathcal{CV}$ into the extrasymmetric matrix M_λ of (8), and use the resulting Pfaffians to generate an ideal of R.

Even though I work mainly with projective varieties and graded rings, the construction itself works on the level of affine cones: φ is usually homogeneous (equivariant for appropriate $\mathbb{G}_m$ actions), but the induced map $\varphi\colon \operatorname{Proj} R \dashrightarrow \mathbb{P}(W)$ need not be a morphism.

1.6 Application to quintic hypersurfaces

We saw that the halfcanonical linear system $A = g^2_5$ of a nonsingular plane quintic C_5 can acquire a base point and become $A = P + 2g^1_2$. The extrasymmetric format (8) also allows the polarising $|\mathcal{O}(1)|$ of a quintic n-fold $V^n_5 \subset \mathbb{P}^{n+1}$ to acquire a transverse base point and $\varphi_{\mathcal{O}(1)}$ to degenerate to a double cover of a rank 3 quadric Q, while V_5 remains nonsingular in codimension 3 (so nonsingular for surfaces and 3-folds).

Theorem 1.3 *Let* $R = k[x_{1\dots n+2}, y_2, z_1, z_2]$ *be the graded polynomial ring with* $\operatorname{wt} x_i, y_2, z_i = 1, 2, 3$. *Let* $b = B_2, y_1 = Y_{1,2}, a = A_5$ *be general forms in* x_i, y_2 *of the stated weights, and write*

$$\mathcal{X} \subset \operatorname{Proj} R \times \mathbb{A}^1_\lambda = \mathbb{P}^{n+4}(1^{n+2}, 2, 3^2) \times \mathbb{A}^1_\lambda \tag{16}$$

for the variety defined by the 4×4 *Pfaffians of the extrasymmetric matrix* M_λ *of* (8). *It is a flat family* X_λ *of projectively Gorenstein codimension* 4 *varieties parametrised by* λ, *and the fibre* $X_{\lambda \neq 0}$ *is projectively equivalent to a general quintic in* $\mathbb{P}^{n+1}$, *lifted to* $\mathbb{P}(1^{n+2}, 2, 3^2)$ *by the forms* b, z_1, z_2 *of* (10) *(*b *contains* y_2*).*

When $\lambda = 0$, *the Pfaffians take the form* (11) *with* $y_1 = Q$; *these equations define an* n*-fold* X_0 *with singular locus of dimension* $n - 3$ *(empty if* $n \leq 2$*). The linear system* $|\mathcal{O}_{X_0}(1)|$ *has a transverse base point.*

Geometry of $F \subset \mathbb{P}(1^{n+2}, 2)$ Consider the involution that acts by -1 on λ, z_1, z_2 and fixes x_i, y_2 and b, y_1, a. Each Pfaffian of M_λ is $\pm$ invariant (compare (8–11)), and this induces an involution on $\mathcal{X}$ that restricts to a "hyperelliptic" involution on X_0. The quotient morphism $X_0 \to F \subset \mathbb{P}(1^{n+2}, 2)$ given by the free linear system $|\mathcal{O}(2)|$ is a finite double cover of the codimension 2 determinantal n-fold F given by

$$\bigwedge^2 N = 0 \quad \text{where} \quad N = \begin{pmatrix} x_1 & x_2 & y_1 \\ x_2 & x_3 & y_2 \end{pmatrix} \tag{17}$$

(and $y_1 = Y_{1,2}(x_i)$ general). This F is singular exactly where $N = 0$, together with the quasismooth point $P_{y_2} \in F$, an isolated $\frac{1}{2}$ orbifold point.

The reader new to all this should concentrate on the surface case $n = 2$, which is familiar from [H1], and relates closely to the relative 2-canonical morphism of a genus 2 fibration at a 2-disconnected fibre as described in [CP]: the 1-canonical image is then the quadric $Q : (x_1 x_3 = x_2^2) \subset \mathbb{P}^3_{\langle x_{1\dots 4}\rangle}$. The blown up base point of $|K_S|$ maps to the x_3-axis $L : (x_1 = x_2 = 0)$, and has two marked point $Y_{1,2} = 0$ (typically, $x_3^2 - x_4^2 = 0$) that are the essential singularities of the branch locus in Horikawa's treatment.

Write $Q : (x_1 x_3 = x_2^2) \subset \mathbb{P}^{n+1}_{\langle x_{1\dots n+2}\rangle}$ for the n-fold quadric of rank 3, the image of X_0 under $\varphi_{\mathcal{O}(1)}$. The birational map $\beta \colon Q \dashrightarrow F$ is given by quadratic forms on Q allowed poles on the fibre $L = \mathbb{P}^{n-1} : (x_1 = x_2)$ but required to vanish on $L \cap Y_{1,2}$, giving $y_2 = x_2 Y_1 / x_1 = x_3 Y_1 / x_2$ in addition to quadratic forms in $x_{1\dots n+2}$. Expressed in birational geometry, β first blows up the vertex $x_1 = x_2 = x_3$ to make the n-fold scroll $\mathbb{F}(2, 0^{n-1})$, then blows up the nonsingular quadric $Y_1 = 0$ in the fibre L, and finally contracts L to a $\frac{1}{2}$ orbifold point at P_{y_2}. The career of the locus $x_1 = x_2 = x_3 = 0$ is also interesting: it starts life as the vertex $\mathbb{P}^{n-2}$ of the quadric, is blown up to the negative locus $E = \mathbb{P}^1 \times \mathbb{P}^{n-2}$ of the scroll. At the fibre L it meets the nonsingular quadric $L \cap Y_1$, and after the blowup of Y_1, is contracted to the $\frac{1}{2}$ orbifold point P_{y_2} of the divisor $\mathbb{P}^{n-1}(1^{n-1}, 2)_{\langle x_4 \dots n+2, y_2 \rangle} \subset F$, given also by $x_1 = x_2 = x_3 = 0$.

The branch locus of the double cover $X_0 \to F$ consists of the divisor

$$D : (ax_1 = bY_1^2, \;\; ax_2 = bY_1 y_2, \;\; ax_3 = by_2^2), \qquad (18)$$

together with the $\frac{1}{2}$ point P_{y_2}; these are disjoint because $y_2 \in b = B_2(x_i, y_2)$. To prove Theorem 1.3, I only need to establish that D is nonsingular outside the singularities of F.

Conjecture: the Type A family is a generic hypersurface if $\lambda \neq 0$. When $\lambda = 0$ it is a birational double cover of a quadric of rank 3, and is singular at a conic in the vertex (e.g. 2 points if $n = 3$). The Type B family when $x_2 \neq x_2'$ is a double cover of a quadric of rank 4, and is singular at a conic in the vertex (e.g., nonsingular if $n = 3$, singular at 2 points if $n = 4$). It would be interesting to know the relation between the topology of the general Type A and the general Type B, e.g., for the Calabi–Yau case. Can do by computer algebra. Should be easy by Bertini. Hypersurface question on affine pieces.

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Horikawa has

I, IIb, IIa

where I and IIa are the irreducible components of moduli,
and IIb is in the closure of both the other two, in
codim 1 in each.

I have

A with la <> 0, A with la = 0, B with x2 = x2', B with x2 <> x2'.

In the curve case there is only on component, given by
the first case A with la <> 0, and the last 3 coincide, and are
a codim 1 subvariety of A.
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In the 3-fold case, I want general case B corresponds to
Horikawa fake quintic Y2, with a single base point and double
cover of quadric of rank 4. Questions: can Y2 be nonsingular,
and is it diffeo to Y1 = quintic in PP^4. Maybe Pic Y2 = 2ZZ?
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1.7 Other applications

The restriction to $y_1 = Y_{1,2}$, that is, only one new variable y_2 in degree 2, was motivated by the application to quintic hypersurfaces in straight $\mathbb{P}^n$. There are many other interesting cases, starting with natural degenerations of hypersurfaces $V_5 \subset \mathbb{P}(1^n, 2)$ to codimension 4.

The key variety $\mathcal{CV}$ of (10) has a 3-parameter family of $\mathbb{C}^\times$ actions with weights:

$$\begin{matrix} x_1 \mapsto n-l \\ x_2 \mapsto n \\ x_3 \mapsto n+l \end{matrix} \quad \begin{matrix} y_1 \mapsto m \\ y_2 \mapsto m+l \end{matrix} \quad \begin{matrix} z_1 \mapsto n+m \\ z_2 \mapsto n+m+l \end{matrix} \quad \begin{matrix} b \mapsto 2n \\ a \mapsto n+2m+l \end{matrix} \tag{19}$$

The determinantal $\bigwedge^2 N = 0$ and its double cover given by (11) apply in other cases. In particular, the same tricks give natural degenerations of K3 and Fano hypersurfaces $V_5 \subset \mathbb{P}(1^n, 2)$ to codimension 4.

To finish.

2 On the BCP construction

Extending Horikawa's work on surfaces with $p_g = 4$, $K^2 = 6$, Bauer, Catanese and Pignatelli [BCP] study deformations of the ring $R(C, \frac{3}{2}P)$, where C is a hyperelliptic curve of genus 3 and $P \in C$ a Weierstrass point viewed as a $\frac{1}{2}$ orbifold point. Start from the hypersurface

$$R(C, \tfrac{1}{2}P) = k[a, b, c]/(c^2 = f_7(a^4, b)) \tag{20}$$

with $\mathrm{wt}(a, b, c) = 1, 4, 14$. Its Proj $C_{28} \subset \mathbb{P}(1, 4, 14)$ has a $\frac{1}{2}$ orbifold point at $P = (1, 0, 0)$ and ample divisor $A = \frac{1}{2}P$ with $K_{C,\mathrm{orb}} = 9A = 2g^1_2 + \frac{1}{2}P$. The ring $R(C, \frac{3}{2}P)$ is the third Veronese truncation $R(C, \frac{1}{2}P)^{[3]}$, and is Gorenstein codimension 4 with generators

$$x = a^3, \quad y = a^2b, \quad z = ab^2, \quad u = b^3, \quad v = ac, \quad w = bc$$

with $\mathrm{wt}(x,y,z,u,v,w) = 1,2,3,4,5,6$ and relations

$$\bigwedge^2 \begin{pmatrix} x & y & z & v \\ y & z & u & w \end{pmatrix} = 0 \quad \text{and} \quad \begin{aligned} v^2 &= [a^2 f], \\ vw &= [abf], \\ w^2 &= [b^2 f], \end{aligned} \tag{21}$$

where the square brackets render a form of weight $3d$ in a,b into a form of weight d in x,y,z,u. There are 2×2 different renditions of (21), that express our ring in four ways as sections of key varieties. The choices are at the two ends of the binary form $f_7(a^4,b)$: terms with high powers of a roll as

$$a^3, a^2b, ab^2 \mapsto x,y,z \qquad \text{or as} \qquad a^6, a^5b, a^4b^2 \mapsto x^2, xy, y^2 \tag{22}$$

and at the other end, terms with high powers of b roll as

$$a^2b^4, ab^5, b^6 \mapsto z^2, zu, u^2 \qquad \text{or as} \qquad a^2b, ab^2, b^3 \mapsto y,z,u \tag{23}$$

Every monomial $a^{28-4j}b^j$ in $f_7(a^4,b)$ has $j \ge 4$ or $28-4j \ge 4$, so choosing renditions at the two ends of f gives:

$$\begin{aligned} v^2 &= Ax + Cy & & Ax - Dz^2 & & Bx^2 + Cy & & Bx^2 + Dz^2 \\ vw &= Ay + Cz & \text{or} \quad & Ay - Dzu & \text{or} \quad & Bxy + Cz & \text{or} \quad & Bxy + Dzu \\ w^2 &= Az + Cu & & Az - Du^2 & & By^2 + Cu & & By^2 + Du^2 \end{aligned} \tag{24}$$

where $A = A_9$, $B = B_8$, $C = C_8$, $D = D_4$ are forms of the stated weights in x,y,z,u. The four cases in (24) are called (I)–(IV).

Case I This is a double Jerry, see [TJ], Section 8. Two projections eliminate x and u to the codimension 2 c.i. ideal

$$yw = zv, \quad vw = Ay + Cz \tag{25}$$

in the product of the ideals $I_x = (z,w,A)$ and $I_u = (y,v,C)$. The relations (25) deform to the apparently more general form

$$\begin{pmatrix} z & w & A \end{pmatrix} M \begin{pmatrix} y \\ v \\ c \end{pmatrix} = 0, \quad \begin{pmatrix} z & w & A \end{pmatrix} N \begin{pmatrix} y \\ v \\ c \end{pmatrix} = 0 \tag{26}$$

with M of weights $\begin{smallmatrix} 3 & 0 & -3 \\ 0 & -3 & -6 \\ -3 & -6 & -9 \end{smallmatrix}$ and N of weights $\begin{smallmatrix} 6 & 3 & 0 \\ 3 & 0 & -3 \\ 0 & -3 & -6 \end{smallmatrix}$. However, there are not many deformation entries of positive degree in these matrixes, and they can be absorbed by coordinate changes. For example, in $yw - zv + m_{11}zw$, the m_{11} is absorbed by $w \mapsto w + m_{11}z$ or $v \mapsto v - m_{11}y$.

So the variety $W_{8,11} \subset \mathbb{P}(1,2,3,4,5,6,8,9)_{\langle y,z,v,w,C,A\rangle}$ given by (25) and its double unprojection $V \subset \mathbb{P}(1,2,3,4,5,6,8,9)_{\langle x,y,z,u,v,w,C,A\rangle}$ is rigid in these degrees.

Case II Unproject from x to get 5 equations fitting together as the Pfaffians of

$$\begin{pmatrix} 0 & y & z & v \\ & z & u & w \\ & & w & -Du \\ & & & A \end{pmatrix}, \quad \text{of weights } \begin{smallmatrix} 0 & 2 & 3 & 5 \\ & 3 & 4 & 6 \\ & & 6 & 8 \\ & & & 9 \end{smallmatrix} \tag{27}$$

with unprojection ideal $I_x = (z,u,w,A)$. The entry $m_{12} = 0$ has weight 0. Except for m_{13}, m_{15} all the entries are in I_x, so we can view this as Tom_1 or Jerry_{23}.

The Tom_1 equations give the extrasymmetric format

$$\begin{pmatrix} 0 & z & x & y & v \\ & u & y & z & w \\ & & v & w & A \\ & & & 0 & Dz \\ & & & & Du \end{pmatrix}, \tag{28}$$

We can deform the zero entries m_{12} and m_{45} in (28) to λ and λD (with λ a variable of degree 0), and the entry $m_{24} = z$ to an independent variable t of weight 3, leading to the matrix

$$\begin{pmatrix} \lambda & z & x & y & v \\ & u & y & t & w \\ & & v & w & A \\ & & & \lambda D & Dz \\ & & & & Du \end{pmatrix} \tag{29}$$

When $\lambda \neq 0$ the equations eliminate v, w, A, D to give affine space $\mathbb{A}^5_{\langle x,y,z,t,u\rangle}$ or $\mathbb{P}(1,2,3,3,4)_{\langle x,y,z,t,u\rangle}$. Putting back the values of the tokens A, D gives the surface codimension 2 c.i.

$$S_{4,9} : (\lambda^2 D = xt - y^2, \lambda A = zw - uv) \subset \mathbb{P}^4(1,2,3,3,4)_{\langle x,y,z,t,u\rangle}, \tag{30}$$

where $v = yz - xu$ and $w = zt - yu$. Since D has the same weight as the variable u, we can think of D as $\mu u + D'$, and for $\mu \neq 0$, this is a general K3 surface $S_9 \subset \mathbb{P}(1,2,3,3)$ with a built-in degeneration.

Case III Eliminating u gives the ring as the unprojection of the ideal of Pfaffians

$$\begin{pmatrix} 0 & x & y & v \\ & y & z & w \\ & & v & C \\ & & & -Bx \end{pmatrix} \quad \text{of weights} \quad \begin{matrix} -1 & 1 & 2 & 5 \\ & 2 & 3 & 6 \\ & & 5 & 8 \\ & & & 9 \end{matrix} \tag{31}$$

in the c.i. ideal $I = (x, y, v, C)$, with the entry $m_{12} = 0$ of weight -1. Except for m_{24}, m_{25}, every entry of the matrix is in I, so the ring can be viewed either as a Tom_2 or Jerry_{13} unprojection.

Because of the -1, these formats do not allow to lose the 2×3 minors.

Case IV Every variable appears quadratically in the equations, so there is no naturally occuring unprojection. The deformation family is the matrix format.

3 Divisor of odd degree in $v_2(\mathbb{P}^2)$

As part of part of the trigonal dichotomy, Castelnuovo, Petri and Mukai tell us that the canonical model $C_{10} \subset \mathbb{P}^5$ of a nonhyperelliptic, nontrigonal curve of genus 6 is either a quadric section of a del Pezzo surface S_5 in the general case, or is the second Veronese embedding $v_2(C_5 \subset \mathbb{P}^2)$ of a plane quintic. In either case there are 6 quadric relations; in the general case these are 5×5 Pfaffians intersect a quadric hypersurface, leading to a 6×10 resolution. In the plane quintic case, C needs 3 further cubic equations. The equations of C are $\bigwedge^2 M = 0$ and $(A_1, A_2, A_3)M = 0$ where

$$M = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{pmatrix} \tag{32}$$

corresponds to the Veronese embedding $x_1 = u^2$, $x_2 = uv$, $x_3 = uw$, $x_4 = v^2$, $x_5 = vw$, $x_6 = w^2$, and the three cubic equations are renditions $[uf_5]$, $[vf_5]$, $[wf_5]$ where $f_5(u, v, w)$ defines $C_5 \subset \mathbb{P}^2$.

As before, the key is a seemingly trivial trick with this rendition: observing that every monomial in u, v, w of degree 5 is divisible either by u, or by v^3 or w^3, I write the equation of C_5 as $f_5 = uA + v^3B + w^3C$ (with A quadratic in the x_i and B, C linear) and the renditions as

$$\begin{aligned} uf_5 &= x_1A + x_2x_4B + x_3x_6C \\ vf_5 &= x_2A + x_4^2B + x_5x_6C \\ wf_5 &= x_3A + x_4x_5B + x_6^2C \end{aligned} \tag{33}$$

Now the set of all 9 equations defining C can be written as 4×4 Pfaffians of

$$M = \begin{pmatrix} 0 & 0 & x_1 & x_2 & x_3 \\ & 0 & x_2 & x_4 & x_5 \\ & & x_3 & x_5 & x_6 \\ & & & x_6C & -x_4B \\ & & & & A \end{pmatrix} \tag{34}$$

The promising appearance of this as a 6×6 extrasymmetric matrix is a deception: the top left-hand block cannot become nonzero while preserving the format. I work instead by projecting out x_1. Then

$$M = \begin{pmatrix} 0 & x_2 & x_4 & x_5 \\ & x_3 & x_5 & x_6 \\ & & x_6C & -x_4B \\ & & & A \end{pmatrix} \mapsto \begin{pmatrix} \lambda & x_2 & x_4 & x_5 \\ & x_3 & x_5 & x_6 \\ & & x_6C & -x_4B \\ & & & A \end{pmatrix} \tag{35}$$

is a Jerry_{45} with unprojection ideal (x_4, x_5, x_6, A). In this format, the three top left entries are free, so I can replace $0 \mapsto \lambda$; since A is a token, I can also project him out to get the equations

$$\begin{pmatrix} x_3 & -x_2 & \lambda C \\ \lambda B & -x_3 & x_2 \end{pmatrix} \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} = 0 \quad \Longrightarrow \quad x_1 \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} x_2^2 - \lambda x_3C \\ x_2x_3 - \lambda^2BC \\ x_3^2 - \lambda x_2B \end{pmatrix}. \tag{36}$$

Thus the equations without A are

$$\begin{pmatrix} x_1 & x_2 & x_3 & \lambda B \\ & \lambda C & x_2 & x_3 \\ & & x_4 & x_5 \\ & & & x_6 \end{pmatrix}. \tag{37}$$

The final *long equation* is

$$x_1 A + x_2 x_4 B + x_3 x_6 C - \lambda BC x_5 = 0. \tag{38}$$

I do not how to write it as a Pfaffian in any useful way. This is characteristic of Jerry.

It is interesting to understand how the deformation $\lambda \neq 0$ allows the 9 equations defining the special g^2_5 curve to pass to just 6 equations defining the general curve. The effect of the λ in (37) is to give 5 general equations defining a del Pezzo surface S_5; the other quadratic equation $f = \lambda A - x_4 x_6 + x_5^2$ defines the general curve C as quadratic section of S_5. Having done this, the three cubic equations $x_i f$ for $i = 1, 2, 3$ become combinations of these 6.

Wenfei's case: $v_2(C_{15} \subset \mathbb{P}(1,3,5))$ **deforms to** $C_{6,6} \subset \mathbb{P}(1,2,3,3)$ The general curve $C_{15} \subset \mathbb{P}(1,3,5)_{\langle u,v,w \rangle}$ is nonsingular, with $K_C = 6P$ where $P : (v^5 + w^3 = 0) \in \mathbb{P}^1(3,5)$. Polarising the same curve by $2P$ gives the second Veronese $v_2(C_{15} \subset \mathbb{P}(1,3,5))$; as before, write $x = u^2$, $y = uv$, $z_1 = uw$, $z_2 = v^2$, $s = vw$ and $t = w^2$ for the generators, dividing degrees by 2 so that $\operatorname{wt} x, y, z_1, z_2, s, t = 1, 2, 3, 3, 4, 5$; also write $f_{15} = uA_1 + vA_2 + wA_3$, and render the A_i (temporarily) as forms in x, y, z_1, z_2, s, t of weights $7, 6, 5$. Then as before, the ring $R(C, 2P)$ is related by $\bigwedge^2 M = 0$ and $(A_1, A_2, A_3)M = 0$, giving 9 relations of weights $4, 5, 6,\ 6, 7, 8,\ 8, 9, 10$. I can write them as the 4×4 Pfaffians of

$$\begin{pmatrix} 0 & 0 & x & y & z_1 \\ & 0 & y & z_2 & s \\ & & z_1 & s & t \\ & & & A_3 & -A_2 \\ & & & & A_1 \end{pmatrix} \quad \text{of weights} \quad \begin{matrix} -1 & 0 & 1 & 2 & 3 \\ & 1 & 2 & 3 & 4 \\ & & 3 & 4 & 5 \\ & & & 5 & 6 \\ & & & & 7 \end{matrix} \tag{39}$$

This extrasymmetric format does not as it stands allow me to deform C to the c.i. $C_{6,6} \subset \mathbb{P}(1,2,3,3)$. As before, a rendition trick is the key: the monomials of $f_{15}(u,v,w)$ not divisible by v are $u^{15}, u^{10}w, u^5w^2, w^3$; therefore, every monomial in f is divisible by u^3 or v or w^3, giving the rendition

$$f_{15} = Du^3 + Bv + Ew^3 \quad \text{with} \quad \operatorname{wt} D, B, E = 6, 6, 0. \tag{40}$$

The fact that E is a nonzero scalar is the thing that will express s, t as functions of x, y, z_1, z_2 when $\lambda \neq 0$.

Now rewrite the equations not involving z_2 as the Pfaffians of

$$\begin{pmatrix} 0 & x & y & z_1 \\ & z_1 & s & t \\ & & Et & -B \\ & & & Dx \end{pmatrix} \mapsto \begin{pmatrix} \lambda & x & y & z_1 \\ & z_1 & s & t \\ & & Et & -B \\ & & & Dx \end{pmatrix}. \tag{41}$$

This is a Jerry$_{35}$ matrix: the entries in Rows 3 and 5 are in the unprojection ideal (x, z_1, t, B). Almost the same calculation as above give the unprojection equations for z_2 as the Pfaffians of

$$\begin{pmatrix} \lambda E & x & y & z_1 \\ & y & z_2 & s \\ & & s & t \\ & & & \lambda D \end{pmatrix} \quad \text{of weights} \quad \begin{matrix} 0 & 1 & 2 & 3 \\ & 2 & 3 & 4 \\ & & 4 & 5 \\ & & & 6 \end{matrix} \tag{42}$$

together with a long equation for Bz_2. When $\lambda E \neq 0$, these equations eliminate s, t

Full set of equations

$$\begin{matrix} y^2 - xz_2 + \lambda Es & 4 \\ yz_1 - xs + \lambda Et & 5 \\ xt - z_1^2 + \lambda B & 6 \\ z_1z_2 - ys + \lambda^2 DE & 6 \\ z_1 s - yt + \lambda Dx & 7 \end{matrix} \qquad \begin{matrix} Dx^2 + By + Ez_1t & 8 \\ s^2 - z_2t + \lambda Dy & 8 \\ Bz_2 + Dxy + Est + \lambda DEz_1 & 9 \\ Dxz_1 + Bs + Et^2 & 10 \end{matrix} \tag{43}$$

Notice that if $\lambda = E = 1$ (which I can take wlog) then the first 4 equations express s, t, B, D as simple polynomial expressions in x, y, z_1, z_2, and one checks that the remaining 5 equations then hold identically, so that the variety defined by these equation is just the graph over $\mathbb{A}^4_{\langle x,y,z_1,z_2\rangle}$ of s, t, B, D. Substituting general sextics in x, y, z_1, z_2 for B, D defines a complete intersection $C_{6,6} \subset \mathbb{P}(1, 2, 3, 3)$.

Scrap K3 example: take the hypersurface $X_5 \subset \mathbb{P}(1, 1, 1, 2)$; it is a hypersurface with a $\frac{1}{2}$ orbifold point. If you want to treat it by resolving, you get nonsingular K3 S with fractional divisor $D = B + \frac{1}{2}\Gamma$, where B is ample and $B^2 = 2$, so defines a double cover $S \to \mathbb{P}^2$, and $B \cdot \Gamma = 1$, so maps to a bitangent line.

The case $|2D|$ is in the literature as part of the trigonal dichotomy – the curves in $|2D|$ have a g^2_5, so the image of φ_{2D} is contained in the Veronese cone $\mathbb{P}(1,1,1,2) = P * v_2(\mathbb{P}^2) \subset \mathbb{P}^6$. Instead of being an intersection of quadrics, its ideal contains the 6 quadric cones through $v_2(\mathbb{P}^2)$ and three cubics corresponding to the rendered products $[x_i g_5]$.

The ring $R(X, 2D)$ is the following codimension 4 structure with 9×16 resolution. Take the symmetric matrix

$$M = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{pmatrix} \quad \text{and set} \quad \bigwedge^2 M = 0 \quad \text{and} \quad \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} M = 0. \tag{44}$$

If you give x_i, y_i weight one, this is a projectively Gorenstein 4-fold in straight $\mathbb{P}^8$ contained in the cone $\mathbb{P}^2 * v_2(\mathbb{P}^2)$. It can be viewed as the degeneration $\lambda \to 0$ of the extrasymmetric Pfaffian variety

$$\begin{pmatrix} \lambda y_3 & -\lambda y_2 & x_1 & x_2 & x_3 \\ & \lambda y_1 & x_2 & x_4 & x_5 \\ & & x_3 & x_5 & x_6 \\ & & & y_3 & -y_2 \\ & & & & y_1 \end{pmatrix}. \tag{45}$$

Setting y_i to be quadratic in the x_i gives the canonical curve $C \subset \mathbb{P}^5$ or the second Veronese of the K3 $X_5 \subset \mathbb{P}(1,1,1,2)$. The degeneration with $\lambda \to 0$ in (37) does not explain how to deform it to a general canonical curve or K3 surface. For this, we need to factorise the y_i some more.

Let $f_5(u_1, u_2, u_3)$ be the equation of a nonsingular plane quintic. Every monomial in f contains one of u_1^2, u_2^2, u_3^2.

So we can render $u_1 f, u_2 f, u_3 f$ as

$$x_1 A + x_2 B + x_3 C$$

A similar weighted structure should handle many 2nd Veronese embedding of a hypersurface in $\mathbb{P}(a,b,c)$ or $\mathbb{P}(a,b,c,d)$ with three variables a, b, c of odd weight.

4 Horikawa Dicks case

Surfaces with $p_g = 3$, $K^2 = 4$. Family II_a: assume the general $C \in |K_S|$ is a hyperelliptic curve of genus 5 polarised by the halfcanonical divisor $A =$

$P_0 + g^1_2 + P_\infty$, where P_0, P_∞ are Weierstrass points. Take coordinates on $\mathbb{P}^1$ so that $P_0, P_\infty \mapsto 0, \infty$ and $f_{10}(t_1, t_2)$ gives the other 10 branch points. Write $u\colon \mathcal{O}_C \hookrightarrow \mathcal{O}_C(P_0 + P_\infty)$ and $v\colon \mathcal{O}_C \hookrightarrow \mathcal{O}_C(P_1 + \cdots + P_{10})$ for the constant sections, so $u^2 = t_1t_2$ and $v^2 = f_{10}(t_1, t_2)$. Then $R(C, A)$ is generated by

$$\begin{aligned} x_1 = ut_1,\ \ x_2 = ut_2 &\quad \text{in degree 1,} \\ y_1 = t_1^4,\ \ y_2 = t_2^4 &\quad \text{in degree 2,} \\ z_1 = vt_1,\ \ z_2 = vt_2 &\quad \text{in degree 3,} \end{aligned} \tag{46}$$

and related by

$$\operatorname{rank}\begin{pmatrix} y_1 & x_1 & x_2^2 & z_1 \\ x_1^2 & x_2 & y_2 & z_2 \end{pmatrix} \le 1 \quad \text{and} \quad \begin{aligned} z_1^2 &= [t_1^2 f_{10}], \\ z_1z_2 &= [t_1t_2f_{10}], \\ z_2^2 &= [t_2^2 f_{10}], \end{aligned} \tag{47}$$

where, as before, the brackets [ ] render the right-hand side as sextics in x_i, y_i. The point is to understand the different ways of doing this.

Remark 4.1 Note that $A = g^1_4$ on a curve of $g = 5$ has Brill–Noether number 1, so imposes 1 condition on the moduli of C, and C, A has 11 moduli. The hyperelliptic guy has $2g - 1 = 9$ moduli, and the trigonal guy with $K_C = 2(g^1_3 + P_\infty)$ has 10 moduli. The result for curves is that the two fixed points imposes transversal nonsingular divisorial conditions on C, A.

4.1 Deforming away the base point P_0

The curve C deforms to lose the fixed point P_0, so that $A = P_0 + g^1_2 + P_\infty \mapsto g^1_3 + P_\infty$. It seems elegant to treat this deformation first in terms of the following bigger variety

$$V \subset \mathbb{A}^{10}_{\langle x_1,x_2,c,y_1,y_2,D,z_1,z_2,a,\beta\rangle} \tag{48}$$

defined by

$$\bigwedge^2 \begin{pmatrix} y_1 & x_1 & D & z_1 \\ cx_1 & x_2 & y_2 & z_2 \end{pmatrix} = 0 \quad \text{and} \quad \begin{aligned} z_1^2 &= Ay_1 + bD^2, \\ z_1z_2 &= Acx_1 + bDy_2, \\ z_2^2 &= Acx_2 + by_2^2. \end{aligned} \tag{49}$$

This corresponds to choosing a rendition, and tokenising the features that make possible the subsequent deformation (massage based on hindsight). In detail, any monomial in f_{10} is divisible either by t_1^2 or t_2^6, giving $f_{10} = At_1^2 + bt_2^6$, with $A = A_8(t_1, t_2)$ and $b = b_4(t_1, t_2)$. I multiply by t_1^2, t_1t_2, t_2^2, then substitute

$$(t_1^4, t_1^3t_2, t_1^2t_2^2) \mapsto (y_1, x_1^2, x_1x_2) \tag{50}$$

in the first summand, and

$$(t_1^2t_2^6, t_1t_2^7, t_2^8) \mapsto (x_2^4, x_2^2y_2, y_2^2) \tag{51}$$

in the second. After this, I tokenise x_1^2 as cx_1 and $x_2^2 = D$.

The resulting set of 9 equations has the following two interpretations as unprojections, where I introduce a deformation parameter λ:

$$y_1 \cdot (x_2, y_2, z_2, A) \quad \text{and Tom}_1 \text{ matrix} \quad M_1 = \begin{pmatrix} \lambda & x_1 & D & z_1 \\ & x_2 & y_2 & z_2 \\ & & z_2 & -by_2 \\ & & & Ac \end{pmatrix} \tag{52}$$

and

$$x_2 \cdot (y_1, D, z_1, A) \quad \text{and Tom}_2 \text{ matrix} \quad M_2 = \begin{pmatrix} \lambda c & y_1 & D & z_1 \\ & cx_1 & y_2 & z_2 \\ & & z_1 & -bD \\ & & & A \end{pmatrix}. \tag{53}$$

The two sets of Pfaffians overlap in two equations for y_2z_1 and z_1z_2; putting them together and coloning out a or d or y_2 or z_1 or z_2 gives the "long equation"

$$x_2y_1 = cx_1^2 + \lambda^2bc. \tag{54}$$

Thus the 9 equations are

$$\begin{array}{lll} x_1y_2 = Dx_2 + \lambda z_2, & x_1z_2 = x_2z_1 - \lambda by_2, & z_1^2 + Ay_1 + bD^2, \\ x_2y_1 = cx_1^2 + \lambda^2bc, & y_1z_2 = cx_1z_1 - \lambda bcD, & z_1z_2 + Acx_1 + bDy_2, \\ y_1y_2 = cDx_1 + \lambda cz_1, & y_2z_1 = Dz_2 - \lambda Ac, & z_2^2 + Acx_2 + by_2^2. \end{array} \tag{55}$$

When $\lambda \neq 0$ these eliminate z_2, leaving the codimension 3 variety generated by the Pfaffians of

$$\begin{pmatrix} 0 & x_2 & c & y_2 \\ & x_1^2 + \lambda^2b & y_1 & \lambda z_1 + Dx_1 \\ & & Dx_1 - \lambda z_1 & \lambda^2A \\ & & & -D^2 \end{pmatrix} \quad \text{of weights} \quad \begin{smallmatrix} 0 & 1 & 1 & 2 \\ & 2 & 2 & 3 \\ & & 3 & 4 \\ & & & 4 \end{smallmatrix} \tag{56}$$

4.2 Moving the base point P_∞

A parallel interpretation of the original nine equations (47) allows the other base point P_∞ to move. I keep $x_1^2 = C$ and $y_2^2 = B$ as tokens (instead of factoring them as cx_1 and by_2), but factor the quantities D and A as $D = dx_2$ and $A = ay_1$. This gives

$$\bigwedge^2 \begin{pmatrix} y_1 & x_1 & dx_2 & z_1 \\ C & x_2 & y_2 & z_2 \end{pmatrix} = 0 \quad \text{and} \quad \begin{aligned} z_1^2 &= ay_1^2 + Bdx_1, \\ z_1z_2 &= aCy_1 + Bdx_2, \\ z_2^2 &= aC^2 + By_2. \end{aligned} \tag{57}$$

The μ deformation comes from the unprojection interpretations:

$$y_2 \cdot (x_1, y_1, z_1, B) \quad \text{and Tom}_2 \text{ matrix} \quad M_1 = \begin{pmatrix} \mu & y_1 & x_1 & z_1 \\ & C & x_2 & z_2 \\ & & z_1 & -Bd \\ & & & ay_1 \end{pmatrix} \tag{58}$$

and

$$x_1 \cdot (y_2, C, z_2, B) \quad \text{and Tom}_1 \text{ matrix} \quad M_2 = \begin{pmatrix} \mu d & y_1 & dx_2 & z_1 \\ & C & y_2 & z_2 \\ & & z_2 & -B \\ & & & aC \end{pmatrix}. \tag{59}$$

The two sets of Pfaffians overlap in two equations for y_1z_2 and z_1z_2; coloning out gives the "long equation"

$$x_1y_2 = dx_2^2 + \mu^2 ad. \tag{60}$$

Thus the 9 equations are

$$\begin{aligned} x_1y_2 &= dx_2^2 + \mu^2 ad, & x_1z_2 &= x_2z_1 + \mu ay_1, & z_1^2 &+ ay_1^2 + Bdx_1, \\ x_2y_1 &= Cx_1 + \mu z_1, & y_1z_2 &= Cz_1 - \mu dB, & z_1z_2 &+ ay_1C + Bdx_2, \\ y_1y_2 &= dCx_2 + \mu dz_2, & y_2z_1 &= dx_2z_2 - \mu adC, & z_2^2 &+ aC^2 + By_2. \end{aligned} \tag{61}$$

4.3 Putting together the λ and μ deformations

My λ and μ deformation families depend on choices and assumptions that are a priori incompatible if f_{10} has a nonzero term in $t_1^5t_2^5$. Ignoring this for the moment, assume that $f_{10} = a_4t_1^6 + bt_2^6$. With a little trial and error, one checks that the λ and μ deformations (55) and (61) fit together, somewhat miraculously, with only a single $\lambda\mu$ term in the z_1z_2 equation:

$$\begin{aligned}
x_1y_2 &= dx_2^2 + \lambda z_2 + \mu^2 ad,\\
x_2y_1 &= cx_1^2 + \lambda^2 bc + \mu z_1,\\
y_1y_2 &= cdx_1x_2 + \lambda cz_1 + \mu dz_2,\\
x_1z_2 &= x_2z_1 - \lambda by_2 + \mu ay_1,\\
y_1z_2 &= cx_1z_1 - \lambda bcdx_2 - \mu bdy_2,\\
y_2z_1 &= dx_2z_2 - \lambda acy_1 - \mu acdx_1,\\
&z_1^2 + ay_1^2 + bdx_1y_2 - \lambda bdz_2,\\
&z_1z_2 + acx_1y_1 + bdx_2y_2 - \lambda\mu abcd,\\
&z_2^2 + acx_2y_1 + by_2^2 - \mu acz_1.
\end{aligned} \tag{62}$$

I assert that setting μ or λ to zero gives back the known λ and μ deformation families, and that these equations define a flat deformation over $\mathbb{A}^2_{\langle\lambda,\mu\rangle}$. To check flatness, it is enough to check that the 16 syzygies (65) hold (with $e = 0$).

Finally, I deal with the missing term in $a_5t_1^5t_2^5$ in $f_{10}(t_1,t_2)$ by setting $f_{10} = a_4t_1^6 + et_1t_2 + b_4t_2^6$ where $e = a_5y_1y_2$, and render it as $L_7 \mapsto L_7 + ex_1^2$, $L_8 \mapsto L_8 + ex_1x_2$, $L_9 \mapsto L_9 + ex_2^2$ or

$$\begin{aligned}
t_1^2f_{10} &= ay_1^2 + ex_1^2 + bdx_1y_2,\\
t_1t_2f_{10} &= acx_1y_1 + ex_1x_2 + bdx_2y_2,\\
t_2^2f_{10} &= acx_2y_1 + ex_2^2 + by_2^2.
\end{aligned} \tag{63}$$

The equations become

$$\begin{aligned}
L_1 :\ & x_1y_2 = d(x_2^2 + \mu^2 a) + \lambda z_2,\\
L_2 :\ & x_2y_1 = c(x_1^2 + \lambda^2 b) + \mu z_1,\\
L_3 :\ & y_1y_2 = cdx_1x_2 + \lambda cz_1 + \mu dz_2 - \lambda\mu e\\
&\qquad \equiv c(dx_1x_2 + \lambda z_1) + \mu(dz_2 - \lambda e)\\
&\qquad \equiv d(cx_1x_2 + \mu z_2) + \lambda(cz_1 - \mu e),\\
L_4 :\ & x_1z_2 = x_2z_1 - \lambda by_2 + \mu ay_1,\\
L_5 :\ & y_1z_2 = (cz_1 - \mu e)x_1 - bd(\lambda cx_2 + \mu y_2),\\
L_6 :\ & y_2z_1 = (dz_2 - \lambda e)x_2 - ac(\lambda y_1 + \mu dx_1),\\
L_7 :\ & z_1^2 + ay_1^2 + ex_1^2 + bdx_1y_2 - \lambda b(dz_2 - \lambda e) = 0,\\
L_8 :\ & z_1z_2 + acx_1y_1 + ex_1x_2 + bdx_2y_2 - \lambda\mu abcd = 0,\\
L_9 :\ & z_2^2 + acx_2y_1 + ex_2^2 + by_2^2 - \mu a(cz_1 - \mu e) = 0.
\end{aligned} \tag{64}$$

This set of equations comes neatly from $I_0 = (L_1, L_2, L_4, L_8)$ (unchanged from (62) except for the unsurprising term ex_1x_2 in L_8) by coloning out

$x_1x_2y_1y_2$; its syzygy matrix M is

$$
\begin{array}{ccccccccc}
y_1 & dx_2 & -x_1 & -\mu d & \lambda & . & . & . & . \\
cx_1 & y_2 & -x_2 & \lambda c & . & \mu & . & . & . \\
. & -z_1 & \lambda b & -y_1 & x_1 & . & \mu & . & . \\
\lambda bc & -z_2 & . & -cx_1 & x_2 & . & . & \mu & . \\
-z_1 & \mu ad & . & dx_2 & . & x_1 & . & \lambda & . \\
-z_2 & . & \mu a & y_2 & . & x_2 & . & . & \lambda \\
. & -ay_1 & . & z_1 & . & -\lambda b & -x_2 & x_1 & . \\
by_2 & . & . & z_2 & \mu a & . & . & -x_2 & x_1 \\
 & & & & & & & & \\
. & \mu ae & . & acy_1+ex_2 & . & -by_2 & -\mu ac & z_2 & -z_1 \\
\lambda be & . & . & -bdy_2-ex_1 & -ay_1 & . & -z_2 & z_1 & -\lambda bd \\
acy_1+ex_2 & . & . & -\mu acd & . & z_2 & . & y_2 & -dx_2 \\
-ex_1 & . & -ay_1 & dz_2-\lambda e & . & -z_1 & -y_2 & . & dx_1 \\
. & -ex_2 & -by_2 & -cz_1+\mu e & -z_2 & . & cx_2 & . & -y_1 \\
. & bdy_2+ex_1 & . & \lambda bcd & z_1 & . & -cx_1 & y_1 & . \\
-cz_1+\mu e & . & z_2 & cdx_2 & -y_2 & . & . & . & -\mu d \\
. & dz_2-\lambda e & -z_1 & cdx_1 & . & y_1 & \lambda c & . & .
\end{array}
\tag{65}
$$

(Or ad lib, apply opposite row operation to Rows 1–8 and Rows 9–16, or swap Rows i and $i+8$.) One checks that it satisfies ${}^tMJM=0$ where $J=\left(\begin{smallmatrix}0&I\\I&0\end{smallmatrix}\right)$ is the standard quadratic form, so that essentially the same matrix M also provides the second syzygies, giving the projective resolution

$$
\mathcal{O}_X \leftarrow \mathcal{O} \leftarrow P_1 \leftarrow P_2 \leftarrow P_3 \leftarrow P_4 \leftarrow 0, \tag{66}
$$

with

$$
\begin{aligned}
P_1 &= 2\mathcal{O}(-3)\oplus 2\mathcal{O}(-4)\oplus 2\mathcal{O}(-5)\oplus 3\mathcal{O}(-6),\\
P_2 &= 2\mathcal{O}(-5)\oplus 4\mathcal{O}(-6)\oplus 4\mathcal{O}(-7)\oplus 4\mathcal{O}(-8)\oplus 2\mathcal{O}(-10),\\
P_4 &= \mathcal{O}(-14) \quad\text{and}\quad P_3=\mathcal{H}om(P_2,P_4)=P_2^\vee\otimes\mathcal{O}(-14).
\end{aligned}
\tag{67}
$$

4.4 Set $\lambda \neq 0$ and eliminate z_2

If λ is invertible, L_1 gives $z_2=((x_2^2+\mu^2a)d-x_1y_2)/\lambda$, and (64) boil down to the Pfaffians of

$$
\begin{pmatrix}
\mu & x_2 & c & y_2 \\
 & x_1^2+\lambda^2 b & y_1 & \lambda z_1+dx_1x_2 \\
 & & z_1 & -a(\lambda y_1+\mu dx_1) \\
 & & & \lambda e-dz_2
\end{pmatrix},
\tag{68}
$$

the first 4 of which give

$$\begin{aligned}
&c(x_1^2+\lambda^2 b) - x_2y_1 + \mu z_1,\\
&\mu a(\lambda y_1+\mu d x_1) + x_2(\lambda z_1 + d x_1 x_2) - (x_1^2+\lambda^2 b)y_2,\\
&\mu\lambda e - \mu d z_2 - \lambda c z_1 - c d x_1 x_2 + y_1 y_2,\\
&\lambda e x_2 - d x_2 z_2 + ac(\lambda y_1 + \mu d x_1) + y_2 z_1,
\end{aligned} \tag{69}$$

whereas

$$\mathrm{Pf}_{23.45} = \mu a d x_1 y_1 + d x_1 x_2 z_1 - \lambda^2 b d z_2 - d x_1^2 z_2 + \lambda a y_1^2 + \lambda z_1^2 + \lambda e(x_1^2 + \lambda^2 b). \tag{70}$$

After subtracting dx_1L_4, this is divisible by λ and gives
$L_7 = \mu^2 abd^2 + bd^2x_2^2 + ay_1^2 + z_1^2 + \lambda^2 be + ex_1^2$.
Similarly if μ is invertible, set $z_1 = ((x_1^2+\lambda^2 b)c - x_2y_1)/\mu$

$$\begin{pmatrix}
\lambda & x_1 & d & y_1 \\
 & x_2^2+\mu^2 a & y_2 & \mu z_2 + c x_1 x_2 \\
 & & z_2 & -b(\mu y_2 + \lambda c x_2) \\
 & & & \mu e - c z_1
\end{pmatrix} \tag{71}$$

The two matrixes have a common Pfaffian 12.45, and (after cancelling λ and μ judiciously), their Pfaffians together generate the ideal (62). Check that
$-\lambda L[4] = x_1 \,\mathrm{Pf}_{12.34}(M_\lambda) + \mathrm{Pf}_{12.35}(M_\mu)$,
$\mu L[4] = x_2 \,\mathrm{Pf}_{12.34}(M_\mu) + \mathrm{Pf}_{12.35}(M_\lambda)$.
$\lambda\mu L[7] = \lambda z_1 \,\mathrm{Pf}_{12.34}(M_\mu) + (\lambda^2 b + x_1^2)\,\mathrm{Pf}_{12.45}(M_\lambda) - (y_1 + dx_1)\,\mathrm{Pf}_{12.35}(M_\lambda)$;

4.5 OLDER STUFF

4.6 Dicks matrix

$$\begin{pmatrix} t & x_1 & y_1 & x_2^2 & z_1 \\ & x_2 & x_1^2 & y_2 & z_2 \\ & & -z_1 + t\mu x_1 y_2 & -z_2 - t\lambda x_2 y_1 & -\lambda y_1^2 - \mu y_2^2 \\ & & & -th - x_1 z_1 + x_2 z_2 & -\mu x_1 x_2^2 y_2 + \mu x_2 y_2^2 + x_1 h \\ & & & & -\lambda x_1^2 x_2 y_1 + \lambda x_1 y_1^2 + x_2 h \end{pmatrix} \tag{72}$$

This is a flat deformation of the Type III curve to a Type I curve, not passing via Type II. Setting $t = 0$ gives

$$\begin{pmatrix} 0 & x_1 & y_1 & x_2^2 & z_1 \\ & x_2 & x_1^2 & y_2 & z_2 \\ & & -z_1 & -z_2 & -\lambda y_1^2 - \mu y_2^2 \\ & & & -x_1 z_1 + x_2 z_2 & -\mu x_1 x_2^2 y_2 + \mu x_2 y_2^2 + x_1 h \\ & & & & -\lambda x_1^2 x_2 y_1 + \lambda x_1 y_1^2 + x_2 h \end{pmatrix} \tag{73}$$

Mess around a bit. Change $x_1, x_2 \mapsto -x_1, -x_2$ gives

$$\begin{pmatrix} 0 & x_1 & y_1 & x_2^2 & z_1 \\ & x_2 & x_1^2 & y_2 & z_2 \\ & & z_1 & z_2 & \lambda y_1^2 + \mu y_2^2 \\ & & & x_1 z_1 - x_2 z_2 & \mu x_2 y_2 (x_1 x_2 - y_2) + x_1 h \\ & & & & \lambda x_1 y_1 (x_1 x_2 - y_1) + x_2 h \end{pmatrix} \tag{74}$$

Add a few row-column operations to get

$$\begin{pmatrix} 0 & x_1 & y_1 + x_1 x_2 & x_1^2 + x_2^2 & z_1 \\ & x_2 & x_1^2 + x_2^2 & y_2 + x_1 x_2 & z_2 \\ & & z_1 & z_2 & y_1 A + y_2 B \\ & & & 0 & x_1 C \\ & & & & x_2 C \end{pmatrix} \tag{75}$$

where $C = x_1^2 A + x_2^2 B$. This gives the equations

$$\begin{aligned} z_1^2 &= -x_1^2 C + (y_1 + x_1 x_2)(y_1 A + y_2 B) \\ z_1 z_2 &= -x_1 x_2 C + (x_1^2 + x_2^2)(y_1 A + y_2 B) \\ z_2^2 &= -x_2^2 C + (x_1 x_2 + y_2)(y_1 A + y_2 B) \end{aligned} \tag{76}$$

4.7 Dicks deformation

Putting back in the t, one get

$$\begin{pmatrix} t & x_1 & y_1+x_1x_2 & x_1^2+x_2^2 & z_1+\mu t x_1 y_2 \\ & x_2 & x_1^2+x_2^2 & x_1x_2+y_2 & z_2-\lambda t x_2 y_1 \\ & & z_1+\mu t x_1 y_2 & z_2-\lambda t x_2 y_1 & \lambda y_1(y_1-x_1x_2)+\mu y_2(y_2-x_1x_2) \\ & & & tC & x_1C \\ & & & & x_2C \end{pmatrix} \tag{77}$$

where C is given by

$$C = -\mu x_1^2 y_2 - \lambda x_2^2 y_1 - h \tag{78}$$

4.8 The conclusion

Consider the skew 6×6 matrix

$$\begin{pmatrix} t & x_1 & y_1 & X & z_1 \\ & x_2 & X & y_2 & z_2 \\ & & z_1 & z_2 & A \\ & & & tC & x_1C \\ & & & & x_2C \end{pmatrix} \tag{79}$$

with $\operatorname{wt} t = 0$, $\operatorname{wt} x, y, z = 1, 2, 3$, $\operatorname{wt} X = 2$, $\operatorname{wt} A = 4$, $\operatorname{wt} C = 4$.

This is a standard extrasymmetric format, and is a Tom unprojection from y_1, y_2 or A. One checks by computer algebra that the substitution

$$X, A, C \mapsto \text{general polynomials of weight } 2, 4, 4 \tag{80}$$

of $k[x_0, x_1, x_2, y_1, y_2, z_1, z_2]$ gives nonsingular surfaces with $p_g = 3$, $K^2 = 4$. These are in family III if $t = 0$, and family I if $t \neq 0$.

However, not every surface of type III is in this format.

4.9 Dicks matrix 2

My calculation in (64) is equivalent to the following result in Dicks' thesis.

Dicks gives a mysterious way of getting a 2-parameter deformation of the curve over the ring, allowing the two base points of $A = P_1 + g^1_2 + P_3$ to be deformed away independently. For this, consider the 6×6 skew matrix

$$\begin{pmatrix} 1 & x_1 & y_1 & x_2^2 & z_1 \\ & x_2 & x_1^2 & y_2 & z_2 \\ & & -rz_1 + s^2 x_1 y_2 & -sz_2 - r^2 x_2 y_1 & -ry_1^2 - sy_2^2 \\ & & & -sx_1 z_1 + rx_2 z_2 - rsh & r(x_1 h + x_2 y_2^2) - sx_1 x_2^2 y_2 \\ & & & & -rx_1^2 x_2 y_1 + s(x_1 y_1^2 + x_2 h) \end{pmatrix} \tag{81}$$

Because of the 1, it has only 6 independent Pfaffians $\mathrm{Pf}_{12.ij}$:

$$\begin{aligned} \mathrm{Pf}_1 &= -x_1^3 + s^2 x_1 y_2 + x_2 y_1 - rz_1, \\ \mathrm{Pf}_2 &= -x_1 y_2 + x_2^3 - r^2 x_2 y_1 - sz_2, \\ \mathrm{Pf}_3 &= -x_1 z_2 + x_2 z_1 - ry_1^2 - sy_2^2, \\ \mathrm{Pf}_4 &= x_1^2 x_2^2 - sx_1 z_1 + rx_2 z_2 - y_1 y_2 - rsh, \\ \mathrm{Pf}_5 &= x_1^2 z_1 - sx_1 x_2^2 y_2 + rx_1 h + rx_2 y_2^2 - y_1 z_2, \\ \mathrm{Pf}_6 &= -rx_1^2 x_2 y_1 + sx_1 y_1^2 - x_2^2 z_2 + sx_2 h + y_2 z_1 \end{aligned} \tag{82}$$

Cancelling r and s gives also the following three relations

$$\begin{aligned} Q_0 &:= (z_1 \mathrm{Pf}_1 - y_1 \mathrm{Pf}_3 + sy_2 \mathrm{Pf}_4 + x_1 \mathrm{Pf}_5)/r; \\ &= x_1^2 h + x_1 x_2 y_2^2 + sx_2 y_2 z_2 + y_1^3 - s^2 y_2 h - z_1^2 \\ Q_1 &:= (z_2 \mathrm{Pf}_1 + x_1 y_2 s \mathrm{Pf}_2 - x_1^2 \mathrm{Pf}_3 + x_2 \mathrm{Pf}_5)/r; \\ &= x_1^2 y_1^2 + x_2^2 y_2^2 - rsx_1 x_2 y_1 y_2 + x_1 x_2 h - z_1 z_2 \\ Q_2 &:= (z_2 \mathrm{Pf}_2 - y_2 \mathrm{Pf}_3 + ry_1 \mathrm{Pf}_4 + x_2 \mathrm{Pf}_6)/s; \\ &= x_1 x_2 y_1^2 - rx_1 y_1 z_1 + x_2^2 h - r^2 y_1 h + y_2^3 - z_2^2 \end{aligned} \tag{83}$$

5 From my Dicks paper

Theorem 5.1 *Let X be a surface in case (III). Then $R(X) = R(X, K_X) = k[x_0, x_1, x_2, y_1, y_2, z_1, z_2]/I$, where the ideal I is generated by 9 relations in the rolling factors format described above. In detail:*

Case (III) *Set*

$$\begin{aligned} X_1 &= x_1^2 + a_2x_0x_1 + e_1x_0^2, \\ X_2 &= x_2^2 - b_4x_0x_2 - f_2x_0^2, \end{aligned} \qquad \textit{and} \qquad \begin{aligned} Y_1 &= y_1 + d_1x_0x_1 + i_1x_0^2, \\ Y_2 &= y_2 + d_2x_0x_2 + i_2x_0^2, \end{aligned} \tag{84}$$

and write $A = \begin{pmatrix} y_1 & x_1 & X_2 & z_1 \\ X_1 & x_2 & y_2 & z_2 \end{pmatrix}$. *Then the first* 6 *relations are given by* $\operatorname{rank} A \le 1$, *and the last* 3 *by*

$$\left.\begin{array}{rcllll} z_1^2 &=& y_1^2Y_1 & + x_1^2H & + X_2^2Y_2 \\ z_1z_2 &=& y_1X_1Y_1 & + x_1x_2H & + y_2X_2Y_2 \\ z_2^2 &=& X_1^2Y_1 & + x_2^2H & + y_2^2Y_2 \end{array}\right\} +$$

$$+\, x_0^3 \left\{\begin{array}{ll} +2l_1x_1y_1 & + 2l_2x_1X_2 \\ +l_1(x_1X_1 + x_2y_1) & + l_2(x_1y_2 + x_2X_2) \\ +2l_1x_2X_1 & + 2l_2x_2y_2 \end{array}\right\} +$$

$$+\, x_0^4 \left\{\begin{array}{l} +n_1y_1 + n_3x_1x_2 - n_3b_4x_0x_1, \\ +n_1X_1 + n_3x_2^2 - n_3b_4x_0x_2 \\ \qquad (= n_1x_1^2 + n_3X_2 + n_1a_2x_0x_1), \\ +n_1x_1x_2 + n_3y_2 + n_1a_2x_0x_2. \end{array}\right.$$

Here $H = h + x_0h' + \cdots$ *is a quartic, and the undefined symbols* a_2, e_1 *etc. are just constants in* k *that can be chosen freely, except that* (wake up, this is important!) $n_1e_1 + n_3f_2 = 0$ *must hold; plugging in the definition of* X_1, X_2, *one sees that this is equivalent to the bracketed equality in the last line of the display.*

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6 Appendix: Spinor coordinates

```
Gorenstein codimension 4

Assume 9 x 16 resolution. Write the matrix of first syzygies
as a 16 x 9 matrix M satisfying
 Transpose(M)*J*M = 0,
where J = (0 I \\ I 0) is the standard quadratic form.

Supose that M is thus. The columns of M are 9 vectors that are
```

isotropic, so at most 8 of them are linearly independent. Assume M has rank 8, so that its columns are maximal isotropic subspaces V(M) in the orthogonal Grassmannian OGr(8,16).

Define the spinors of M as the spinor coordinates of V(M). This means the following: let e1.. e8, f1.. f8 be the basis of 16 space so that the quadratic form J is <ei, fj> = delta{i,j}. Making 8 choices between ei and fi for i in 1.. 8 gives 256 subsets of the basis that base maximal isotropic subspace. Corresponding to the two components OGr and OGr^- of the orthogonal Grassmannian, these fall into two sets of 128 subsets called even and odd. An even subset contains evenly many of e1..e8 and f1.. f8, whereas an odd one contains oddly many of each. Write I cup J for these subsets. e.g.
 I = [e1.. e8], J = [] is even;
 I = [e1.. e7], J = [f8] is odd.
J is determined as the complement of I.

Lemma Write M(I) for the 8 x 9 submatrix of M consisting of the rows of I cup J.

(I) If I is even, the submatrix M(I) has odd rank. In particular, its 8 x 8 minors are zero.

(II) If I is odd, the submatrix M(I) has even rank; it must equal 8 for some I because I assume rank M = 8. Assume rank M(I) = 8. Then its 8 x 8 minors factor as Li * QI^2 where

L1.. L9 bases the kernel of M (according to Cramer's rule), and

The square root QI is an element of the ideal (L1.. L9).

Remark

This may explain why the 8 x 8 minors of M are in the cube of the ideal (L1.. L9), and why the rank of M drops by 3 at a time. It may also begin to explain the apparently major differences between (k+1) x 2k resolution for k even and odd.

Background

The symmetry of this setup is the root system D8 of SO(16), namely:

make the same permutation of the 8 elements [e1..e8] and [f1..f8];

swap evenly many ei <-> fi.

The spinor representation of the spin double cover of SO(16) is
  U + wedge^3 U + wedge^5 U + wedge^7 U.
The Pluecker representation wedge^8 (U plus Udual) is the symmetric square of the spinor representation.

See Corti and Reid, Weighted Grassmannians, arXiv:math/0206011 for the general idea in the D5 case. However, Dn for odd and even differ in one key respect: the spinor coordinates are indexed by "half" the vertices of an n-cube. For n even this means that we choose one half of a bipartite graph. For n odd, we divide the cube by antipodal reflection.

Conjectures

The square root QI is sum aIj*Lj with coefficients aIj=a_{I,j} determined

by different mechanisms in individual cases.

NOT? from the complementary rows of M in terms of the Weil group symmetry. ? always part of a syzygy?

(Searching for aIj to express QI as this
sum is possibly the wrong question, like defining Pfaffian as sqrt of determinant, instead of a quantity in its own right.)

There should be a formula for P1.. P9 as quadratic functions of the entries of M.

At present, in individual cases, this is an extended puzzle in the style of Sudoku. See below for a significant particular case treated in Magma.

```
RRgr<la,mu,a,b,c,d,x1,x2,y1,y2,z1,z2,e> :=
PolynomialRing(Rationals(),[0,0,2,2,1,1,1,1,2,2,3,3,4]);
 // PolynomialRing(Rationals(),[1,1,6,4,3,2,3,4,5,7,8,9,10]);
J16 := ZeroMatrix(RRgr,16,16); for i in [1..8] do J16[i,i+8] := 1; J16[i+8,i] :

L := [
-x1*y2+d*(x2^2+mu^2*a)+la*z2,
-x2*y1+c*(x1^2+la^2*b)+mu*z1,
-y1*y2+c*d*x1*x2+la*c*z1+mu*d*z2-la*mu*e,
-x1*z2+x2*z1-la*b*y2+mu*a*y1,
-y1*z2-(mu*e-c*z1)*x1-b*d*(la*c*x2+mu*y2),
-y2*z1-(la*e-d*z2)*x2-a*c*(la*y1+mu*d*x1),
z1^2+a*y1^2+e*x1^2+b*d*x1*y2+la*b*(la*e-d*z2),
z1*z2+a*c*x1*y1+e*x1*x2+b*d*x2*y2-la*mu*a*b*c*d,
z2^2+a*c*x2*y1+e*x2^2+b*y2^2+mu*a*(mu*e-c*z1)
];

M := Matrix([
[y1,d*x2,-x1,-mu*d,la,0,0,0,0],
[c*x1,y2,-x2,la*c,0,mu,0,0,0],
[0,-z1,la*b,-y1,x1,0,mu,0,0],
[la*b*c,-z2,0,-c*x1,x2,0,0,mu,0],
[-z1,mu*a*d,0,d*x2,0,x1,0,la,0],
[-z2,0,mu*a,y2,0,x2,0,0,la],
[0,-a*y1,0,z1,0,-la*b,-x2,x1,0],
[b*y2,0,0,z2,mu*a,0,0,-x2,x1],
[0,mu*a*e,0,a*c*y1+x2*e,0,-b*y2,-mu*a*c,z2,-z1],
[la*b*e,0,0,-b*d*y2-x1*e,-a*y1,0,-z2,z1,-la*b*d],
[a*c*y1+x2*e,0,0,-mu*a*c*d,0,z2,0,y2,-d*x2],
[-x1*e,0,-a*y1,d*z2-la*e,0,-z1,-y2,0,d*x1],
[0,-x2*e,-b*y2,-c*z1+mu*e,-z2,0,c*x2,0,-y1],
```

```
[0,b*d*y2+x1*e,0,la*b*c*d,z1,0,-c*x1,y1,0],
[-c*z1+mu*e,0,z2,c*d*x2,-y2,0,0,0,-mu*d],
[0,d*z2-la*e,-z1,c*d*x1,0,y1,la*c,0,0]
]);

I1 := SetToSequence(Subsets({1..8},1));
I1s := [SetToSequence(m) : m in I1];
I3 := SetToSequence(Subsets({1..8},3));
I3s := [SetToSequence(m) : m in I3];
I5 := SetToSequence(Subsets({1..8},5));
I5s := [SetToSequence(m) : m in I5];
I7 := SetToSequence(Subsets({1..8},7));
I7s := [SetToSequence(m) : m in I7];
Iodd := Sort(I1s) cat Sort(I3s) cat Sort(I5s) cat Sort(I7s);

for I in Iodd[1..5] do I;
  J := [i+8 : i in [1..8] | i notin I];
  X := Determinant(Submatrix(M,I cat J,[1..8])) div L[9];
  if IsSquare(Terms(X)[1]) then Factorization(SquareRoot(X))[1][1];
  else Factorization(SquareRoot(-X))[1][1]; end if;
end for;

The 52 cases hit by a single term are 6 predictable octads
where MI has 7 zeros in Col3,5,6,7,8,9, plus the 4 exceptions
[1,3,7], [3,5,7], [2,6,8], [4,6,8] for which MI has 5 zeros in
Col1 and Col2.

[1,3,7,10,12,13,14,16] -> -y1*L[7]
[3,5,7,9,10,12,14,16] -> -z1*L[7]
[2,6,8,9,11,12,13,15] -> -y2*L[9]
[4,6,8,9,10,11,13,15] -> -z2*L[9]

// a fishing operation to find the coefficients aIi of QI.
// The 8 edges out of a given EVEN octad have related coefficients.

// In this case there are 6 even octads corresponding to
// where cols3, 5, 6, 7, 8, 9 of M have zero. The 8 edges out
// of these give 48 of the entries as predictable monomial times
```

```
// L8, L6, L5, L1, L3, L2 respectively.

Col3 has zeros at [4,5,7,8,9,10,11,14]. Its 8 odd neighbours are given
by swapping one ei and fi, that is, one pair of rows i and i+8, starting
with 4 -> 12.

8 odd neighbours of [4,5,7,8,9,10,11,14] where Col3 has zeros
[5,7,8,9,10,11,12,14] -> M[12,3]*L[8] with M[12,3] = -a*y1 = -a*M[14,8]
[4,7,8,9,10,11,13,14] -> M[13,3]*L[8] with M[13,3] = -b*y2 = -b*M[11,8]
[4,5,8,9,10,11,14,15] -> M[15,3]*L[8] with M[15,3] = z2 = M[9,8]
[4,5,7,9,10,11,14,16] -> M[16,3]*L[8] with M[16,3] = -z1 = -M[10,8]
[1,4,5,7,8,10,11,14] -> M[1,3]*L[8] with M[1,3] = -x1 = -M[7,8]
[2,4,5,7,8,9,11,14] -> M[2,3]*L[8] with M[2,3] = -x2 = M[8,8]
[3,4,5,7,8,9,10,14] -> M[3,3]*L[8] with M[3,3] = la*b = b*M[5,8]
[4,5,6,7,8,9,10,11] -> M[6,3]*L[8] with M[6,3] = mu*a = a*M[4,8]

The complement of [4,5,7,8,9,10,11,14] is [1,2,3,6,12,13,15,16]
where Col8 has zeros:

8 odd neighbours of [1,2,3,6,12,13,15,16] where Col8 has zeros
[2,3,6,9,12,13,15,16] -> M[9,8]*L[3] with M[9,8] = z2
[1,3,6,10,12,13,15,16] -> M[10,8]*L[3] with M[10,8] = z1
[1,2,6,11,12,13,15,16] -> M[11,8]*L[3] with M[11,8] = y2
[1,2,3,12,13,14,15,16] -> M[14,8]*L[3] with M[14,8] = y1
[1,2,3,4,6,13,15,16] -> M[4,8]*L[3] with M[4,8] = mu
[1,2,3,5,6,12,15,16] -> M[5,8]*L[3] with M[5,8] = la
[1,2,3,6,7,12,13,16] -> M[7,8]*L[3] with M[7,8] = x1
[1,2,3,6,8,12,13,15] -> M[8,8]*L[3] with M[8,8] = -x2

8 odd neighbours of [2,5,6,7,9,11,12,16] where Col5 has zeros
[5,6,7,9,10,11,12,16] -> M[10,5]*L[6] with M[10,5] = -a*y1
[2,6,7,9,11,12,13,16] -> M[13,5]*L[6] with M[13,5] = -z2
[2,5,7,9,11,12,14,16] -> M[14,5]*L[6] with M[14,5] = z1
[2,5,6,9,11,12,15,16] -> M[15,5]*L[6] with M[15,5] = -y2
[1,2,5,6,7,11,12,16] -> M[1,5]*L[6] with M[1,5] = la
[2,3,5,6,7,9,12,16] -> M[3,5]*L[6] with M[3,5] = x1
[2,4,5,6,7,9,11,16] -> M[4,5]*L[6] with M[4,5] = x2
[2,5,6,7,8,9,11,12] -> M[8,5]*L[6] with M[8,5] = mu*a
```

```
8 odd neighbours of [1,3,4,8,10,13,14,15] where Col6 has zeros
[3,4,8,9,10,13,14,15] -> M[9,6]*L[5] with M[9,6] = -b*y2
[1,4,8,10,11,13,14,15] -> M[11,6]*L[5] with M[11,6] = z2
[1,3,8,10,12,13,14,15] -> M[12,6]*L[5] with M[12,6] = -z1
[1,3,4,10,13,14,15,16] -> M[16,6]*L[5] with M[16,6] = y1
[1,2,3,4,8,13,14,15] -> M[2,6]*L[5] with M[2,6] = mu
[1,3,4,5,8,10,14,15] -> M[5,6]*L[5] with M[5,6] = x1
[1,3,4,6,8,10,13,15] -> M[6,6]*L[5] with M[6,6] = x2
[1,3,4,7,8,10,13,14] -> M[7,6]*L[5] with M[7,6] = -la*b

8 odd neighbours of [1,2,4,5,6,8,11,15] where Col7 has zeros
[2,4,5,6,8,9,11,15] -> M[9,7]*L[1] with M[9,7] = -mu*a*c
[1,4,5,6,8,10,11,15] -> M[10,7]*L[1] with M[10,7] = -z2
[1,2,5,6,8,11,12,15] -> M[12,7]*L[1] with M[12,7] = -y2
[1,2,4,6,8,11,13,15] -> M[13,7]*L[1] with M[13,7] = c*x2
[1,2,4,5,8,11,14,15] -> M[14,7]*L[1] with M[14,7] = -c*x1
[1,2,4,5,6,11,15,16] -> M[16,7]*L[1] with M[16,7] = la*c
[1,2,3,4,5,6,8,15] -> M[3,7]*L[1] with M[3,7] = mu
[1,2,4,5,6,7,8,11] -> M[7,7]*L[1] with M[7,7] = -x2

8 odd neighbours of [1,2,3,4,5,7,14,16] where Col9 has zeros
[2,3,4,5,7,9,14,16] -> M[9,9]*L[2] with M[9,9] = -z1
[1,3,4,5,7,10,14,16] -> M[10,9]*L[2] with M[10,9] = -la*b*d
[1,2,4,5,7,11,14,16] -> M[11,9]*L[2] with M[11,9] = -d*x2
[1,2,3,5,7,12,14,16] -> M[12,9]*L[2] with M[12,9] = d*x1
[1,2,3,4,7,13,14,16] -> M[13,9]*L[2] with M[13,9] = -y1
[1,2,3,4,5,14,15,16] -> M[15,9]*L[2] with M[15,9] = -mu*d
[1,2,3,4,5,6,7,16] -> M[6,9]*L[2] with M[6,9] = la
[1,2,3,4,5,7,8,14] -> M[8,9]*L[2] with M[8,9] = x1

The "long expressions" for Y = sqrt.
  52 of the Y are product mij * Lk of a single Lk
  64 of the remainder have an expresssion as sum of two such
terms. The 12 remaining terms are most interesting. They are
the 8 singleton [i] and 4 triplets [1,5,6], [1,5,8], [2,3,4],
[2,4,7]. Each of these has two different expressions as a sum
of 3 terms.
```

```
[1,10,11,12,13,14,15,16]
  [ 9, 4, 1 ],
  [ 7, 5, 3 ]
  -> e*y1*L[1] - c*d*z1*L[4] - d*y1*L[9]
  = (b*d*y2+x1*e)*L[3] + (d*z2-la*e)*L[5] - c*d*x2*L[7]  cf col4
(by syzygy e*si1 + d*si13)

[2,9,11,12,13,14,15,16]
  [ 7, 4, 2 ],
  [ 9, 6, 3 ]
  -> y2*e*L[2] + c*d*z2*L[4] - c*y2*L[7]
  = (a*c*y1+x2*e)*L[3] + (c*z1-mu*e)*L[6] - c*d*x1*L[9]
(by syzygy e*si2 + c*si12)

[3,9,10,12,13,14,15,16]
  [ 7, 4, 2 ],
  [ 7, 5, 3 ]
  -> e*z1*L[2] + e*y1*L[4] - c*z1*L[7]
  = la*b*e*L[3] + e*x1*L[5] + (mu*e-c*z1)*L[7]
(by syzygy e*si3)

[4,9,10,11,13,14,15,16]
  [ 7, 4, 2 ],
  [ 9, 8, 5 ]
  -> e*z2*L[2] - b*c*d*y2*L[4] -c*z2*L[7]
  = (a*c*y1+x2*e)*L[5] - (c*z1-mu*e)*L[8] + la*b*c*d*L[9]
(by syzygy -e*si4 + c*si10)

[5,9,10,11,12,14,15,16]
  [ 9, 4, 1 ],
  [ 8, 7, 6 ]
  -> -e*z1*L[1] - a*c*d*y1*L[4] + d*z1*L[9]
  = -(b*d*y2+x1*e)*L[6] - mu*a*c*d*L[7] + (d*z2-la*e)*L[8]  cf col4
(by syzygy -e*si5 + d*si9)

[6,9,10,11,12,13,15,16]
  [ 9, 4, 1 ],
```

```
  [ 9, 6, 3 ]
  -> e*z2*L[1] -e*y2*L[4] - d*z2*L[9]
  = mu*a*e*L[3] + e*x2*L[6] + (la*e-d*z2)*L[9]
(by syzygy e*si6)

[7,9,10,11,12,13,14,16]
  [ 7, 4, 2 ],
  [ 8, 7, 6 ]
  -> a*e*y1*L[2] - e*z1*L[4] - a*c*y1*L[7]
  = -la*b*e*L[6] - (a*c*y1+e*x2)*L[7] + e*x1*L[8]  cf col1
(by syzygy e*si7)

[8,9,10,11,12,13,14,15]
  [ 9, 4, 1 ],
  [ 9, 8, 5 ]
  -> -mu*a*e*L[5] + e*x2*L[8] - (b*d*y2+e*x1)*L[9]
  = e*b*y2*L[1] + e*z2*L[4] - b*d*y2*L[9]
(by syzygy e*si8)

[1,5,6,10,11,12,15,16]
  [ 6, 3, 1 ],
  [ 9, 4, 1 ]
  -> la*e*L[1] - d*y2*L[4] - la*d*L[9]
  = -(d*z2-la*e)*L[1] + mu*a*d*L[3] +d*x2*L[6]
(by syzygy d*si6)

[1,5,8,10,11,12,14,15]
  [ 9, 4, 1 ],
  [ 8, 5, 1 ]
  -> (e*x1+b*d*y2)*L[1] + mu*a*d*L[5] - d*x2*L[8]
  = e*x1*L[1] - d*z2*L[4] - d*x1*L[9]
(by syzygy d*si10)

[2,3,4,9,13,14,15,16]
  [ 5, 3, 2 ],
  [ 7, 4, 2 ]
  -> mu*e*L[2] + c*y1*L[4] - mu*c*L[7]
  = (mu*e-c*z1)*L[2] + la*b*c*L[3] + c*x1*L[5]
```

```
(by syzygy c*si3)

[2,4,7,9,11,13,14,16]
  [ 7, 4, 2 ],
  [ 8, 6, 2 ]
  -> (x2*e+a*c*y1)*L[2] + la*b*c*L[6] - c*x1*L[8]
  = x2*e*L[2] + c*z1*L[4] - c*x2*L[7]
(by syzygy c*si7)

[1,4,5,7,8,10,11,14]
 -> x1*L[8]  col3 has only x1, part of si7
 = a*y1*L[2]-z1*L[4]+la*b*L[6]+x2*L[7]

[1,2,3,4,6,13,15,16]
 -> -mu*L[3] col8 has only mu, not part of any si

[1,2,4,7,8,11,13,14]
 -> -c*x1*L[4]+x2*L[5]              part of si4, cf col7
 = -la*b*c*L[1] + z2*L[2] - mu*L[8]    cf col6

[2,3,4,6,7,9,13,16]
 -> -z2*L[2] - c*x1*L[4]            part of si4, cf col1
 = -la*b*c*L[1] - x2*L[5] - mu*L[8]    cf col5

[1,4,5,6,8,10,11,15]
 -> -z2*L[1]  part of si6           z2 is only entry of col7
 -mu*a*L[3] - y2*L[4] - x2*L[6] - la*L[9]  cf col5, col8

[1,2,4,11,13,14,15,16] -> c*(-y1*L[1] + mu*d*L[4])
  = c*(d*x2*L[2] - x1*L[3] + la*L[5])  (by si1)
  It is identically divisible by c because everything in Col7 is.

[1,2,5,11,12,14,15,16] -> d*(-y2*L[2]-la*c*L[4]);
  = d*(c*x1*L[1] - x2*L[3] + mu*L[6])  (by si2)

[1,2,5,11,12,14,15,16]
 -> -d*y2*L[2]-la*c*d*L[4]         d*(part of si2)
 = c*d*x1*L[1]-d*x2*L[3]+mu*d*L[6]     cf col4
```

```
The 64 hit by a double term are
[
    [ 1, 2, 4 ],
    [ 1, 2, 5 ],
    [ 1, 2, 7 ],
    [ 1, 2, 8 ],
    [ 1, 3, 5 ],
    [ 1, 4, 5 ],
    [ 1, 4, 6 ],
    [ 1, 4, 7 ],
    [ 1, 5, 7 ],
    [ 1, 6, 7 ],
    [ 1, 6, 8 ],
    [ 1, 7, 8 ],
    [ 2, 3, 5 ],
    [ 2, 3, 7 ],
    [ 2, 3, 8 ],
    [ 2, 4, 5 ],
    [ 2, 4, 6 ],
    [ 2, 4, 8 ],
    [ 2, 5, 8 ],
    [ 2, 7, 8 ],
    [ 3, 4, 5 ],
    [ 3, 4, 6 ],
    [ 3, 4, 7 ],
    [ 3, 5, 6 ],
    [ 3, 5, 8 ],
    [ 3, 6, 7 ],
    [ 3, 6, 8 ],
    [ 3, 7, 8 ],
    [ 4, 5, 6 ],
    [ 4, 6, 7 ],
    [ 5, 6, 8 ],
    [ 6, 7, 8 ],
    [ 1, 2, 3, 5, 8 ],
    [ 1, 2, 3, 7, 8 ],
    [ 1, 2, 4, 6, 7 ],
```

```
    [ 1, 2, 4, 7, 8 ],
    [ 1, 2, 5, 7, 8 ],
    [ 1, 2, 6, 7, 8 ],
    [ 1, 3, 4, 5, 6 ],
    [ 1, 3, 4, 6, 7 ],
    [ 1, 3, 5, 6, 7 ],
    [ 1, 3, 5, 6, 8 ],
    [ 1, 3, 5, 7, 8 ],
    [ 1, 3, 6, 7, 8 ],
    [ 1, 4, 5, 6, 7 ],
    [ 1, 4, 6, 7, 8 ],
    [ 1, 5, 6, 7, 8 ],
    [ 2, 3, 4, 5, 6 ],
    [ 2, 3, 4, 5, 8 ],
    [ 2, 3, 4, 6, 7 ],
    [ 2, 3, 4, 6, 8 ],
    [ 2, 3, 4, 7, 8 ],
    [ 2, 3, 5, 6, 8 ],
    [ 2, 3, 5, 7, 8 ],
    [ 2, 3, 6, 7, 8 ],
    [ 2, 4, 6, 7, 8 ],
    [ 3, 4, 5, 6, 7 ],
    [ 3, 4, 5, 6, 8 ],
    [ 3, 4, 6, 7, 8 ],
    [ 3, 5, 6, 7, 8 ],
    [ 1, 2, 3, 4, 6, 7, 8 ],
    [ 1, 2, 3, 5, 6, 7, 8 ],
    [ 1, 3, 4, 5, 6, 7, 8 ],
    [ 2, 3, 4, 5, 6, 7, 8 ]
]

Those only hit by triple terms are
[
    [ 1 ],
    [ 2 ],
    [ 3 ],
    [ 4 ],
    [ 5 ],
```

```
    [ 6 ],
    [ 7 ],
    [ 8 ],
    [ 1, 5, 6 ],
    [ 1, 5, 8 ],
    [ 2, 3, 4 ],
    [ 2, 4, 7 ]
]

TripleTerm := []; for I in [I : I in Iodd | I notin SingleTerm and I notin Doub
J := [i+8 : i in [1..8] | i notin I];
MI := Submatrix(M,I cat J,[1..9]);
X := Determinant(Submatrix(MI,[1..8],[1..8])) div L[9];
if IsSquare(Terms(X)[1]) then Y := SquareRoot(X);
else Y := SquareRoot(-X); end if;
triple := [[i,j,k] : i in [j+1..9], j in [k+1..8], k in [1..7] | Y in Ideal([L[
if triple ne [] then Append(~TripleTerm, I);
end if;
end for;

Cols1 and 2 have only 6 zeros, and the corresponding octads
are not so convincing.

Col1 has zeros at [3,7,9,13,14,16].
Adding 2 of (2 or 10) and (4 or 12) to [3,7,9,13,14,16] gives
two odd octads
[2,3,7,9,12,13,14,16] -> x1*(c*L[7]-e*L[2])  NB c*x1 = M[2,1], e*x1 = M[12,1]
[3,4,7,9,10,13,14,16] -> la*b*(c*L[7]-e*L[2])  la*b*c = M[4,1], la*b*e = M[10,1

Instead, swap one in 6 ways, then add (2 choices) gives
12 more odd octads:

[2,4,7,9,11,13,14,16] -> -c*z1*L[4]+x2*(c*L[7]-e*L[2])
   = b*y2*L[3]-mu*e*L[4]+z2*L[5]+y1*L[9]
[7,9,10,11,12,13,14,16] -> e*z1*L[4] + a*y1*(c*L[7]-e*L[2])
   = (a*c*y1+e*x2)*L[7] + la*b*e*L[6] - e*x1*L[8]

[2,3,4,9,13,14,15,16] -> mu*(c*L[7]-e*L[2]) - c*y1*L[4]
```

```
[3,9,10,12,13,14,15,16] -> z1*(c*L[7]-e*L[2])-e*y1*L[4]

[1,2,3,4,7,13,14,16] -> y1*L[2]
[1,3,7,10,12,13,14,16] -> y1*L[7]

[2,3,4,5,7,9,14,16] -> z1*L[2]
[3,5,7,9,10,12,14,16] -> z1*L[7]

[2,3,4,6,7,9,13,16] -> c*x1*L[4]+z2*L[2]
[3,6,7,9,10,12,13,16] -> z2*L[7] + e*x1*L[4]

[2,3,4,7,8,9,13,14] -> la*b*c*L[4] + b*y2*L[2]
[3,7,8,9,10,12,13,14] -> b*y2*L[7]+la*e*b*L[4]

Similarly, Col2 has zeros at [6,8,10,11,12,15]
Add (1/9) and (5/13) to [6,8,10,11,12,15] to get
[1,6,8,10,11,12,13,15] -> x2*(d*L[9]-e*L[1])
[5,6,8,9,10,11,12,15] -> a*mu*(d*L[9]-e*L[1])

[1,5,8,10,11,12,14,15] -> x1*(d*L[9]-e*L[1]) + d*z2*L[4]
[8,9,10,11,12,13,14,15] -> b*y2*(d*L[9]-e*L[1]) - e*z2*L[4]

[1,5,6,10,11,12,15,16] -> la*(d*L[9]-e*L[1])+d*y2*L[4]
[6,9,10,11,12,13,15,16] -> z2*(d*L[9]-e*L[1])+e*y2*L[4]

[1,2,5,6,8,11,12,15] -> y2*L[1]
[2,6,8,9,11,12,13,15] -> y2*L[9]

[1,3,5,6,8,10,12,15] -> -z1*L[1]+d*x2*L[4]
[3,6,8,9,10,12,13,15] -> z1*L[9]-e*x2*L[4]

[1,4,5,6,8,10,11,15] -> z2*L[1]
[4,6,8,9,10,11,13,15] -> z2*L[9]

[1,5,6,7,8,10,11,12] -> a*y1*L[1]-mu*a*d*L[4]
[6,7,8,9,10,11,12,13] -> a*y1*L[9] - mu*a*e*L[4]

8 odd octads out of [] // 4 of these already covered
```

```
[1,10,11,12,13,14,15,16] -> y1*(d*L[9]-e*L[1])+c*d*z1*L[4]
[2,9,11,12,13,14,15,16] -> y2*(c*L[7]-e*L[2])-c*d*z2*L[4]
[3,9,10,12,13,14,15,16] -> z1*(c*L[7]-e*L[2])-y1*e*L[4]
[4,9,10,11,13,14,15,16] -> z2*(c*L[7]-e*L[2])+b*c*d*y2*L[4]
[5,9,10,11,12,14,15,16] -> -z1*(d*L[9]-e*L[1])+a*c*d*y1*L[4]
[6,9,10,11,12,13,15,16] -> z2*(d*L[9]-e*L[1])+e*y2*L[4]
[7,9,10,11,12,13,14,16] -> a*y1*(c*L[7]-e*L[2])+e*z1*L[4]
[8,9,10,11,12,13,14,15] -> b*y2*(d*L[9]-e*L[1])-e*z2*L[4]

8 odd octads out of [1,2,3,4,5,6,7,8] // the 4 short ones are already covered
[2,3,4,5,6,7,8,9] -> mu*a*L[2]+x2*L[4]
[1,3,4,5,6,7,8,10] -> la*b*L[1]-x1*L[4]
[1,2,4,5,6,7,8,11] -> -x2*L1
[1,2,3,5,6,7,8,12] -> x1*L[1]+la*L[4]
[1,2,3,4,6,7,8,13] -> x2*L[2]-mu*L[4]
[1,2,3,4,5,7,8,14] -> x1*L2
[1,2,3,4,5,6,8,15] -> mu*L1
[1,2,3,4,5,6,7,16] -> la*L2

Ad hoc guys

[1,4,8,10,11,13,14,15] -> z2*L[5];
[2,3,5,6,7,9,12,16] -> M[3,5]*L[6] with M[3,5] = x1

[1,2,4,11,13,14,15,16] -> -c*y1*L[1]+mu*c*d*L[4]
[3,5,6,7,8,9,10,12] -> a*(x1*L[5]+la*b*L[3]-z1*L[2]) = a*y1*L[4]-a*mu*L[7]

I0 := [1..8]; M0 := Submatrix(M,I0,[1..9]); Rank(M0) eq 7;
Determinant(Submatrix(M0,Exclude(I0,6),Exclude([1..8],1)));

Determinant(Submatrix(M0,Exclude(I0,6),Exclude([1..8],1))) eq -L[1]* L[2]*x1*(m
Determinant(Submatrix(M0,Exclude(I0,6),Exclude([1..8],2))) eq L[2]^2*x1*(mu^2*a
Determinant(Submatrix(M0,Exclude(I0,6),Exclude([1..8],3))) eq -L[2]*x1*((mu^2*a
Determinant(Submatrix(M0,Exclude(I0,6),Exclude([1..8],4))) eq L[2]*x1*(mu^2*a+x
Determinant(Submatrix(M0,Exclude(I0,6),Exclude([1..8],5))) eq -x1*L[2]*((mu^2*a
Determinant(Submatrix(M0,Exclude(I0,6),Exclude([1..8],6))) eq L[2]*x1*((mu^2*a+
Determinant(Submatrix(M0,Exclude(I0,6),Exclude([1..8],7))) eq -L[2]*x1*((mu^2*a
Determinant(Submatrix(M0,Exclude(I0,6),Exclude([1..8],8))) eq x1*L[2]*((mu^2*a+
```

```
> K := Basis(Kernel(Submatrix(M0,I0,[1..8])))[1];
K[1] eq x2*L[4] + mu*a*L[2];
K[2] eq -x1*L[4] + la*b*L[1];
K[3] eq x2*L[1];
K[4] eq -la*L[4] - x1*L[1];
K[5] eq mu*L[4]-x2*L[2];
K[6] eq x1*L[2];
K[7] eq mu*L[1];
K[8] eq -la*L[2];

That is, K is
[L1,L2,L4] * Matrix(8,[0,la*b,x2,-x1,0,0,mu,0, mu*a,0,0,0,-x2,x1,0,-la, x2,-x1,
cf L1, L2, L4 related respectively to cols 6,5,8 of M0 read upside down with +

I0 := [1..6] cat [15,16]; M0 := Submatrix(M,I0,[1..9]); Rank(M0) eq 7;
K := Basis(Kernel(Submatrix(M0,[1..8],[1..8])))[1];

K[1]eq -la*c*L[4]-mu*L[6];
K[2] eq -mu*d*L[4] + la*L[5];
K[3] eq -la*c*L[1];
K[4] eq la*L[3];
K[5] eq -mu*L[3];
K[6] eq mu*d*L[2];
K[7] eq la*L[2];
K[8] eq mu*L[1];

L1 relates to Col7, L2 to Col9, L3 to Col8, then
less convincingly L4 to Col4, L5 to Col6 and L6 to Col5

I0 := [1,2,3,4,5,7] cat [14,16]; M0 := Submatrix(M,I0,[1..9]); Rank(M0) eq 7;
K := Basis(Kernel(Submatrix(M0,[1..8],[1..8])))[1];
[z1, la*b*d, d*x2, -d*x1, y1, mu*d, -la, -x1]
These are the nonzero entries of Col9

I124 := [2,3,4,7,9,13,14,16];
M124 := Submatrix(M,I124,[1..9]);
K := Basis(Kernel(M124))[1];
```

```
K[1] eq la*b*c*L[7] - la*b*e*L[2];
K[2] eq c*x2*L[7] - c*z1*L[4] - x2*e*L[2];
K[3] eq e*x1*L[2] -c*x1*L[7];
K[4]eq -c*x1*L[5]+(c*z1-mu*e)*L[2]-la*b*c*L[3];
K[5] eq -y1*L[2];
K[6] eq z1*L[2];
K[7] eq c*x1*L[4]+z2*L[2];
K[8] eq -b*y2*L[2]-la*b*c*L[4];

// doesn't yet work

II := Ideal(L); JJ := Ideal([M[i,4] : i in [1..16]]);
for I in Iodd do I;
J := [i+8 : i in [1..8] | i notin I];
MI := Submatrix(M,I cat J,[1..9]);
X := Determinant(Submatrix(MI,[1..8],[1..8])) div L[9];
if IsSquare(Terms(X)[1]) then Y := SquareRoot(X);
else Y := SquareRoot(-X); end if;
Y in II*JJ;
end for;

> I := Random(Iodd); J := [i+8 : i in [1..8] | i notin I]; MI := Submatrix(M,I
> if IsSquare(Terms(X)[1]) then Y := SquareRoot(X); else Y := SquareRoot(-X); e

  so of these Y are of the form mij*L[k] where ij is an entry of M (MI?) and L[
  one of the relations, with i,j,k determined by some magic.
  Others are sums that are part of a syzygy.
```

Various bits of Magma code are contained in the TeX file after the `\end{document}`