LMS/EPSRC Short Instruction Course. Methods of non-equilibrium Statistical Mechanics in Turbulence.

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1 Aims of the course.

Understanding of turbulence is one of the most challenging problems of modern mathematical and theoretical physics. Turbulence can be described as a chaotic, highly non-equilibrium state of a non-linear physical system. Defined this way, turbulence embraces a broader class of examples than the chaotic Navier-Stokes flow, - a system for which the concept of turbulence was originally introduced and developed. In particular, turbulence appears as far-from-equilibrium states in plasmas, solids, Bose-Einstein condensates and even in nonlinear optics. The characteristic feature of turbulence is the presence of strong interaction between many degrees of freedom, which renders most attempts of perturbative treatment of the problem useless. Still mathematicians, physicists and engineers invest a large effort in understanding of turbulence due to the unprecedented importance of turbulence both for theoretical and applied science.

Thanks to fundamental works of Richardson, Taylor, Kolmogorov and Obukhov, we have a phenomenology of turbulent cascades leading to the famous Kolmogorov spectrum. This theory has been successfully applied to a wide range of turbulent systems. What is still lacking, however, is a fundamental theory of turbulence, which would allow both finding a rigorous mathematical foundation for the Kolmogorov theory and understanding of its limitations.

The list of non-equilibrium statistical models the turbulent regime of which has been understood theoretically, has been growing very slowly over the past 60 years. The examples of important breakthroughs are the theory of wave turbulence, passive scalar advection, the theory of kinematic magnetic dynamo and Burgers turbulence. The success in solving all these problems came from the intensive use of machinery of non-equilibrium statistical physics, such as kinetic theory, quantum field theory-inspired instanton analysis of rare fluctuations and the theory of non-equilibrium critical phenomena.

Non-equilibrium statistical mechanics will play an increasingly important role in further progress of turbulence. Unfortunately, the range of tools and methods of non-equilibrium statistical mechanics used in modern turbulence research is so wide and they are developing so fast that there is not a single text book which could introduce a graduate student to this area of research. Our goal is to fill this gap by organising an LMS course on the non-equilibrium statistical mechanics of turbulence. The course will be given immediately before a week-long international workshop devoted to the same subject, and it will aim to prepare the graduate students and young researchers for this workshop.

Three world class experts in statistical physics and turbulence, professors John Cardy (Oxford University), Gregory Falkovich (Weizmann Institute of Science) and Krzysztof Gawedzki (ENS Lyon) have kindly agreed to give introductory courses on the most important topics of modern theoretical turbulence.

2 Objectives of the course.

The course will introduce PhD students and young researchers to the methods of non-equilibrium statistical physics used in modern theory of turbulence. It will provide them with a basis for understanding the content of the following workshop on non-equilibrium statistical mechanics. It will equip them with powerful research tools, which will hopefully be useful in their future study of turbulence.

We expect this course to attract students in applied mathematics, theoretical physics and engineering from around the UK and from abroad.

3 Pr. John Cardy: "Field Theory and Non-Equilibrium Statistical Mechanics."

Outline

1.
   - Brownian motion; Langevin equation; Einstein relation; Fokker-Plank equation.
   - Correlation functions and response functions; fluctuation-dissipation relation.
   - Response function formalism.
   - The master equation; detailed balance.
2.
   - Reaction-diffusion problems; rate equations; role of fluctuations in low dimensions.
   - Master equation and the Doi-Peliti formalism.
3.
   - Coherent state path integral representation.
   - representation in terms of a Langevin-type equation
   - diagrammatic expansion and Feynman rules.
4.
   - renormalization group analysis of $A + A \to \emptyset$.
5.
   - systems with conservation laws: $A + B \to \emptyset$; $A + A \rightleftharpoons C$

Problems.

Mon. This exercise is to revise your knowledge of the annihilation and creation operator formalism which we'll be using in lecture 2. These satisfy the commutation relations $[a, a^\dagger] = 1$. The state $|0\rangle$, satisfying $a|0\rangle = 0$, is used to build up the Fock space spanned by the states $|n\rangle \equiv {a^\dagger}^n|0\rangle$, with $n = 1, 2, \ldots$. [Hint: it is sometimes useful to think of a as $d/da^\dagger$.]

1. if $\langle 0|0\rangle = 1$, what is $\langle n|n'\rangle$?
2. show that $|n\rangle$ is an eigenstate of $N \equiv a^\dagger a$
3. what is $a\, e^{\lambda a^\dagger}|0\rangle$?
4. show that $e^{\lambda a}\, f(a^\dagger) = f(a^\dagger + \lambda)\, e^{\lambda a}$
5. show that
$$\int \frac{d^2z}{\pi}\, e^{-|z|^2}\, e^{za^\dagger}|0\rangle\langle 0|e^{z^* a} = 1$$
where the integral is over the whole complex plane
6. (harder) find the eigenvalues and right eigenstates of
$$H \equiv 2a^\dagger a^2 + {a^\dagger}^2 a^2$$
What about the left eigenstates?

Tues.

1. Write down the Doi-Peliti hamiltonian in the shifted form for the reactions $A+A \to \emptyset$ and $A + A \to A$. Show that they are related by a simple canonical transformation. What is the consequence for, e.g., the asymptotic mean particle density in the the two cases? Generalise your results to the processes $3A \to \emptyset$, $3A \to A$, $3A \to 2A$.
2. show that, at least formally, we can define conjugate variables (θ, n), where $[\theta, n] = 1$, through the transformation
$$a = e^{-\theta}\, n \qquad a^\dagger = e^{\theta}$$
Consider a model in which particles can hop between nearest neighbour sites ($j \to j + 1$) on a ring. The hopping rate $R_{j\to j+1}$ is a function $f(n_j)$ of the number of particles at site j *before* the hop. Write down the hamiltonian in terms of the above variables. Show that the stationary state (which is a right eigenstate of H with eigenvalue zero) has the product form $\prod_j \psi(n_j)$, and work out the function $\psi(n)$ in terms of $f(n)$. Is this the most general such hopping hamiltonian whose steady state has this simple product form?

Thurs. 1. For the reaction $2A \to \emptyset$ considered in the lecture, with initial mean density n_0, draw some of the diagrams which contribute to the density-density correlation function $\overline{n(x,t)n(x',t')}$. Can you sum explicitly all the diagrams which contain no loops?

2. For the reaction $3A \to \emptyset$, draw the diagrams which renormalise the rate constant. What is the critical dimension? Evaluate the 1-loop correction to the rate constant renormalisation and hence sum all the vertex corrections to all orders. Evaluate the RG β-function and hence compute the asymptotic density in the critical dimension. [All these computations are a straightforward generalisation of the case $2A \to \emptyset$ considered in lectures 3 & 4.]

Fri. 1. Catch up on or review the previous problems.

2. Consider the irreversible reaction-diffusion process $A + B \to C$, where the products C of the reaction are inert. Suppose that a constant current $+J$ of A particles is imposed at $x = -\infty$, with an equal but opposite current $-J$ of B particles at $x = +\infty$, so that the system reaches a steady state. Write down the inhomogeneous rate equations for the steady state densities $a(x)$, $b(x)$, (assume that A and B have equal diffusion constants) and show that the reaction rate $R(x) = \lambda a(x)b(x)$ has a scaling solution as function of x, D, λ and J. How does the width of the reaction zone scale with J in this approximation? What is the upper critical dimension d_c for the fluctuations in this problem? *Without doing any explicit calculation*, use RG ideas to argue how this scaling should be modified for $d < d_c$.

3.1 Preparatory reading list.

Although I'll be discussing classical statistical mechanics, I'll be using some of the mathematical formalism of quantum mechanics. Thus if you are un- familiar with this (or have forgotten it), it would be worth reading up on:

- any standard elementary quantum mechanics reference on the properties of number raising and lowering operators (a, $a^\dagger$), as applied, for example, to the simple harmonic oscillator. This is dealt with in almost all undergraduate quantum mechanics books.

- the use of these operators in representing hamiltonians in many-body quantum mechanics. This appears again in any (slightly old-fashioned) introductory textbook on quantum field theory, for example pp. 19-24 of *Introduction to Quantum Field Theory* by M. Peskin and D. Schroeder (Addison-Wesley, 1995.)

- a simple account of the use of the Feynman path integral formal- ism in quantum mechanics, for example in pp. 275-282 of Peskin and Schroeder above, although once again this appears in most introductory texts. We shall actually be using a version of this called the coherent state path integral, which is described in L. Schulman, *Techniques and Applications of Path Integration*, (New York :Wiley, 1981).

Further reading.

- my lectures will be an updated version of part of some earlier lectures which are available at

  `http://www-thphys.physics.ox.ac.uk/user/JohnCardy/notes.ps`

  However note that there is a fair number of typos in these, which will (hopefully) be corrected in these lectures.

- for a short review of some of these ideas, see J.L. Cardy, *Renormalisation group approach to reaction-diffusion problems*, in: Proceedings of Mathematical Beauty of Physics, Ed. J.-B. Zuber, Adv. Ser. in Math. Phys. **24**, 113 (1997); cond-mat/9607163.

- a more complete and recent review is in U.C. Tauber, M.J. Howard, and B.P. Vollmayr-Lee, *Applications of field-theoretic renormalization group methods to reaction-diffusion problems*, J. Phys. A 38, R79 (2005); cond-mat/0501678.

- to see why this has anything to do with turbulence, see C. Connaughton, R. Rajesh and O. Zaboronski, *Cluster-Cluster Aggregation as an Analogue of a Turbulent Cascade: Kolmogorov Phenomenology, Scaling Laws and the Breakdown of self-similarity* , cond-mat/0510389.

4 Pr. Krzysztof Gawedzki: "Soluble models of turbulent transport."

1. Turbulent flow as a random dynamical system
2. Multiplicative ergodic theory, multiplicative large deviations and multiplicative fluctuation theorem
3. Kraichnan model of passive advection
4. Dissipative anomaly and generalized flows
5. Zero mode scenario of intermittency

4.1 Preparatory reading list.

1. Some very vague knowledge on the concept of chaotic dynamical systems and on the relation between dynamical systems and (non-)equilibrium statistical mechanics and turbulence would be helpful but not indispensable. For the first topic, see Chapt. I of Ott's book *Chaos in dynamical systems*, for the second one, Sects. 1.1-1.6 of Dorfman's book *An introduction to chaos in nonequilibrium statistical mechanics* and for the third one, Sects. 1.1, 1.2.1, 1.3 and 8.1 and 8.2 of the book *Dynamical Systems Approach to Turbulence* by Bohr-Jensen-Paladin-Vulpiani.
2. Some familiarity with probabilistic concepts: random variables, probability measures, Gaussian integrals and processes, white noise and Brownian motion will be needed. I shall also introduce stochastic integrals and stochastic differential equations so some rudimentary familiarity with those concepts (Ito versus Stratonovich conventions) on an informal level will be welcome. A low key introduction may be Chapt. 4 of Gardiner's *Handbook of stochastic methods* also, possibly, Sect. 3.3.3 of Risken's *The Fokker-Planck equation.* A much more advanced text would be Sects. 2.2-4.2 of Oksendal's book *Stochastic differential equations.*

Some texts with the content overlapping with this course:

1. Sections II and III of Chetrite-Delannoy-Gawedzki's *Kraichnan flow in a square: an example of integrable chaos*, arXiv:nlin.CD/0606015

2. A review talk *Simple models of turbulent transport* in XIVth International Congress on Mathematical Physics" ed. J.-C, Zambrini, World Scientific 2005, pp. 38-49

3. Lecture notes *Soluble models of turbulent advection* in Random Media 2000, ed. J. Wehr, Wydawnictwa ICM, available also as arXiv:nlin.CD/0207058

4.2 Further (after course) reading:

1. A good discussion of the Multiplicative Ergodic Theorem is in two J.Kelliher's texts on his web-page

   `http://www.ma.utexas.edu/users/kelliher/Geometry/Geometry.html`

2. For recent complementary reviews of fluctuation theorems in non-equilibrium statistical mechanics see Gallavotti's *Stationary nonequilibrium statistical mechanics*, arXiv:cond-mat/0510027 and Kurchan's *Non-equilibrium work relations*, arXiv:cond-mat/0511073

3. Mathematical theory of Kraichnan model generalized flows has been set up in Le Jan-Raimond's *Flows, coalescence and noise*, Annals of Probability 32 (2004), 1247-1315, available also as arXiv:math.PR/0203221

4. Details on the Kraichnan model zero modes are reviewed in Sect. II.E of Falkovich-Gawedzki-Vergassola's *Particles and fields in fluid turbulence*, Rev. Mod. Phys. 75 (2001), 913-975 available also as arXiv:cond-mat/0105199

5 Pr. Gregory Falkovich: "Turbulence theory as part of statistical physics."

1. Weak wave turbulence
2. Strong wave turbulence
3. Incompressible turbulence
4. Statistical conservation laws
5. Broken and restored symmetries in turbulence

5.1 Preparatory reading list.

1. G. Falkovich and K.R. Sreenivasan, *Lessons from hydrodynamic turbulence*, Physics Today **59**(4), 43 (2006).

   `http://www.weizmann.ac.il/home/fnfal/KRSPhysTodayApr2006.pdf`

2. G. Falkovich, *Introduction to turbulence theory*,

   `http://www.weizmann.ac.il/home/fnfal/introturb06.pdf`