CODY, Report on Scientific Achievements

January 28, 2009

Work package I: Conformal structures, analytic and geometric background and view; Potential theory, analytic tools (A1-A2).

This workpackage is one of the engines of the proposal and includes for example the aim to generalize classical theory about quasiconformal mappings to more general settings, applications of holomorphic motions, removability of sets. Coordinator for this workpackage is partner 3 [University of Helsinki].

Task I.1. Study mappings with unbounded but controlled quasiconformal dilatation.

Here several fundamental properties of mappings of finite distortion were established. The work [7] solves the long-standing open problem, proposed by Iwaniec and Sbordone, of describing the optimal Sobolev-regularity for mappings of exponential distortion, in terms of the degree of integrability of the distortion function. As a bi-product, the work also gives a constructive and modern approach to the theorem of David on the existence and uniqueness of homeomorphic solutions to the degenerate Beltrami equation. This result is fundamental in establishing the quasiconformal surgery methods at non-hyperbolic situations. With the ER-fellow A. Clop at Helsinki, removability results for quasiregular mappings were established in the works [1]-[2] and [4]-[6], and further regularity results, for Sobolev-smooth coefficients, in the work [9]. During this period also the monograph [8] was completed, providing a modern and rather exhaustive exposition of quasiconformal methods required for this task.

Task I.2. To study mappings of finite distortion with random coefficients.

The work [12] presents the first model and a general approach to the Beltrami equation with random coefficients. Here one first defines the Beltrami coefficient on a lattice, with independent random distributions in each cell, and normalizes the mapping as the identity at infinity. Letting the lattice size tend to zero, it is then proven that almost surely the mappings converge to a unique deterministic limit. If the random distributions are symmetric, the limit mapping turns out to be the identity; however it is also shown that for general random distributions the limit is a genuine nonlinear quasiconformal mapping or a mapping of finite distortion.

For the future work and aiming at different non-trivial conformally invariant limit configurations, the goal is to apply the methods developed for correlated coefficient distributions, and for random mappings obtained by compositions of maps in the different lattice levels.

Task. I.3. Singular integrals on general settings including metric measure spaces.

The works [3] and [4] (by Clop, **[ER]** at Helsinki) develop the quaisconformal parametrization of rough hypersurfaces in higher dimensional spaces, and geometric properties and regularity for minimizers of functionals such as the Mumford-Shah functional. The work [13] developes in the n-dimensional setting the singular integrals for measures which are of dimension n-1. Assuming that the measure are supported of graphs of Lipschitz functions it is shown that one has control of the relevant maximal functions and convergence of the truncated integral operators.

List of publications by CODY participants related to Work Package I:

Papers which are published:

[1] K. Astala, A. Clop., J. Mateu, J. Orobitg and I. Uriarte-Tuero, "Improved Painlevé removability for bounded planar quasiregular mappings", in "Harmonic analysis, partial differential equations, and related topics", Contemp. Math. 428 (2008), 1-12. [Journal][**ER**]

[2] K. Astala, A. Clop., J. Mateu, J. Orobitg and I. Uriarte-Tuero, "Distortion of Hausdorff measures and improved Painlevé removability for bounded quasiregular mappings", Duke Math. J. 141 (2008), 539-571. [Journal] [**ER**]

[3] Guy David, Quasiminimal sets for Hausdorff measures, in Recent Developments in Nonlinear PDEs, Proceeding of the second symposium on analysis and PDEs (June 7-10, 2004), Purdue University, D. Danielli editor, 81–99, Contemp. Math. 439, Amer. Math. Soc., Providence, RI, 2007. [Article in Conf Proc]

[4] A. Clop, "Removable singularities for Hölder continuous quasiregular mappings in the plane" Ann. Acad. Sci. Fenn. Ser. A I Math. 32 (2007), 171-178. [Journal] [**ER**]

[5] A. Clop, "Nonremovable sets for Hölder continuous quasiregular mappings in the plane" Michigan Math. J. 55 (2007) no.1, 195-208. [Journal] [**ER**]

[6] A. Clop and X. Tolsa, "Analytic capacity and quasiconformal mappings with $W^{1,2}$Beltrami coefficient", Math. Res. Lett. 15 (2008), 779-793. [Journal] [**ER**]

Papers accepted for publication:

[7] Astala, K., J. Gill, S. Rohde and E. Saksman. Optimal regularity for planar mappings of finite distortion. Ann. Inst. H. Poincaré Anal. Non Linéaire. [Accepted for Publ]

[8] Astala, K., Iwaniec, T. and G. Martin. Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane. Mathematical Series, Princeton University Press, 2009. [Book]

[9] A. Clop, D. Faraco, J. Mateu, J. Orobitg and X. Zhong, "Beltrami equations with coefficient in the Sobolev space $W^{1,p}$", Publ. Mat. [Accepted for Publ] [**ER**]

[10] Guy David, Thierry De Pauw and Tatiana Toro, A generalization of Reifenberg's Theorem in $\mathbb{R}^3$, Geometric And Functional Analysis. [Accepted for Publ]

[11] Guy David, Hölder regularity for almost-minimal sets in $\mathbb{R}^n$, Annales de la Faculté des Sciences de Toulouse. [Accepted for Publ]

Preprints:

[12] Astala, K., Rohde, S., Saksman, E. and T. Tao. Random Quasiconformal Mappings. [Preprint]

[13] Guy David, C^{1+a}-regularity for almost-minimal sets in $\mathbb{R}^n$. [Preprint]

[14] V. Chousionis and P. Mattila, Boundedness and convergence for singular integrals of measures separated by Lipschitz graphs. [Preprint]

Work Package II : Topics in Fractals and Multifractal Analysis (A3, see also A2, P1-2 and themes D)

This workpackage includes to work on a general theory for computing fractal dimensions and multifractal description of various measures associated to dynamical systems. Coordinators for this workpackage are partners 2 [IMPAN, Warsaw] and 8 [TEI, Greece].

Task II.1: Hausdorff dimension and multi-fractal analysis of singular measures.

During the 4 month appointment of an **ESR** (Nina Snigireva) at [TEI], the Hausdorff and the packing dimension of sets of divergence points of self similar measures was investigated, see [2]. The purpose was to extend previous results and to prove that the dimension remains the same if one takes some subsets determined by the normality to several bases. This can be done by the construction of suitable measures. Simultaneously this also gives a simpler proof of previous results and an extension of some other results. It turns out that in this approach the construction of some suitable measures is far easier than by previous methods and one works directly with a concrete measure. In future work, in-homogenous measures will be constructed by suitable contracting similarities. The existence of such measures is well known. In a previous result the Fourier transforms of in-homogenous self-similar measures have been studied and have given relations about the asymptotic behaviour of the Fourier transform. Some constructions, which are given with precise asymptotic behaviour, have the property that the measures are discrete and thus easy to do. The aim of the further work is to see that these constructions are possible with purely continuous measures using well know classes of measures as are the infinite convolution measures.

Task II.2: Building multiscale transforms based on wavelet-type techniques on fractal spaces generated by anomalous scaling laws.

Research undertaken by the team at [TEI] addresses the problem of constructing generalized Riesz Products derived from a multiscale construction, reminiscent of a multiresolution analysis and allowing for the use of generating functions that are either piecewise constants, or trigonometric polynomials as well. This setting includes well known examples of Riesz Products such as the Rademacher and Walsh Riesz Products and recent results about pyramidal Riesz Products as well. The original Rieszs construction was the source of powerful ideas for producing many examples of measures with desired properties (singularity being the goal of the original construction). Being motivated by the development of new measures induced by representations of the Cuntz relations, the main rationale is to build a construction, starting from a set of generating orthonormal functions whose corresponding dilation operators satisfy the Cuntz relations. It then turns out that generalized Riesz Products arise naturally from this construction. The target is to examine in detail the support properties of the corresponding measures induced by these Riesz Products, their fractal properties and compute the Hausdorff dimension of Markovian processes derived from these constructions. A school 'Fractals and multifractal structure, topics in turbulence', on this topic was held in West Macedonia, Greece, from December 2-5, 2008.

Task II.3: Strange attractors and thermodynamical formalism.

See Task V.3 in this report.

List of publications by CODY participants related to the Work Package II (more publications can be found under Work Package V):

[1] Mouratidis Christoforos, Geometric characterization for the Blaschke products, [Preprint]

[2] Antonis Bisbas and Nina Snigireva, Divergence points and normal numbers, [Manuscript in Preparation][**ESR in TEI, Greece**]

[3] S. Smirnov, *Dimension of quasicircles*, Acta Mathematica, to appear.

Work Package III : Iteration of interval and circle maps, and the complexification, weak hyperbolicity and physical measures and Beyond Dimension One

This work package includes also the applications of quasiconformal surgery, holomorphic motions and to deepen the understanding of holomorphic dynamical systems. Coordinators for this workpackage are partners 1 [Warwick, UK] and 4 [CNRS, France].

Task III.1. Physical measures.

The purpose of this project is to consider families of real polynomials, and prove that within such families for Lebesgue almost all parameters the systems have a physical measure. The main problem here is to transfer information in state space to information in parameter space. In the case when one has one critical point the situation has been clarified, see [1], [2] (and also earlier work of Avila). One of the ingredients in the proofs of these results are the combinatorial rigidity of polynomials, see [3] and [4] (which also deals with non-real polynomials with several critical points, provided they are non-renormalisable). A very weak condition for the existence of physical measures was established in [5].

We should also mention the remarkable recent results of Buff and Cheritat on Julia set of positive Lebesgue measure, see [6] and [7].

Task III.2. Size of tongues.

Given a family f_t of circle diffeomorphisms with rotation number rho(t). Determine the size of the tongues associated to rational values of rho(t). The purpose of this project is to generalize the results obtained by Cheritat concerning the asymptotic tail of the analytic parabolic germ through the small divisor approach.

Given a one-periodic analytic function ϕ on the real line, with $-1 < \min \phi' < \max \phi' < 1$, we consider the family of maps $F_t(x) = x + t + phi(x)$ with t a real parameter and the induced circle diffeomorphisms $f_t : R/Z- > R/Z$. Suppose that f_0 has a Hermann ring of modulus m with rotation

number and let p_n/q_n be the approximants and $l(p_n/q_n n)$ the length of the intervals with ratation numbe p_n/q_n. Then $\limsup 1/q_n \log l(p_n/q_n n) \leq \pi m$. This was proved by Kuntal ([ESR] in France) who continues to work on this topic and is aiming to prove $\phi(x) = a/2\pi \sin(2\pi x)$ with $|a| < 1$ that $\lim 1/q_n \log l(p_n/q_n) = \pi m$, and also studies when $a \to 0$ within the family $F_{a,t}(x) = x + t + a/2pi \sin(2\pi x)$.

Task III.3. Higher dimension.

The project *D5: Beyond dimension 1* aims to lay further foundations for a iteration theory in higher dimensions.

CODY has organized a summer school in Göttingen aiming to build a a solid basis for this task. It concentrated on three topics:

1. Geometry and Dynamics on Flat Surfaces
2. Dynamics of Holomorphic Surface Maps
3. Random Graphs and Equivalence Relations

The dynamics of complex maps in higher dimensions has been studied in various contexts: Newton's method (Hubbard and Papadopol), mapping class groups on surfaces (Markovic, solving a conjecture of Thurston affirmatively if the genus of the surface is at least five), relation between Kleinian groups and complex dynamics (Bullet, Freiberger), conformal iterated function systems (e.g. dimension theory) (Biś, Roy, Urbański), hyperbolic skew products (Mihailescu, Urbański), simplicity of Lyapunov spectra (Avila, Viana: Proving the Zorich-Kontsevich conjecture), quadratic differentials (Zorich), degree growth in two-dimensional holomorphic dynamics (Favre, Jonsson), plurisubharmonic functions (Jonsson et al.).

One of the areas were some progress has also been achieved is on the ergodic theoretic foundations for studying higher dimensional dynamics of skew product type. The main question on the existence of conformal measures for a given potential (here fibrewise) has been solved by M. Denker, M. Kifer and M. Stadlbauer, see [9] and [10]. This study also leads to a new type of decomposition of dynamical systems: factorization over orbits (equivalence relations). It also leads to a complete analogue to the non-fibred situation: Good spectral analysis and the existence of absolutely continuous invariant measure having good mixing properties. It is evident that the result

can be used to answer dimension questions (and others) for skew products polynomial maps in $\mathbb{C}^2$, or more generally, for certain holomorphic maps on surfaces. It is evident that this research also lays foundations for project D6.

Weakly dependent random sequences (which arise in the previous context) have been studied with the aim to establish a version of the Erdös-Renyi law. It is expected that it gives some insight into averaging properties along backward orbits.

Another aspect of this aim is make some progress on a no-wandering domain theorem in dimension two. For example, can there be a C^k diffeomorphism which is semi-conjugate to a rotation or an Anosov map on T^2 (where we assume that the fibers of the semi-conjugacy are connected)? In the case that k=0 (and the maps are merely homeomorphisms and semi-conjugate to a rotation), Ferry Kwakkel (**ESR** in Warwick), has given a comprehensive description of what can happen, see [11]. Moreover, if one assumes that the fibers of the semi-conjugacy are discs with bounded geometry, then he has shown this is not generally possible (if the smoothness is at least C^3 and the map is semi-conjugate to an Anosov map, see [13].

Kwakkel has also a result which shows that the Riemannian metric on a torus is determined by combinatorial data (in terms of the focal points), see [12].

List of Publications by CODY participants related to Work Package III

[1] Henk Bruin, Weixiao Shen and Sebastian van Strien, Existence of unique SRB-measures is typical for unimodal families, Annales Scientifiques de l'ENS, 4e série, 39, 2006, 381-414. [Journal]

[2] A. Avila, M. Lyubich and W. Shen, Parapuzzles of the Multibrot set and typical dynamics of unimodal maps. To appear in the Journal of the European Mathematical Society. [Accepted for publication]

[3] A. Avila, M. Lyubich and W. Shen, Combinatorial rigidity for unicritical polynomials, to appear in the Annals of Mathematics. [Accepted for publication]

[4] Oleg Kozlovski and Sebastian van Strien, Local connectivity and quasiconformal rigidity of non-renormalizable polynomials. Preprint Sept 2006, http://arxiv.org/abs/math/0609710, 30 pages. Accepted for publication in the Proceedings of the LMS. [Accepted for publication]

[5] Henk Bruin, Juan Rivera-Letelier, Weixiao Shen and Sebastian van Strien, Large derivatives, backward contraction and invariant densities for interval maps, Inventiones Math. 172, (2008), 509-533. [Journal]

[6], A. Blokh, X Buff, Chéritat and L. Oversteegen, The solar Julia sets of basic quadratic Cremer polynomials. to appear in Erg. Th. and Dyn. Sys. [Accepted for Publ]

[7] X. Buff and A. Chéritat, Quadratic Julia sets with positive area. [Submitted for Publ]

[8] B. Kuntal, [Manuscript in Preparation] **[ESR]**

[9] M. Denker, Y. Kifer and M. Stadlbauer, Conservativity of random Markov systems, Ergodic Th. & Dyn. Sys. 28 (2008),6785. [Journal]

[10] M. Denker, Y. Kifer and M. Stadlbauer Thermodynamic formalism for random countable Markov shifts, Disc. Cont. Dyn. Sys. 22 (2008), 131164. [Journal]

[11] F.H. Kwakkel, Toral automorphisms sem-conjugate to a rotation with wandering domains, [Manuscript in Preparation] **[ESR at Warwcik]**

[12] F.H. Kwakkel, M. Martens and M.M. Peixoto, Rigidity of the torus, [Submitted for Publ] **[ESR at Warwick]**

[13] F. Kwakkel and V. Markovic, Topological entropy and diffeomorphisms of surfaces with wandering domains, [Manuscript in Preparation] **[ESR at Warwick]**

Work Package IV : Geometry of dynamical and parameter space

Coordinators for this workpackage are partners 4 [CNRS, France] and 7 [RUC, Denmark].

Task IV.1. Yoccoz puzzle and para-puzzle in the multi-critical case.

The purpose of this task is to obtain combinatorial and analytical/geometric descriptions of the cubic connectedness locus and corresponding dynamics.The simplest multi-critical maps are the bi-critical (rational) maps. These come in several flavors: Quadratic rational maps have precisely two critical points always and the set of Moebius conjugacy classes form a 2-complex dimensional family of bi-critical dynamical systems on the Riemann sphere. More generally the Moebius conjugacy classes of bi-critical rational maps of degree d¿2 form a 2-complex dimensional sub-family of the space of all degree d dynamical systems on the Riemann sphere. The degree 3 cubic polynomials generically are bi-critical dynamical systems.

Many properties of and ideas for studying any one of these three spaces have generalizations to the other spaces. So far we have mainly made contributions to the study of the two first spaces.

Eva Uhre, a member of partner 7 [RUC, Denmark] has in her thesis [1] studied and modeled the spaces or complex lines of quadratic rational dynamics with a parabolic fixed point. Her work completely classifies the relatively hyperbolic components in these lines. She shows how these relatively hyperbolic components relate to the hyperbolic components of the neighboring complex lines of quadratic rational maps with an attracting fixed point. Her work moreover suggests how to construct puzzles and para-puzzles for those maps. Variants of such (para-)puzzles have been defined and studied for quadratic rational dynamics with a super attracting 2-cycle by Magnus Aspenberg, presently an ER of partner 6 [Kiel] and M. Yampolsky, see [2].

Anja Kabelka, ER at [RUC, Denmark] has for any fixed degree $d > 2$ studied properties of the space of bi-critical rational dynamics. In particular she has studied the structure of these space near the sphere at infinity where the dynamics degenerates to the dynamics of Moebius transformations, see [3]. This paper proves a conjecture by John Milnor from 1993. Anja Kabelka

is moreover preparing a paper [4] on Bounded hyperbolic components of bicritical rational maps, in which she shows, that any hyperbolic component associated with two disjoint attracting cycles of periods at least 2 is bounded. This generalizes a result of Adam Epstein from degree 2 to any degree d¿1.

In addition, Anja Kabelka is preparing a paper [5] on Quadratic rational maps: Dynamical limits and the disappearing limb of the real Mandelbrot set. In this paper Kabelka proves that though all real maps in the ?-limb of the Mandelbrot set M diverges to infinity in moduli space, when the multiplier of the prefered fixed point at infinity is driven to -1 along the real axis. Proper normalizations of the second iterate of those maps converge locally uniformly off the prefered fixed point and its preimage, to real maps of the connectednes locus M^1 in the space of quadratic rational dynamics with a parabolic fixed point of multiplier 1. This proves a special case of a more general conjecture by Adam Epstein that the q-th iterate of quadratic rational maps in the -p/q-limb in the Mandelbrot se converge to maps in M^1 with "corresponding combinatorics", when the multiplier at ∞ converge to $\exp^{i2\pi p/q}$ say radially. Carsten Lunde Petersen [RUC] and Pascale Roesch [CNRS] have published a paper [P-R] in which they transform the Yoccoz para-puzzles into a Combinatorial-Analytic invariant which encodes the Mandelbrot set M. They use this to prove that any two Yoccoz-parameters are connected by a unique Douady-Hubbard regular arc in M. This paper also lays the foundations for comparing M to M^1. This is taken further in [7] in which they have developed a notion of parabolic rays and parabolic puzzles to study the dynamical systems in M^1.

The ER Aspenberg, [8],[10] shows that to every rational Misiurewicz map f one can find a hyperbolic rational map g arbitrarily close to f, and also what the size of the set of Misiurewicz parameters is. He also has some results on how to mate non-renormalizable polynomials, see [9], and some other papers are in preparation.

List of publications of CODY members related to Task IV.1

[1] Eva Uhre, On the Parabolic lines Per1 (e2pip/q) in the Moduli Space of Quadratic Rational Maps, Thesis RUC, 2008. [Monograph]

[2] Magnus Aspenberg, and M. Yampolsky, Mating non-renormalizable quadratic polynomials, arXiv:math/0610343v4. [Submitted for Publ] **[ER]**

[3] Anja Kabelka, Bicritical rational maps: Dynamical limits and ideal limit points of period m curves, accepted for publication in the issue of Journal of Difference Equations and Applications (celebrating the 60-th birthday of Robert Devaney). [Accepted for Publ] **[ER]**

[4] Anja Kabelka, Bounded hyperbolic components of bicritical rational maps, in preparation. [Accepted for Publ] **[ER]**

[5] Anja Kabelka, Quadratic rational maps: Dynamical limits and the disappearing limb of the real Mandelbrot set, in preparation. [Manuscript in Preparation] **[ER]**

[6] C. L. Petersen and P. Roesch, *The Yoccoz Combinatorial Analytic Invariant*, Fields Institute Communications, Volume **53**, 2008. [Journal] **[Two Teams]**

[7] Carsten Lunde Petersen [RUC] and Pascale Roesch [CNRS], *Parabolic Tools*. [Submitted for publication] **[Two Teams]**

[8] M. Aspenberg, Rational hyperbolic maps cluster around Misiurewicz maps, arXiv:0804.1106 [Submitted for Publ] **[ER]**

[9] Magnus Aspenberg, Michael Yampolsky, "Mating Non-Renormalizable Quadratic Polynomials", Comm Math Phys, 2008. [Journal]**[ER]**

[10] Magnus Aspenberg and J. Graczyk, Measure and dimension for semi-hyperolic rational maps of degree 2. [Submitted for Publ]**[ER]**

[11] Magnus Aspenberg, Yet another formula for the Catalan numbers, [Manuscript in preparation] **[ER]**

[12] Magnus Aspenberg, The dual nest, [Manuscript in preparation] **[ER]**

[13] Magnus Aspenberg, Some matings of cubic polynomials. [Manuscript in preparation] **[ER]**

Task IV.2. Newton's method for entire functions

Walter Bergweiler from partner 6 [Kiel, Germany] has in [B] constructed some examples of entire functions where Newton's methods displays unusual behavior. For example, there exists an entire function for which the Fatou set of the associated Newton method has no invariant component:

List of publications of CODY members related to Task IV.2

[1] W. Bergweiler, Newton's method and Baker domains, arXiv:0710.1147v1. [Submitted for Publ]

Task IV.3. Bifurcation of parabolic points

Arnaud's Cheritat from team 4 [CNRS] has defended his habilitation thesis [1] where he explores and extends the correspondence between the dynamics of holomorphic vector fields in $\mathbb{C}$ and iteration of holomorphic maps with parabolic periodic point(s) or near parabolic periodic points. The correspondence has many applications to the study of maps with an indifferent periodic cycle, e.g. bounds of the size of Siegel disks and bounds on the discripancy between the Bruno estimate and the actual size of Siegel disks. It has the potential of many further applications. Xavier Buff [CNRS] and Carsten Lunde Petersen [RUC] have published a paper [B-P] in which they compare the size of Siegel disks in a general 1 complex parameter family of holomorphic maps parametrized by the fixed point eigenvalue to the size of the corresponding Siegel disk in the quadratic family.

List of publications of CODY members related to Task IV.3

[1] A. Cheritat, Sur l'implosion parabolique, la taille des disque de Siegel & une conjecture de Marmi, Moussa et Yoccoz. [Submjtted for Publ]

[2] X. Buff and C. L. Petersen, On the size of linearization domains, Math. Proc. Camb. Phil. Soc. (2008), **145**, 443. [**Two teams**]

Task IV.4, Geometric limits

This task was reported in the CODY report at the end of year 1.

Task IV.5. Study the geometry of dynamical parameterspaces of families of entire (or meromorphic transcendental functions).

In [1] and [2], memers of the Barcelona team study the family by $F_{\lambda,m}(z) = \lambda z^m \exp(z)$ where $m \geq 2$. All functions $F_{\lambda,m}$ have a superattracting fixed point at $z = 0$, and a critical point at $z = -m$. In the parameter planes we focus on the capture zones, i.e., λ values for which the critical point belongs to the basin of attraction of $z = 0$, denoted by $A(0)$. In particular, we study the main capture zone (parameter values for which the critical point lies in

the inmediate basin, $A^*(0)$) and prove that is bounded, connected and simply connected. All other capture zones are unbounded and simply connected. For each parameter λ in the main capture zone, $A(0)$ consists of a single connected component with non-locally connected boundary. For all remaining values of λ, $A^*(0)$ is a quasidisk. On a different approach, we introduce some families of holomorphic maps which serve as a model for $F_{\lambda,m}$, in the sense that they are related by means of quasiconformal surgery to $F_{\lambda,m}$. These results are contained in [1]. On the other hand, using symbolic dynamics we investigate the connection between the dynamics near zero and the dynamics near infinity in the boundary of the immediate basin of attraction of the origin, seeing how polynomial dynamics and exponential combinatorics meet at the boundary of the basin. These results are contained in [2].

In [3] Fagella deals the so-called residual Julia set J_r, defined to be the subset of those points of the Julia set which do not belong to the boundary of any component of the Fatou set. These points are also called buried points of the Julia set and a component of the Julia set which is contained in J_r is called a *buried component*. In this work we survey the most important results related to the residual Julia set for several classes of functions. We also give a new criterion to deduce the existence of buried points and, in some cases, of unbounded curves in the residual Julia set (the so called Devaney hairs). Some examples are the sine family, certain meromorphic maps constructed by surgery and the exponential family.

In [4] Fagella investigates those entire transcendental functions having two singular values: one critical and one asymptotic value, having an invariant Siegel disk. After normalization, and imposing that the asymptotic value has one finite preimage, these functions reduce to the one parameter family $f_a(z) = \lambda a(e^{z/a}(z+1-a) - 1 + a)$, where $\lambda = e^{2\pi i\theta}$, and θ is a fixed Brjuno number. We show that different scenarios that may occur for different parameter values: one of the singular orbits must always accumulate in the boundary of the Siegel disk, while the other one is free. Depending on which orbit is doing what, the Siegel disk might be bounded or unbounded. In the parameter plane, we distinguish different regions in terms of the behaviour of the singular orbits: semihyperbolic components, capture components for the structurally stable parameters. We prove several properties about these different regions. This is work in progress.

In [5]-[8] we are working on a family of rational maps that are singular perturbations of $z^n + c$. These maps are given by $z^n + c + \lambda R(z)$ where λ is a small parameter, $R(z)$ is a suitable rational map and c is a complex

parameter. Mainly, we are interested in the topology and the dynamics on the Julia set. We first study the case when $c = 0$ and consider the family of maps $F_\lambda(z) = z^n + \lambda/z^d$ where $n, d \geq 2$ and λ is small. We investigare the behaviour of the Julia set of F_λ where λ tends to 0, obtaining two very different cases depending on n and d (publication [5]). We also consider different perturbation of z^n with a pole located in a different place. In prepint [7] we consider the complex maps given by $F_{\lambda,a}(z) = z^n + \lambda/(z-a)^d$ where $n, d \geq 1$, λ is small, and the pole a is in the unit circle. We focus on the topological characteristics of the Julia and the Fatou sets of $F_{\lambda,a}$. We also investiagate the case when $c \neq 0$. In this case we consider the family of complex maps given by $F_\lambda(z) = z^n + \lambda/z^n + c$ where $n \geq 2$, λ is small and c is chosen to be the center of a hyperbolic component of the corresponding Multibrot set. In the case when $n \geq 3$ we focus on the structure of the Julia set for a map of this form generalizing a result of McMullen, proving that the Julia set consists of a countable collection of a Cantor sets of closed curves and an uncountable number of points components (publication [6]). In the case when $n = 2$ we obtain a very different situation (preprint [8]), since the Julia set contains Sierpinski carpets.

In [9]-[11] we consider the complex differential equation $z' = f(z)$ where f is an analytic function in $\mathbb{C}$ except, possibly, at isolated singularities. We studied three different problems related to this o.d.e. In the first one we give a unify treatment of well known results and provide new insight into the local normal forms and global properties of the solutions for this family on differential (publication [9]). In the second one we consider a holomorphic differential equation having a center at p, and consider the following perturbation $z' = f(z) + \epsilon R(z, \bar{z})$. We give an integral expression, similar to the Abelian integral, whose zeros control the limit cycles that bifurcate from the periodic orbits of period annulus p. The result is applied to control the simultaneous bifurcation of limit cycles from the two period annuli of $z' = iz + z^2$ after a polynomial perturbation (publication [10]). In the third one we consider a smooth planar autonomous differential equation having a period annulus $\mathcal{P}$. We present a new criterion to ensure that the period function has at most one critical period on $\mathcal{P}$ (preprint [11]).

List of publications by CODY participants related to Work package IV.5

[1] N. Fagella and A. Garijo, The Parameter Planes of $\lambda z^m \exp(z)$ for $m \geq 2$, Comm. Math. Phys., **273** (2007), pp. 755–783. [Journal]

[2] A. Garijo, X. Jarque and M. Moreno Rocha, *Joining polynomial and exponential combinatorics for some entire maps.* [Submitted for publication]

[3] P. Dominguez, N. Fagella, Residual Julia sets of rational and transcendental functions, In Transcendental dynamics and complex analysis, Cambridge University Press, 2007. [Chapter in Book]

[4] R. Berenguel, N. Fagella, An entire transcendental family with an invariant Siegel disk and two singular values. [Submitted for publication]

[5] R. L. Devaney and A. Garijo,*Julia sets coverging to the unit disk* Proc. Amer. Math. Soc., **136** (2008), pp-981–988. [Journal]

[6] P. Blanchard, Robert L. Devaney, A. Garijo and E. Russell,*A Generalized version of the McMullen Domain* Int. J. Bifurcation and Chaos, **18** (2008), pp. 2309–2318. [Journal]

[7] A. Garijo and S. Marotta,*Singular perturbation of z^n with a pole on the unit circle.* Accepted for publication in the J. of Difference Equations and Applications. [Accepted for Publ]

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Work Package V: Hausdorff measure and dimension. Limit sets of Kleinian groups and relations. Iterated Function Systems. (D3,D4,D6)

This workpackage includes questions about Lebesgue measure and conformal measures for Julia sets. In spite of recent progress, much remains unknown about the dimension, measure and conformal measures of Julia sets and Kleinian groups.

Coordinators for this workpackage are partners 6 [Kiel, Germany] and 2 [IMPAN, Warsaw].

Hausdorff dimension and Lebesgue measure of Julia sets: transcendental functions

The purpose of [1] is to extend some results for entire functions to meromorphic functions with a direct or a logarithmic singularity over infinity. In order to this, a new version of Wiman-Valiron theory is developed. For example, it is proved that if f is a meromorphic function with a logarithmic singularity over infinity, then the Hausdorff dimension of the Julia set of f is strictly greater than 1.

The growth of an entire function in the Eremenko-Lyubich class B is related to the Hausdorff dimension of its Julia set. In [2] it is shown that if $\log\log\log|f(z)| < (q+o(1))\log\log|z|$, then the Hausdorff dimension of the Julia set is at least $1+1/q$. This estimate is best possible.

In [3], the authors prove that the boundary of an attracting periodic (period larger than 1) basin for an exponential map has Hausdorff dimension greater than 1 and less than 2. Moreover, the set of points in the boundary which do not escape to infinity has Hausdorff dimension (hyperbolic dimension) greater than 1, while the set of points in the boundary which escape to infinity has Hausdorff dimension 1. The latter differs from Karpinska's result on the fixed point basin.

In [4], Dobbs (ER in Warsaw) shows that for entire maps of the form $z \mapsto \lambda\exp(z)$, such that the orbit of 0 is bounded and such that Lebesgue almost every point is transitive, no absolutely continuous invariant probability measure can exist (though a σ-finite measure exists). This answers a long standing open problem (independently the result was obtained by J.Kotus and G.Swiatek).

In [5], Irene Inoquio (ESR in Warsaw) studies non-compact invariant subsets of the full shift space on a countably infinite number of symbols that are equipped with a metric which does not necessarily extend to the shift. For weakly Hölder continuous potentials she defines the topological pressure and constructs conformal and invariant measures. The main motivation is that these subsets encode the dynamics of some transcendental meromorphic functions. She applies these notions to the system $E_\lambda(z) := \lambda \exp(z)$ where $z \in C$, $0 < \lambda < 1/e$, and to the set of hair endpoints.

V.2. Invariant and conformal measures

(compare Task III.1 for real maps)

a) weak hyperbolicity, inducing, infinite IFS, phase transition, Zeta-function

In [6], the authors consider the thermodynamic formalism of a complex rational map f of degree at least two, viewed as a dynamical system acting on the Riemann sphere. More precisely, for a real parameter t they study the (non-)existence of equilibrium states of f for the potential $-t \ln |f'|$, and the analytic dependence on t of the corresponding pressure function. They give a fairly complete description of the thermodynamic formalism of a rational map that is "expanding away from critical points" and that has arbitrarily small "nice sets" with some additional properties. The results apply in particular to non-renormalizable polynomials without indifferent periodic points, infinitely renormalizable quadratic polynomials with a priori bounds, real quadratic polynomials, topological Collet-Eckmann rational maps, and to backward contracting rational maps. As an application, for these maps they describe the dimension spectrum of Lyapunov exponents, and of pointwise dimensions of the measure of maximal entropy, and obtain some level-1 large deviations results. Puzzles construction contributes to Task IV.1.

In [7] Neil Dobbs (ER at Warsaw) studies ergodic properties of rational maps, generalising the work of F. Ledrappier. A new construction allows for simpler proofs of stronger results. Very general conformal measures are considered. Equivalent conditions are given for an ergodic invariant probability measure with positive Lyapunov exponent to be absolutely continuous with respect to a general conformal measure. If they hold, the author can construct an induced expanding Markov map with integrable return time which

generates the invariant measure.

In [8], Neil Dobbs (ER at Warsaw) develops non-invertible Pesin theory for a new class of maps called cusp maps. These maps may have unbounded derivative, but nevertheless verify a property analogous to $C^{1+\epsilon}$. We do not require the critical points to verify a non-flatness condition, so the results are applicable to $C^{1+\epsilon}$ maps with flat critical points. If the critical points are too flat, then no absolutely continuous invariant probability measure can exist. This generalises a result of Benedicks and Misiurewicz.

In [9] , Neil Dobbs (ER at Warsaw) studies the thermodynamical formalism for renormalisable maps of the interval and the natural potential $-t \log |Df|$. Multiple and indeed infinitely many phase transitions at positive t can occur for some quadratic maps. All unimodal quadratic maps with positive topological entropy exhibit a phase transition in the negative spectrum.

b) Cantor bouquets

For a report on this topic, see Task V.1.

c) Fat Julia for rational maps

For a report on this topic, see tasks III and IV.

d) Continuity of Hausdorff Dimension, Julia-Lavaurs sets and parabolic explosion

This topic is related to Task IV.3.

In [10], the ER Jaksztas considers the following problem. Let $d(c)$ denote the Hausdorff dimension of the Julia set J_c of the polynomial $f_c(z) = z^2 + c$. The function $c \mapsto d(c)$ is real-analytic on the interval $(-3/4, 1/4)$, which is included in the main cardioid of the Mandelbrot set. It is shown by Havard and Zinsmeister, that the derivative $d'(c)$ tends to $+\infty$ as fast as $(1/4 - c)^{d(1/4)-3/2}$ when $c \nearrow 1/4$. Under numerically verified assumption $d(-3/4) < 4/3$, the author prove that $d'(c)$ tends to $-\infty$ as $-(c+3/4)^{3d(-3/4)/2-2}$ when $c \searrow -3/4$.

In [11], the ER Jaksztas considers continuity of the Julia-Lavaurs set: Let $f_0(z) = z^2 + 1/4$. Denote by E_0 the set of parameters $\sigma \in C$, for which the critical point 0 escapes in one step from the filled-in Julia set $K(f_0)$, by the Lavaurs map g_σ. The author proves that if $\sigma_0 \in \partial E_0$, then the Hausdorff

dimension HD of Julia-Lavaurs set J_{0,σ_0} is continuous at σ_0 as the function of the parameter $\sigma_0 \in \overline{E}_0$ if and only if $HD(J_{0,\sigma_0}) \geq 4/3$. Since the condition $HD(J_{0,\sigma_0}) > 4/3$ is satisfied on a dense set of parameters which correspond to preparabolic points, the lower semicontinuity implies the continuity of $HD(J_{0,\sigma_0})$ on an open and dense subset of ∂E_0.

In [12] the hyperbolic Hausdorff dimension d of Julia set for $z \mapsto \lambda(\exp(z)-1)$ is studied. For $\lambda = 1$ the function has a parabolic fixed point. For $\lambda > 1$ it has a sink.. In previous papers the authors (with A.Zdunik) proved real analytic dependence of d on λ on $[1, \infty)$ adn the continuity at $\lambda = 1$ from the right. Here they prove the differentiability at 1 and $d'(1^+) = 0$. They provide also more precies aysmptotics.

V.3. IFS with overlaps, defect in parameters, non-uniqueness of dimension equilibria. (compare II.3)

In [13], the authors investigate properties of the zero of the subadditive pressure used by Falconer, Barreira and Zhang to estimate the box and Hausdorff di- mension of a non-conformal attractor. In the conformal case, and in Falconer's 1-bunched non-conformal case, the contraction rates satisfy bounded distortion and so this zero insensitive to where on each cylinder the contraction is evaluated. In [13] non-linear n-dimensional IFS with triangular maps are studied which do not satisfy bounded distortion but do exhibit the same insensitivity. Moreover this result is the generalization of the result of Karoly Simon and Anthony Manning, who proved this in two dimension. This is also a generalization of K. Falconer and Yun Miao's result, They gave a formula to calculate the Hausdorff-dimension of self-affine sets generated by upper-triangular matrices. The authors notice also some examples whose Hausdorff -dimension is calculable.

In [14], authors show that the L-Y formula is applicable for diagonally self-affine IFSs in R^n by the original methods of Ledrappier and Young. Moreover they show an example to calculate the Hausdorff-dimension of the invariant set which was not known. In this paper there is also a result to prove absolute continuity for the invariant measure of the $\{ax, ax+c, ax+1\}$ IFS for Lebesgue almost every a and c.

In [15], the authors consider the local dimension spectrum of a weak Gibbs measure on a C^1 non-uniformly hyperbolic system of Manneville- Pomeau type. They present the spectrum in three ways: using invariant measures,

uniformly hyperbolic ergodic measures and equilibrium states. They also prove analyticity of the spectrum under additional assumptions. All three presentations are well known for smooth uniformly hyperbolic systems.

In [16], the authors study the dimension spectrum $F(a)$ of Lyapunov exponents a for a general rational map f on the Riemann sphere. In particular as F is related to Legendre transform of the pressure function of $-t \log |f'|$ the analyticity proved in [6] for a large class of functions f, yields the analyticity of F. Also results on sets of irregular points (lower Lyapunov exponent different from the upper) are provided.

In [17], the authors prove the convergence of the velocity field for passive tracer.

In [18], the authors prove that for random set obtained from a percolation satisfying some condition (for Mandelbrot percolation this condition is equivalent to $p > 1/n$, i.e. to the Hausdorff dimension of the random set being greater than 1) with probability 1 the projection of the random set on every line contains an interval.

List of publications of CODY members related to Work Package V

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[4] Neil Dobbs "Non-existence of absolutely continuous invariant probabilities for exponential family". [Submitted for Publ][**ER**]

[5] Irene Inoquio "Thermodynamic formalism for some transcendental maps, symbolic dynamic outlook". [Submitted for Publ][**ESR**]

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[8] Neil Dobbs "On cusps and flat tops", arXiv:0801.3815. [Submitted for Publ] [**ER**]

[9] Neil Dobbs "Renormalisation-induced phase transitions for unimodal maps" Submitted. arXiv:0712.3023v2. [Submitted for Publ] [**ER**]

[10] L. Jaksztas, "On the derivative of the Hausdorff dimension of the quadratic Julia set" [Submitted for Publ] [**ER**]

[11] L. Jaksztas "Continuity of the Hausdorff dimension of the Julia-Lavaurs set", [Submitted for Publ] [**ER**]

[12] G. Havard, M. Urbanski, M. Zinsmeister "Variations of Hausdorff Dimension in the exponential family", [Submitted for Publ]

[13] Balazs Barany, Karoly Simon "Sub-additive pressure for IFS with triangular maps", [Submitted for Publ] [**ESR**]

[14] Balazs Barany, Karoly Simon "Ledrappier-Young formula for diagonally self-affine IFS and an application", [Submitted for Publ] [**ESR**]

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[17] Tomasz Szarek, M. Ślęczka, M. Urbański, "On stability of some Markov semigroups" submitted to J. Statistical Physics. [Submitted for Publ]

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Work Package VI: Scaling limits, conformal invariance and universality. Infinite dimensional systems and Turbulent transport (P1-P3).

This workpackage is coordinated by partners 9 [UNIGE, Switzerland] and 4 [CNRS, France].

During the first two years of the CODY project, three recruited researchers worked on the Work Package VI.

- Konstantin Izyurov, ESR at Geneva node for 12 months in 2007-8, started an ambitious program around tasks VI.1,2,3 and continues his research supported from another grant.
- István Prause, ER at Geneva node for 6 months in 2008, worked on interrelations between Work Packages VI, V, II, I, closest to the task VI.1. He continues his research in the Helsinki node, supported by another grant.
- Hugo Duminil started his ESR position at Geneva node in Ostober 2008, and works on the tasks VI.1 and VI.3, already achieving some progress.

Related research was conducted by several other researchers affiliated with the CODY network.

Work of ESR Konstantin Izyurov: tasks VI.1, VI.2, VI.3

Work of Izyurov during his 12 months stay (2007-08) concerned a number of subjects in the area of two-dimensional statistical mechanics. In 80-s, physicists predicted that there are a lot of lattice models, such as percolation, Ising model, Potts model etc., whose limits (as the lattice mesh goes to zero) are conformally invariant at criticality. Recently, there was some progress in understanding this phenomena, basically due to the SLE machinery. For now, conformal invariance and convergence of interfaces to SLE curves are proven for percolation (Smirnov, 2001 [Smi01]), uniform spanning trees and loop-erased random walks (Lawler-Schramm-Werner, 2001 [LSW04]), harmonic explorer (Schramm-Sheffield [SS05]) and level lines of gaussian free field (Sheffield [SS)6]), Ising model (Smirnov [Smi07], Chelkak-Smirnov [CS08]). In 1999, Kenyon [Ken00] proved the conformal invariance

of severalaspects of the double dimer model, but in the latter case the convergence to SLE is still open.

However, there are predictions made by physicists in this area which intriguingly resist all attempts of rigorous proof. On can mention the Potts model, percolation on quadrangular lattice (or any other but the triangular one), etc.

In this direction Izyurov was attempting to prove conformal invariance of the 4-clockwork Potts model. This model might be easier than other Potts or Z_n models, because in some sense this is a square of the Ising model (for which the conformal invariance is completely understood).

In all above-mentioned cases, the proof of conformal invariance goes as follows. One considers certain observable in a particular model (that is, expectation of certain function of the state of the system and a point on the lattice). One proves that this observable is discretely harmonic or discretely analytic, with some transparent boundary conditions. The existence of such an observable usually follows from some combinatorial relations hidden in the model. Then, one argues that this observable converges as the lattice mesh goes to zero to an analytic function with the same boundary conditions, which, in turn, is conformally invariant.

There are many generalisations of the Ising model where this proof fails at one and the same point. Namely, the analogues of the Ising model observable exist for Potts and Z_n models (Smirnov, Cardy-Riva [RC], Cardy-Radjabpur [RS07], Cardy-Ikhlef [IC08]), but in the latter case they only satisfy a half of Cauchy-Riemann equation (let say, these are no more analytic functions, but divergence-free vector fields). Certainly, this is not enough to prove conformal invariance, since even on the discrete level there is not enough relations to determine the observable.

The 4-clockwork Potts model with free boundary values is the superposition of two independent Ising model. So, one may expect to be able to prove more relations than in other cases. Indeed, one can obtain as many relations as one needs to determine the observable, but the cost is that now equations are nonlinear (of the type "F is a square of discretely holomorphic function", which is, by the contrast to the continuous case, not equivalent to "F is holomorphic"). Moreover, the boundary conditions are quite complicated.

Izyurov also found a number of new divergence-free observables in FK-Potts models, but for now we are unable to employ them for the proof of conformal invariance.

The other topic Izyurov has been working on is 2-dimensional quantum

gravity. Recently Duplantier and Sheffield [DS08] constructed a family of random measures in planar domains, which are conjectured to be scaling limits of the vertex-counting measure in large random triangulations (more precisely, its pullback after uniformization of the triangulation to a planar domain). For this family of measures, they formulated and proved a variant of the KPZ (Knizhnik-Polyakov-Zamolodchikov) formula. But the triangulations are not only measure spaces, they are metric spaces. The properties of their scalig limits (as metric spaces) vere recently studied by Le Gall [LG06], Le Gall-Poulin [LGP06] et al. So, it is natural to ask whether the spaces constructed by Duplantier and Sheffield make sense as metric spaces, and whether we can prove some of its properties (e.g. KPZ formula) in this setting.

The difficulty of this question is that these spaces have an element of riemanian metric given by an exponential of a (random) distribution - namely, multiple of the gaussian free field. So, one needs to approximate that distribution by regular functions and then pass to the limit. Even in somewhat easier case of multiplicative cascades (Schramm, Benjamini [BS08]) we are still unable to prove existence of non-trivial limiting space, nor even determine the correct scaling (the rate of growth of the diameters of the approximating spaces).

Izyurov will stay at Geneva University for the remaining time of the CODY project (supported by other sources) and will keep working on the topics mentioned above.

Work of ER István Prause: task VI.1 in relation to V, II, I

During his 6-months stay in 2008 at the University of Geneva, Prause was working in the area of quasiconformal mappings, holomorphic motions, dimensional distortion of conformal maps which lies in the intersection of the Work Packages VI, V, II, I.

The goal of his research, done in collaboration with K. Astala and S. Smirnov is two-fold. One motivation is a finer understanding of *quasiconformal distortion* and a second one is a connection to *multifractality of harmonic measure*.

The starting point of dimensional properties of harmonic measure is Makarov's theorem [Mak85], which says that on any simply connected planar domain, harmonic measure is supported on a set of Hausdorff dimension one, and puts no mass on a set of dimension less than one, in short $\dim \omega = 1$.

Multifractal analysis of harmonic measure [Mak99] seeks to describe its finer structure by measuring how much deviation is possible from $\dim \omega = 1$ in terms of the local dimension of the measure. One defines the *multifractal spectrum* as

$$f_\Omega(\alpha) = \dim\{z \in \partial\Omega : \omega(B(z,r)) \approx r^\alpha,\ r \to 0\},$$

Here, the only restriction on the scaling exponent α is Beurling's estimate $\alpha \geq 1/2$. Looking for extremal phenomena we define the *universal spectrum* as supremum of individual spectra over all simply connected domains

$$f(\alpha) = \sup_{\Omega \text{ sc.}} f_\Omega(\alpha) \quad \alpha \geq 1/2.$$

In the simply connected case, harmonic measure is the image of length measure on the unit circle under the uniformizing conformal map. It is because of this connection that many classical problems of complex analysis can be formulated in terms of the $f(\alpha)$-spectrum. Examples include Brennan's conjecture, Carleson-Jones conjecture on the coefficient decay of bounded univalent functions or the Hölder domain conjecture. The *universal spectrum conjecture* (sometimes called the Brennan-Carleson-Jones-Krätzer conjecture) incorporates all of these and can be stated as (cf. [Jon05])

$$f(\alpha) = 2 - \frac{1}{\alpha}. \tag{U}$$

In order to understand the phenomenon how Hausdorff dimension changes under a quasiconformal map, Astala [Ast94] used the theory of *holomorphic motions* and techniques from *thermodynamic formalism*.

In the forthcoming work [APS], we use similar techniques to relate this phenomenon to multifractality of harmonic measure. We build a holomorphic motion parametrized by two complex parameters, varying the conformal structure independently on the two sides of the domain. Abundance of symmetries and the interaction of two sides allowed us to obtain some partial results; conjecturally sharp multifractal estimates for a parameter-range in some subclass of domains (quasidisks). In a similar fashion, full multifractal estimate for the (quasisymmetric) welding was developed. One sharp result of this type says, if $\phi\colon \mathbb{D} \to \mathbb{C}$ is a univalent map with K-quasiconformal extension then ϕ' is in $L^p(\mathbb{D})$ up to $p < 2(K+1)/(K-1)$. This corresponds via Sobolev embedding to the well-known improvement of Hölder exponents

from $1/K$ to $2/(K+1)$. The above mentioned results are among the few where conjecturally sharp estimates are established, the first of these being Smirnov's $1+k^2$ bound on the dimension of quasicircles [Smi07]. In the opposite direction, making use of Makarov's theorem we obtained the following quasiconformal compression result: a k-quasiconformal map cannot compress a rectifiable set of positive length lower than to dimension $1-k^2$. An interesting feature of the proofs shows that the sharpness of $1+k^2$ estimate would imply the sharpness of many of our estimates, in particular, this would provide domains with large spectrum.

The complex analytic approach we pursue naturally takes into account *rotational* and *two-sided* effect of the boundary. This is described by the three-parameter spectrum, formally given as

$$\begin{aligned} f(\alpha_-,\alpha_+,\gamma) \;&= \dim\{z : \omega_- B(z,r) \approx r^{\alpha_-}, \omega_+ B(z,r) \approx r^{\alpha_+}, \\ &\qquad \text{boundary rotation } \gamma \text{ at } z\}, \end{aligned}$$

where ω_- and ω_+ denote the harmonic measures from the two sides and the boundary rotation parameter γ describes the rate of spiraling (cf. [Bin97]). Beurling's estimate in its full form describes the relation these parameters have to satisfy for every Jordan curve boundary,

$$\frac{1}{\alpha_-}+\frac{1}{\alpha_+} \le \frac{2}{1+\gamma^2}.$$

We believe the natural extension of (U) to this three-parameter setting is the following *universal bound*

$$f(\alpha_-,\alpha_+,\gamma) \le \frac{2-(1+\gamma^2)\left(\frac{1}{\alpha_-}+\frac{1}{\alpha_+}\right)}{1-\frac{1+\gamma^2}{\alpha_-\alpha_+}}.$$

The research is ongoing and we will communicate our findings in a future publication.

Work of ESR Hugo Duminil: tasks VI.1, VI.3

Hugo Duminil has begun his 12 months stay at the University of Geneva in November 2008. His initial research program is concerned with the phase transitions occuring in 2D lattice models.

As described above, we cannot yet establish the conjectured conformal invariance for majority of the models at criticality. However, we were able to partially establish discrete holomorphicity of certain observables. For such models, it appears that the behavior in the subcritical phase could be described by the so-called massive holomorphic functions.

The goal of the project is to explore more deeply connections between subcriticality (for instance exponential decay of correlations) and massive holomorphicity. The project is to prove some properties of lattice models (mainly connected to the expected conformal invariance) by studying some analytic structures connected to the model.

In particular, we plan to use the available partial knowledge, to rigorously establish the conjectured values of critical temperatures. Partial progress was already achieved, leading to a simplified derivation of the critical temperature in the Ising case.

List of Publications of CODY participants related to Work Package VI

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B. Duplantier and S. Sheffield: Liouville Quantum Gravity and KPZ, arXiv:0808.1560 [Preprint]

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