Theoretical and computational issues in geophysical turbulence

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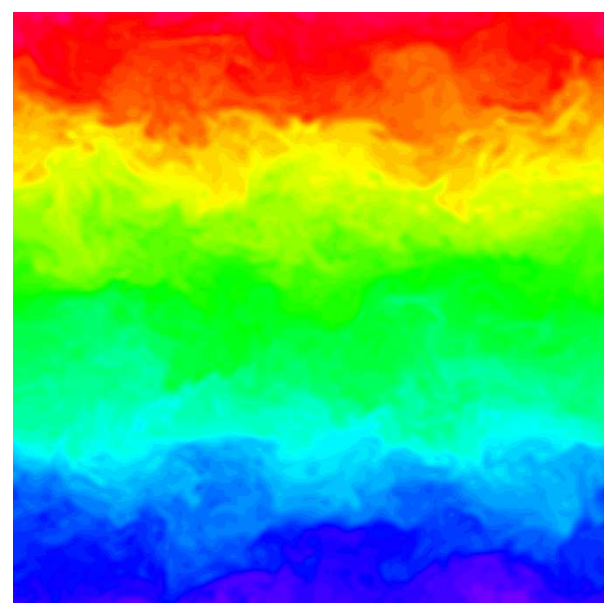


Outline

The Warwick Turbulence Symposium Workshop "Environmental Turbulence from Clouds through the Ocean" and "Theoretical and Computational Issues in Environmental Turbulence", March 2006

- 1. Strong rotation and strong stratification
- 2. Stratification only
  - a) dominated by vortical motion
  - b) dominated by internal waves
- 3. Strong stratification at various rotations
- 4. Rotation only
- 5. Wet turbulence (poster with Kyle Spyksma)

Boussinesq Equations

- Low-budget rotating stratified turbulence

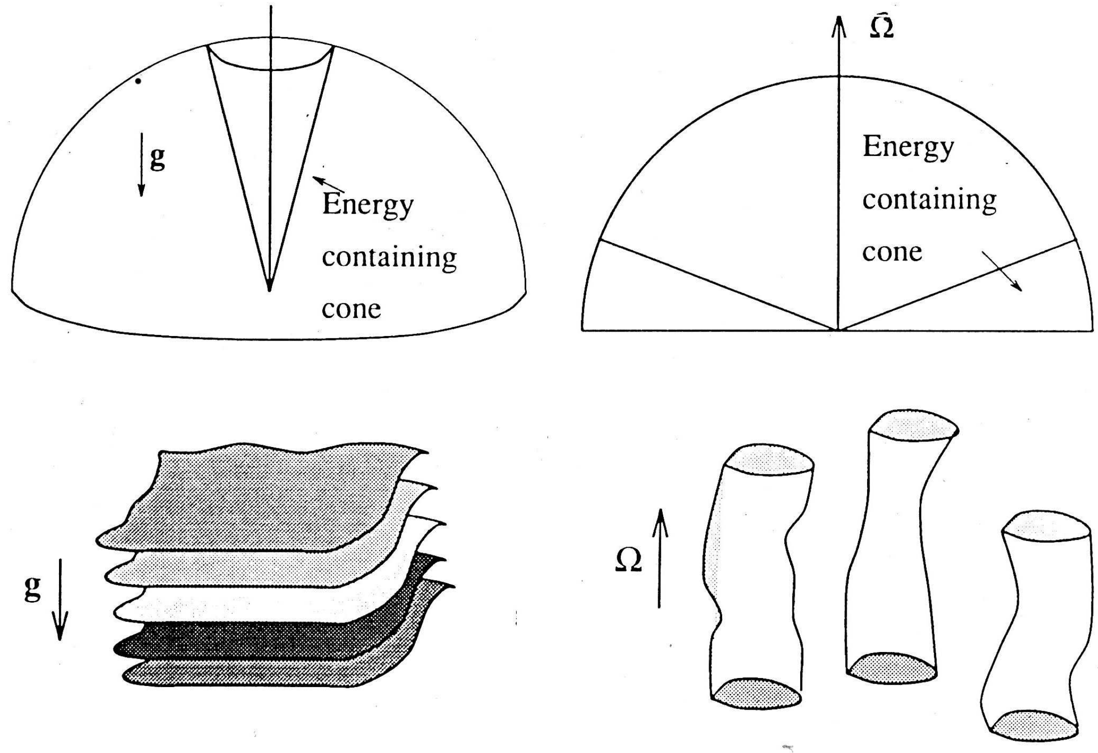
$$
\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + f\hat{\mathbf{z}}\times\boldsymbol{u} &= -\boldsymbol{\nabla}p + b\hat{\mathbf{z}},\\
\boldsymbol{\nabla}\cdot\boldsymbol{u} &= 0,\\
\frac{\partial b}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}b &= -N^2 w.
\end{aligned}
$$

where

$$
b = -\rho g/\rho_o, \quad N^2 = -\frac{g}{\rho_o}\frac{d\overline{\rho}}{dz}
$$

- Simplest set spanning the range from QG to isotropic 3D
- Can be non-dimensionalised about a million different ways

Stratification versus rotation



Stratification => pancakes

Rotation => tubes

Godeferd & Cambon (1993)

Energy migrates to where the high frequencies aren't.

$$\sigma_{\mathbf{k}}^2 = \frac{f^2 k_z^2 + N^2 k_h^2}{k^2}$$

Normal Mode Decomposition

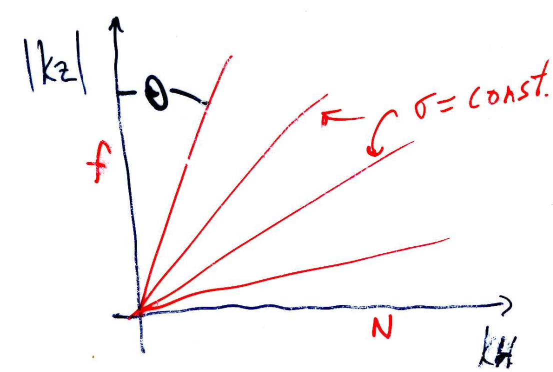
- Use linear normal modes to decompose Fourier coefficients into vortical/slow/geostrophic and wave/fast/ageostrophic parts:

$$\{\hat{\boldsymbol{u}}_{\mathbf{k}}, \hat{w}_{\mathbf{k}}, \hat{b}_{\mathbf{k}}\} \rightarrow \{B_{\mathbf{k}}^{(0)}, B_{\mathbf{k}}^{(+)}, B_{\mathbf{k}}^{(-)}\}$$

- $B_{\mathbf{k}}^{(s)}$'s satisfy:

$$\frac{\partial B_{\mathbf{k}}^{(s)}}{\partial t} + i\lambda_{\mathbf{k}}^{(s)} B_{\mathbf{k}}^{(s)} = \sum_{\substack{\boldsymbol{k}=\boldsymbol{p}+\boldsymbol{q} \\ m,n=\pm,0}} \Gamma_{\mathbf{kpq}}^{smn} B_{\mathbf{p}}^{(m)} B_{\mathbf{q}}^{(n)}.$$

$$\lambda_{\mathbf{k}}^{(0)} = 0, \qquad \lambda_{\mathbf{k}}^{(\pm)} = \pm\sigma_{\mathbf{k}}$$



Normal Mode Decomposition

Let $G_{\mathbf{k}} = B_{\mathbf{k}}^{(0)}$ and $A_{\mathbf{k}}^{(\pm)}(\epsilon t)\mathrm{e}^{\pm i\omega_{\mathbf{k}} t} = B_{\mathbf{k}}^{(\pm)}$, where $\epsilon = \min(Ro, Fr)$.

$$\frac{\partial}{\partial(\epsilon t)}\langle |G_{\mathbf{k}}|^2\rangle = \sum_{\Delta} N_1 \langle G_{\mathbf{k}} G_{\mathbf{p}} G_{\mathbf{q}}\rangle +$$

$$N_2\langle G_{\mathbf{k}} G_{\mathbf{p}} A_{\mathbf{q}}\rangle \mathrm{e}^{i\omega_{\mathbf{q}} t} + N_3 \langle G_{\mathbf{k}} A_{\mathbf{p}} A_{\mathbf{q}}\rangle \mathrm{e}^{i(\omega_{\mathbf{q}}+\omega_{\mathbf{p}})t}$$

$\to 0$

1. conservation

2. resonance

$$\frac{\partial}{\partial(\epsilon t)}\langle |A_{\mathbf{k}}|^2\rangle = \sum_{\Delta} M_1 \langle A_{\mathbf{k}} G_{\mathbf{p}} G_{\mathbf{q}}\rangle \mathrm{e}^{i\omega_{\mathbf{k}} t} + M_2 \langle A_{\mathbf{k}} A_{\mathbf{p}} G_{\mathbf{q}}\rangle \mathrm{e}^{i(\omega_{\mathbf{k}}+\omega_{\mathbf{p}})t}$$

$$+ M_3 \langle A_{\mathbf{k}} A_{\mathbf{p}} A_{\mathbf{q}}\rangle \mathrm{e}^{i(\omega_{\mathbf{k}}+\omega_{\mathbf{p}}+\omega_{\mathbf{q}})t}$$

Energy, $E = \sum_{\mathbf{k}} |G_{\mathbf{k}}|^2 + |A_{\mathbf{k}}|^2 = E_G + E_A$.

Potential Enstrophy, $V \doteq \sum_{\mathbf{k}} \sigma_{\mathbf{k}}^2 k^2 |G_{\mathbf{k}}|^2$ as Ro **or** Fr $\to 0$.

Gravity-wave dispersion relation, $\sigma_{\mathbf{k}} = \left(f^2 \cos^2\theta_{\mathbf{k}} + N^2 \sin^2\theta_{\mathbf{k}}\right)^{1/2}$.

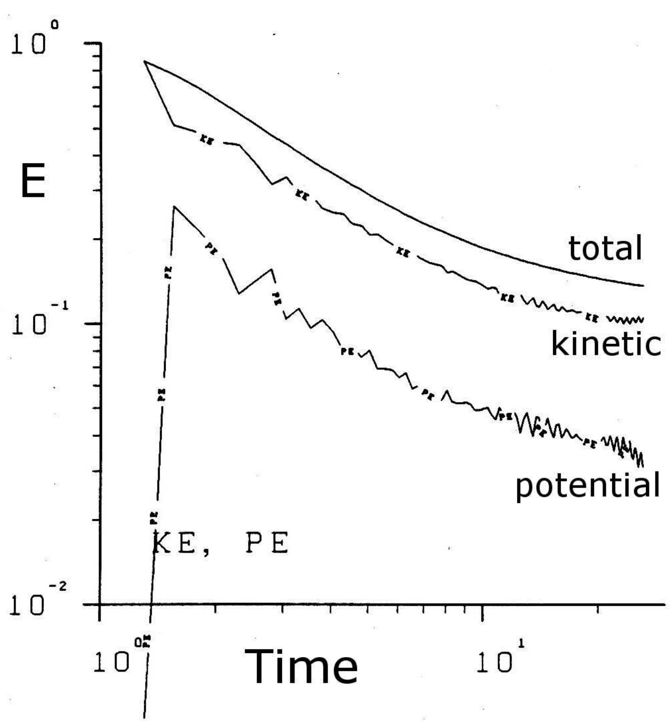
Numerical Approach

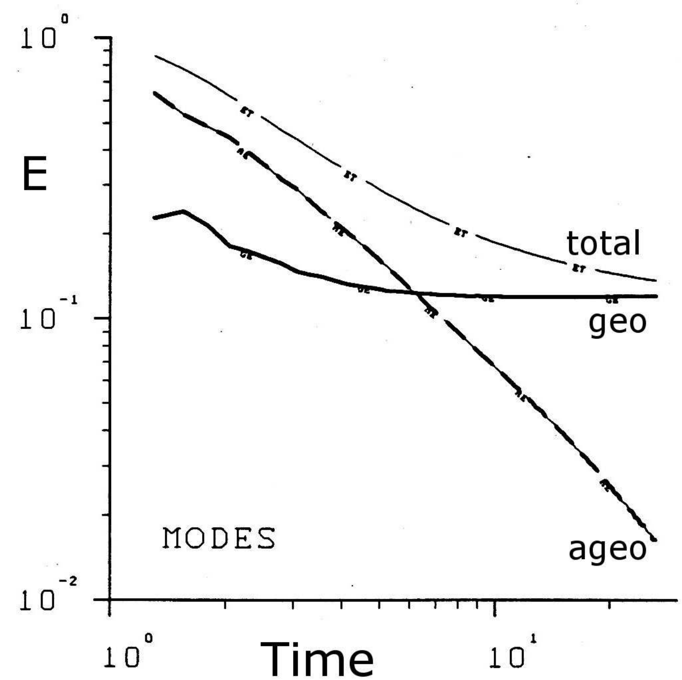
- Use 3D Boussinesq model
  - pseudo-spectral
  - leap-frog time stepping with Robert filter
  - ∇^8 hyperviscosity
  - isotropic domain of size $(2\pi)^3$
  - resolution 90^3 to 240^3 ($\Delta x = \Delta y = \Delta z$)

1. Strong rotation and strong stratification

Decay Simulation with Initial $Ro = Fr = 1$ (Bartello 1995)
If all linear frequencies large, then leading-order dynamics is

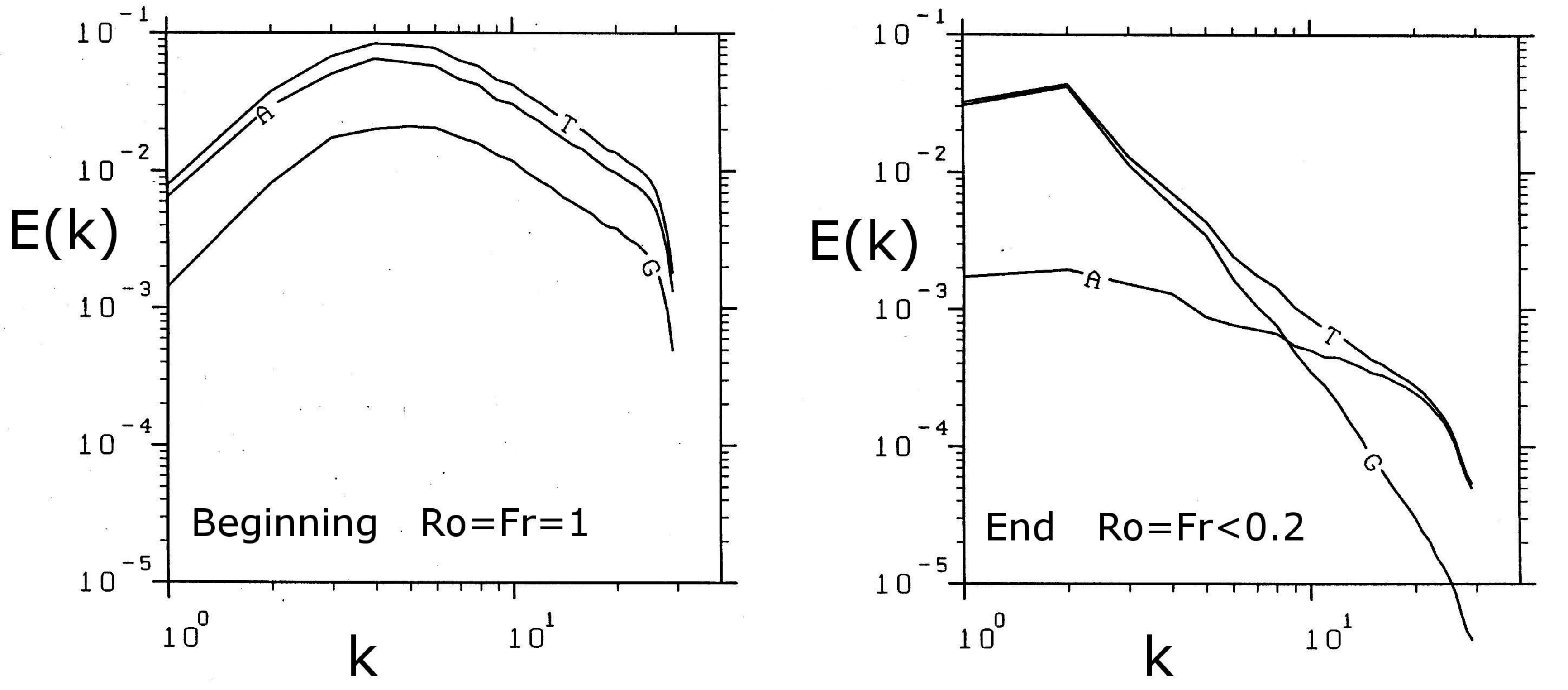
$$\frac{\partial}{\partial t} B_{\mathbf{k}}^{(0)} = \sum_{\Delta} \Gamma_{\mathbf{kpq}}^{000} B_{\mathbf{p}}^{(0)} B_{\mathbf{q}}^{(0)} \quad \text{and} \quad \frac{\partial}{\partial t} B_{\mathbf{k}}^{(\pm)} = \sum_{\Delta} \Gamma_{\mathbf{kpq}}^{\pm 0 \pm} B_{\mathbf{p}}^{(0)} B_{\mathbf{q}}^{(\pm)} .$$





1. Strong rotation and strong stratification

Rotational (G for geostrophic) and wave-mode (A) spectra



Get it? They cross.

2. Stratification only

- Neglect rotation (atmospheric mesoscale, oceanic submesoscale).
- Can still decompose strongly stratified flows into vortical motion (PV) and internal gravity waves.
- We examine stratified turbulence dominated by vortical and wave motion:
  - Limiting dynamics
  - Overturning length scale
  - Gravity wave saturation spectra: $N^2 k_z^{-3}$?

2. Stratification only: Dimensionless equations

- Nondimensionalize the equations, using $\boldsymbol{u} \sim U$, x, $y \sim L$, $z \sim H$, etc. Have two possible time scales (Riley et al 1981, Lilly 1983)
  - Wave time scale: $T \sim L/NH$.
  - Vortical time scale: $T \sim L/U$.
- Which terms are important in the limit of strong stratification?

$$Fr_h = \frac{U}{NL}, \quad Fr_z = \frac{U}{NH} \longrightarrow 0$$

2. Stratification only: Wave scaling

- Using wave time scale $T \sim L/NH$ ($\alpha = H/L$):

$$
\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} + Fr_z\,(\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + w\frac{\partial \boldsymbol{u}}{\partial z}) &= -\boldsymbol{\nabla} p,\\
\alpha^2(\frac{\partial w}{\partial t} + Fr_z\,\boldsymbol{u}\cdot\boldsymbol{\nabla} w + Fr_h\,w\frac{\partial w}{\partial z}) &= -\frac{\partial p}{\partial z} + b,\\
\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{\partial w}{\partial z} &= 0,\\
\frac{\partial b}{\partial t} + Fr_z\,(\boldsymbol{u}\cdot\boldsymbol{\nabla} b + w\frac{\partial b}{\partial z}) &= -w.
\end{aligned}
$$

- Weakly nonlinear internal gravity waves.

2. Stratification only: Vortical scaling

- Use vortical time scale $T \sim L/U$:

$$
\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + Fr_z^2\, w \frac{\partial \boldsymbol{u}}{\partial z} &= -\boldsymbol{\nabla} p, \\
Fr_h^2 \left(\frac{\partial w}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} w + Fr_z^2\, w \frac{\partial w}{\partial z} \right) &= -\frac{\partial p}{\partial z} + b, \\
\boldsymbol{\nabla}\cdot\boldsymbol{u} + Fr_z^2 \frac{\partial w}{\partial z} &= 0, \\
\frac{\partial b}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} b + Fr_z^2\, w \frac{\partial b}{\partial z} &= -w.
\end{aligned}
$$

- Decoupled layers when Fr_h, $Fr_z \to 0$.

- Billant & Chomaz 2001: $H \sim U/N$ and so $Fr_z \equiv 1$.

2. Stratification only: Numerical Approach

- Use same pseudo-spectral Boussinesq model
- Two types of large-scale forcing:
  - vortical forcing: modes around $k_h = 3$, $k_z = 0$
  - wave forcing: around $k_h = 3$, $k_z = 3$
- 10 stratifications, with

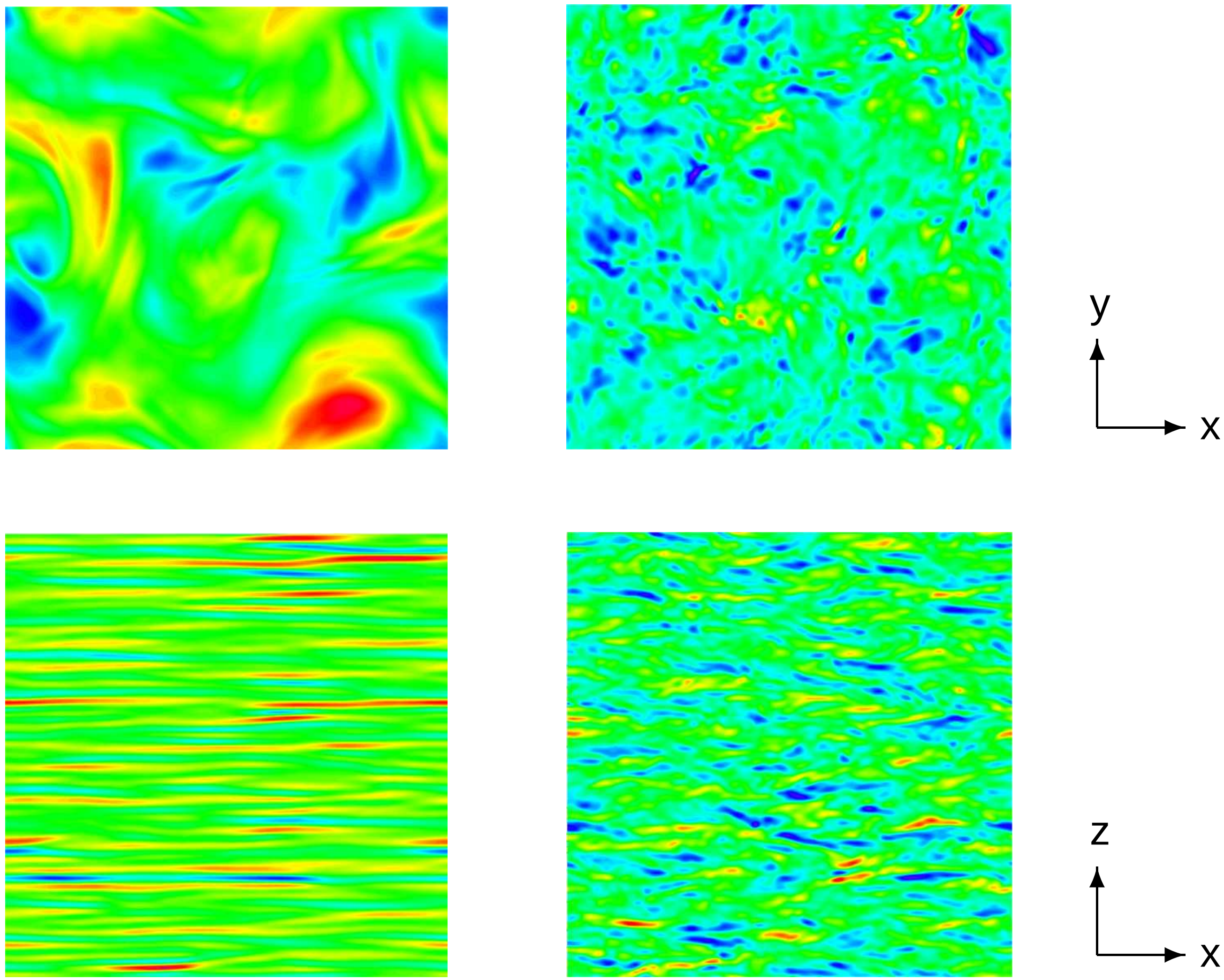
$$0.01 < Fr_h < 10.$$

2. Vorticity slices: $Fr_h \sim 0.01$

Vortical forcing: Wave forcing:

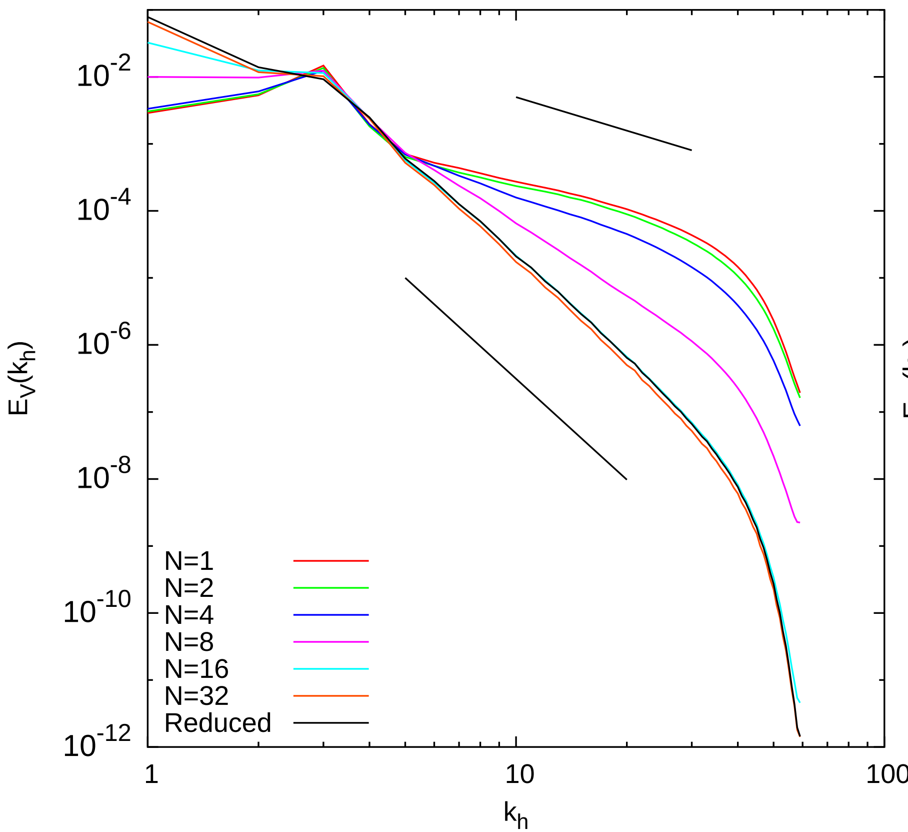
ω_z:

ω_y:

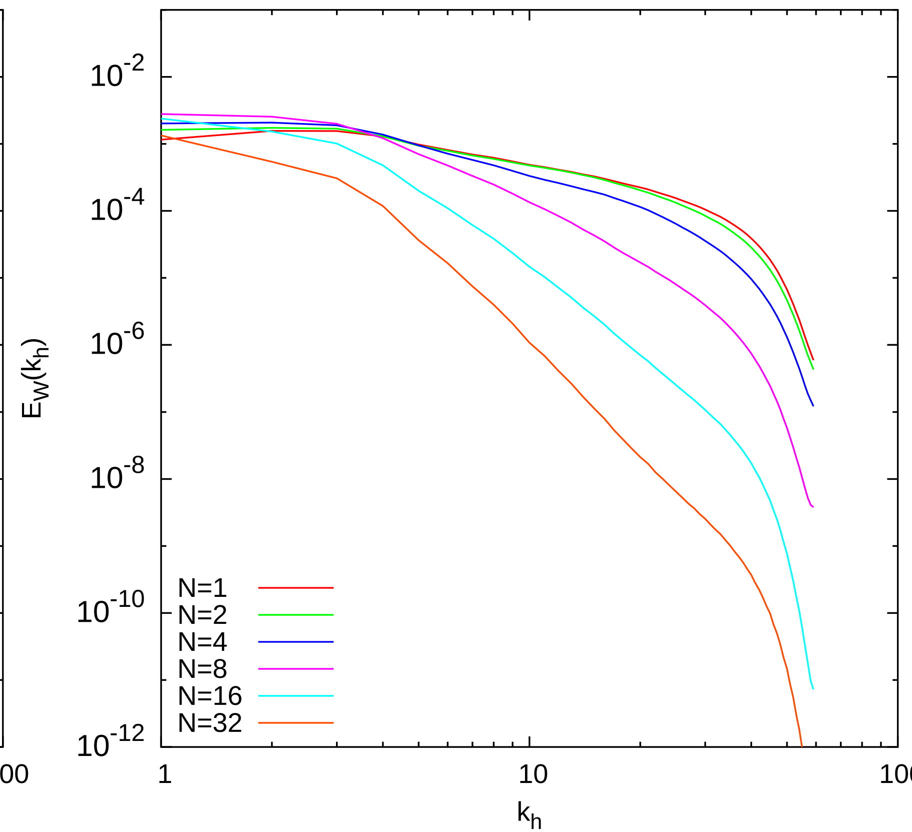


2a. Vortical forcing: k_h spectra

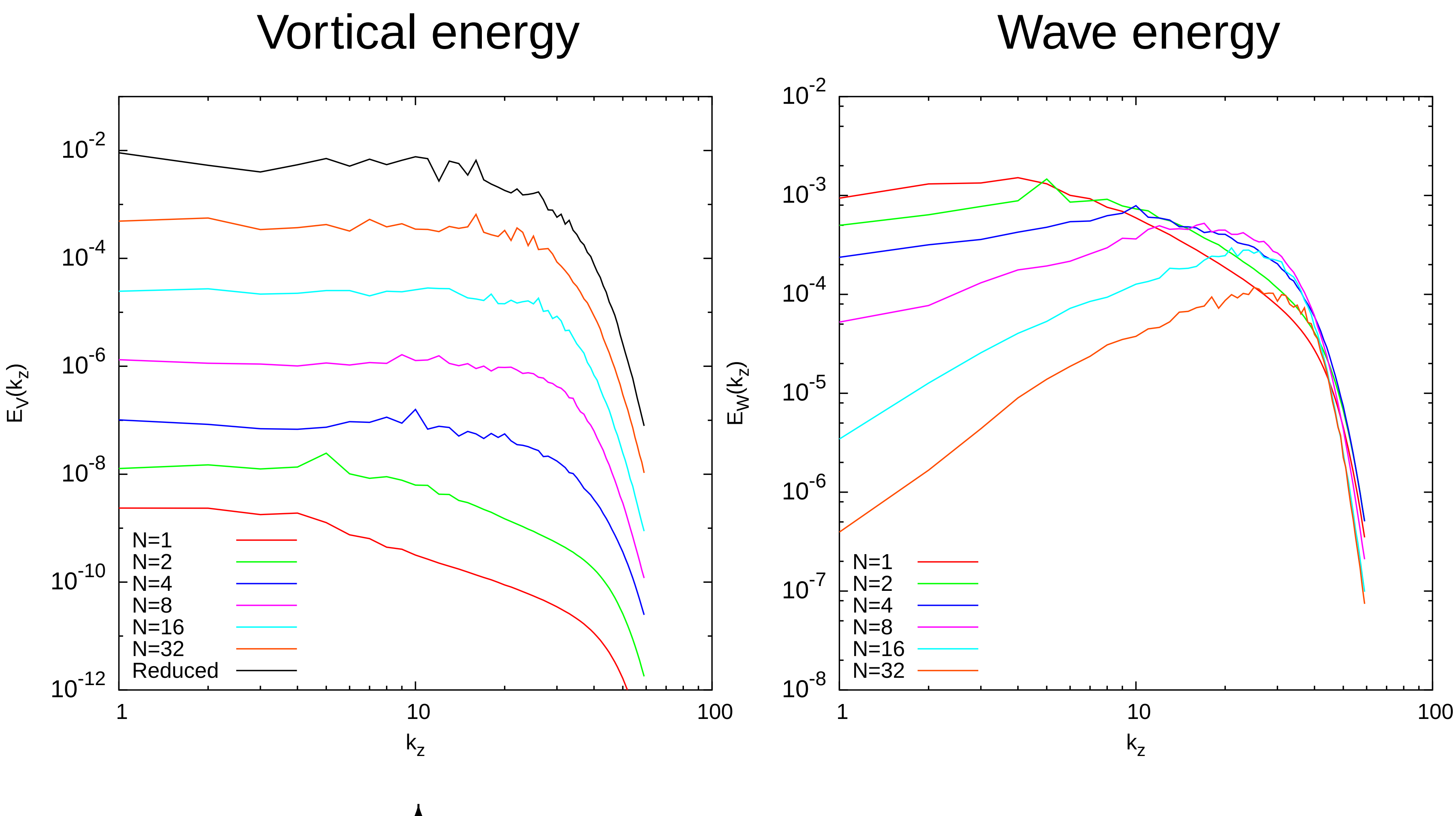
Vortical energy



Wave energy



2a. Vortical forcing: k_z spectra

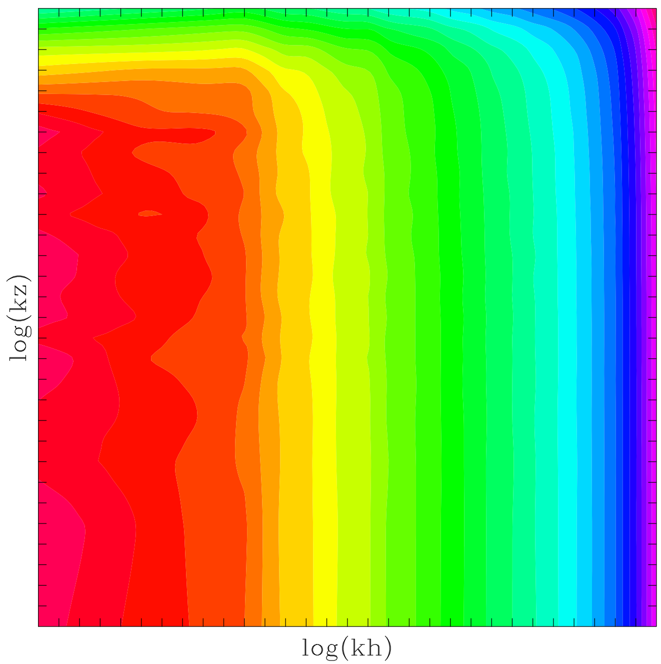


offset by factors of 10

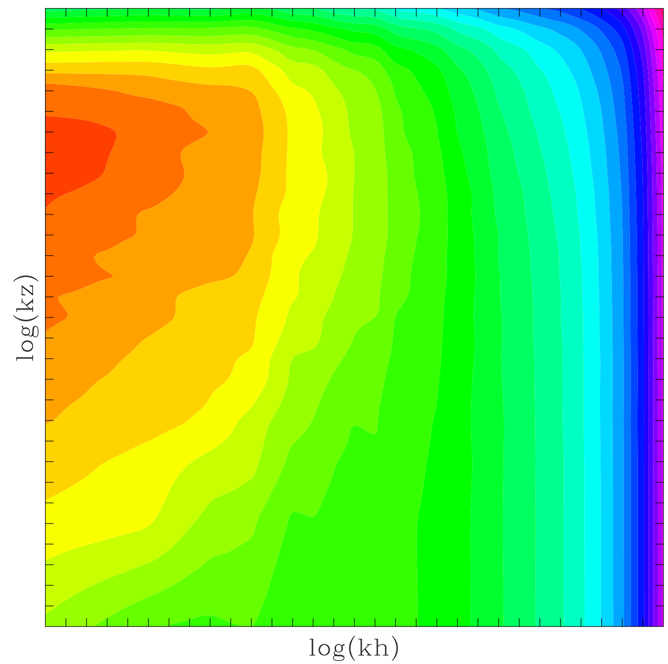
2a. Vortical forcing: k_h–k_z spectra

$Fr_h \sim 0.01$:

Vortical energy



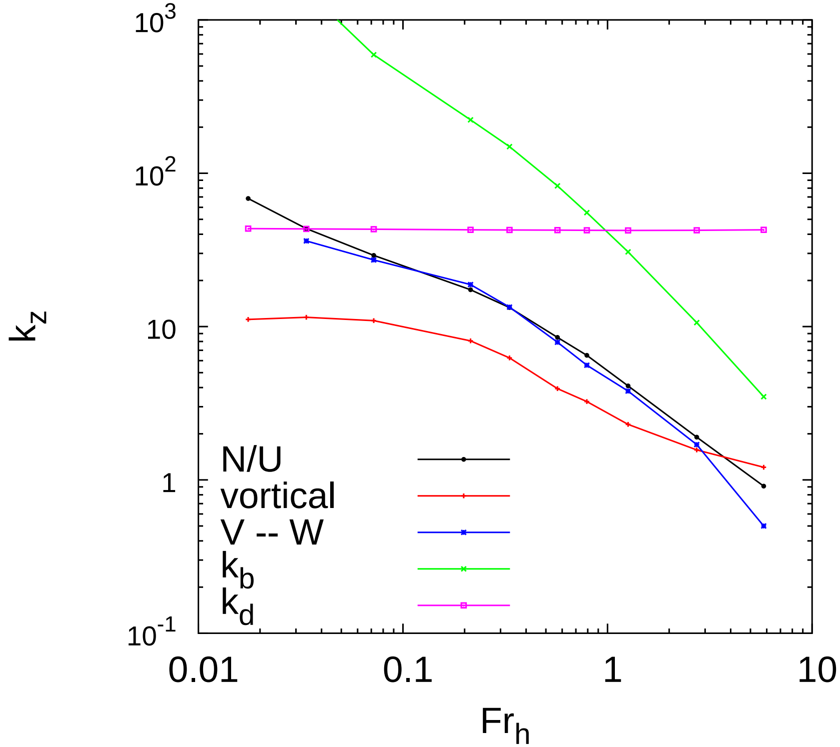
Wave energy



- W-V-V interactions exchange vortical and wave energy.
- Resonant interactions: $\boldsymbol{k} = \boldsymbol{p} + \boldsymbol{q}, \quad \omega_k = \omega_p + \omega_q$

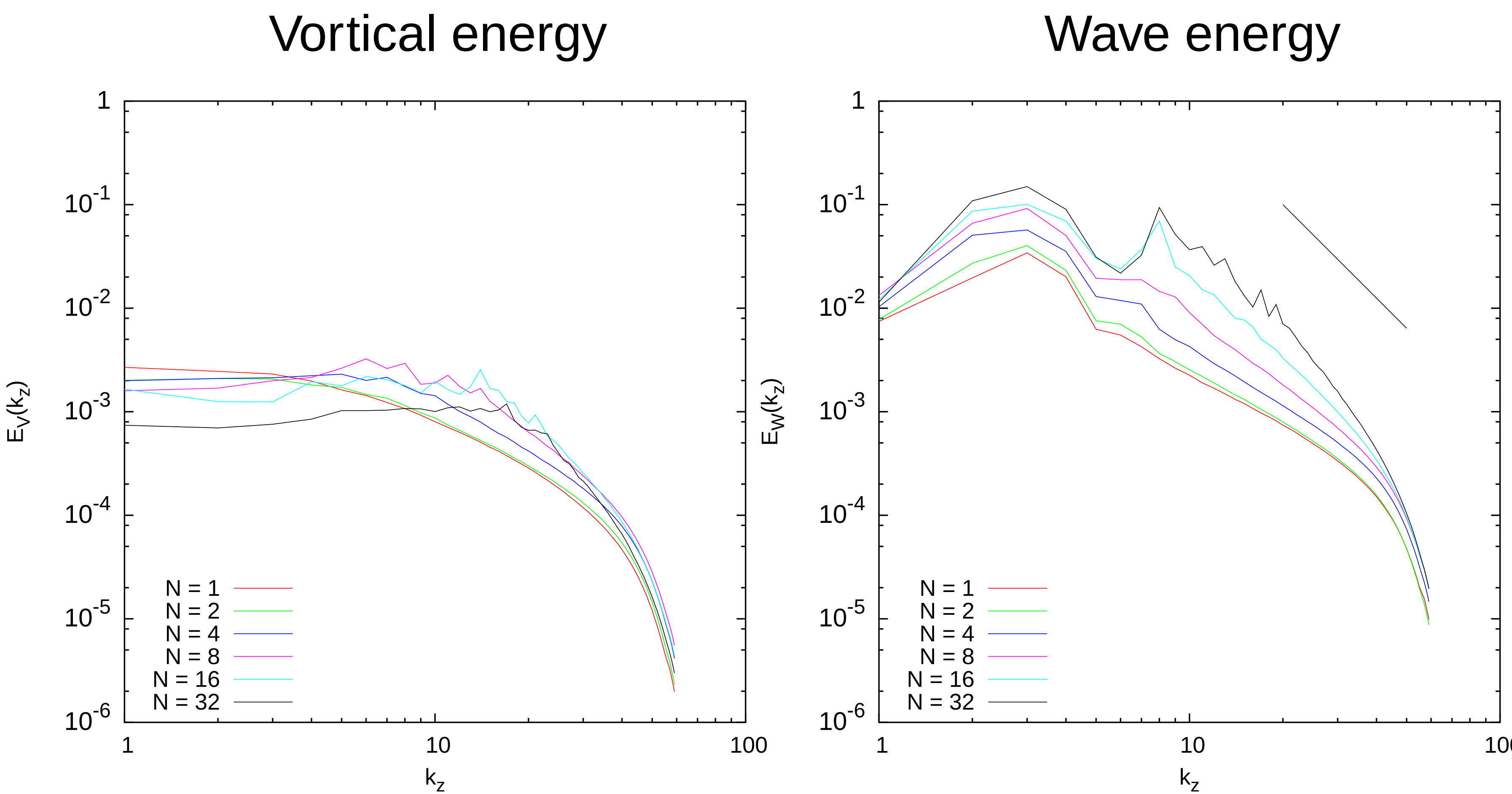
2a. Vortical Forcing: Length scales

Vortical forcing:



- $k_b \sim \sqrt{N^3/\epsilon}$
- k_d: dissipation

2b. Wave forcing: k_z spectra



2b. Wave forcing: $N^2\ k_z^{-3}$ spectra?

Dashed line: $0.1N^2\ k_z^{-3}$ (Bouruet-Aubertot, Sommeria & Staquet 1996)

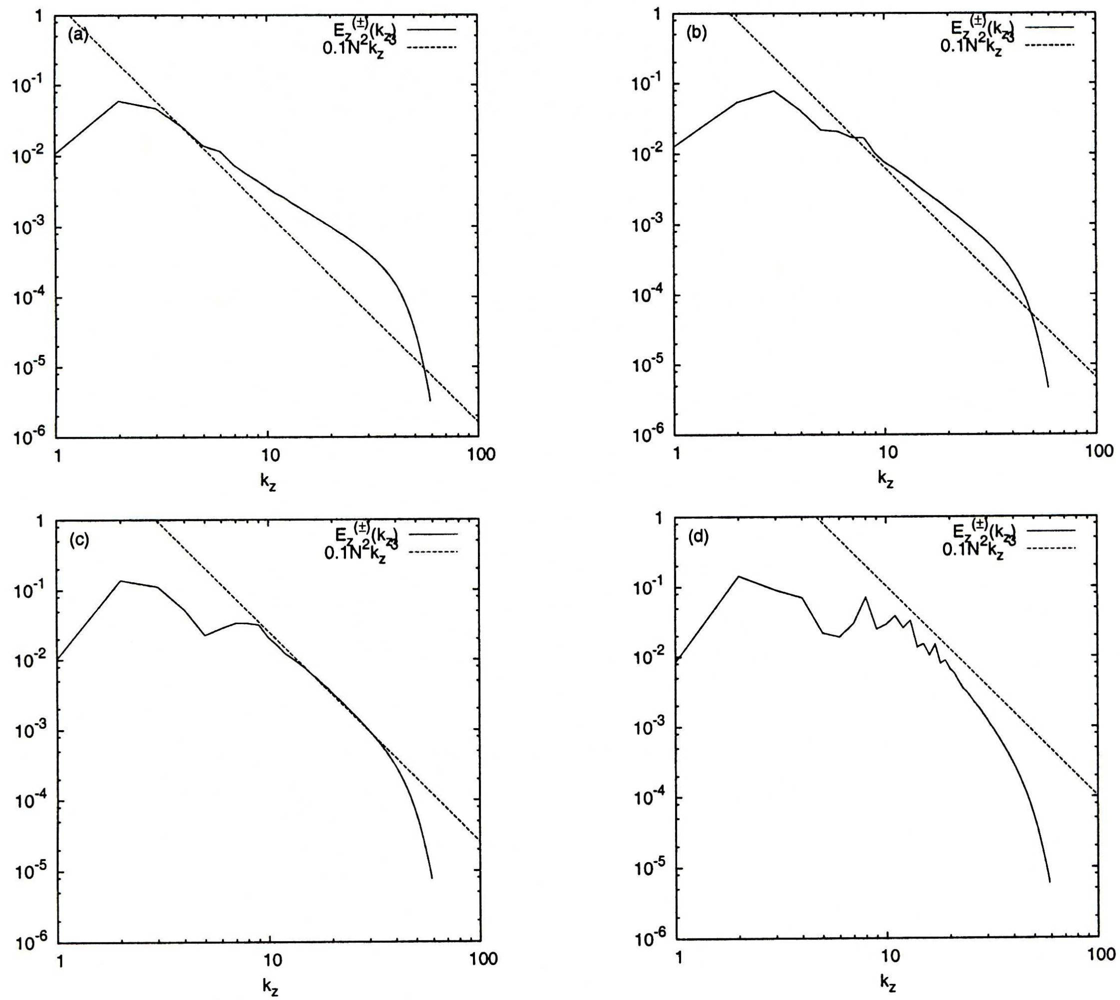
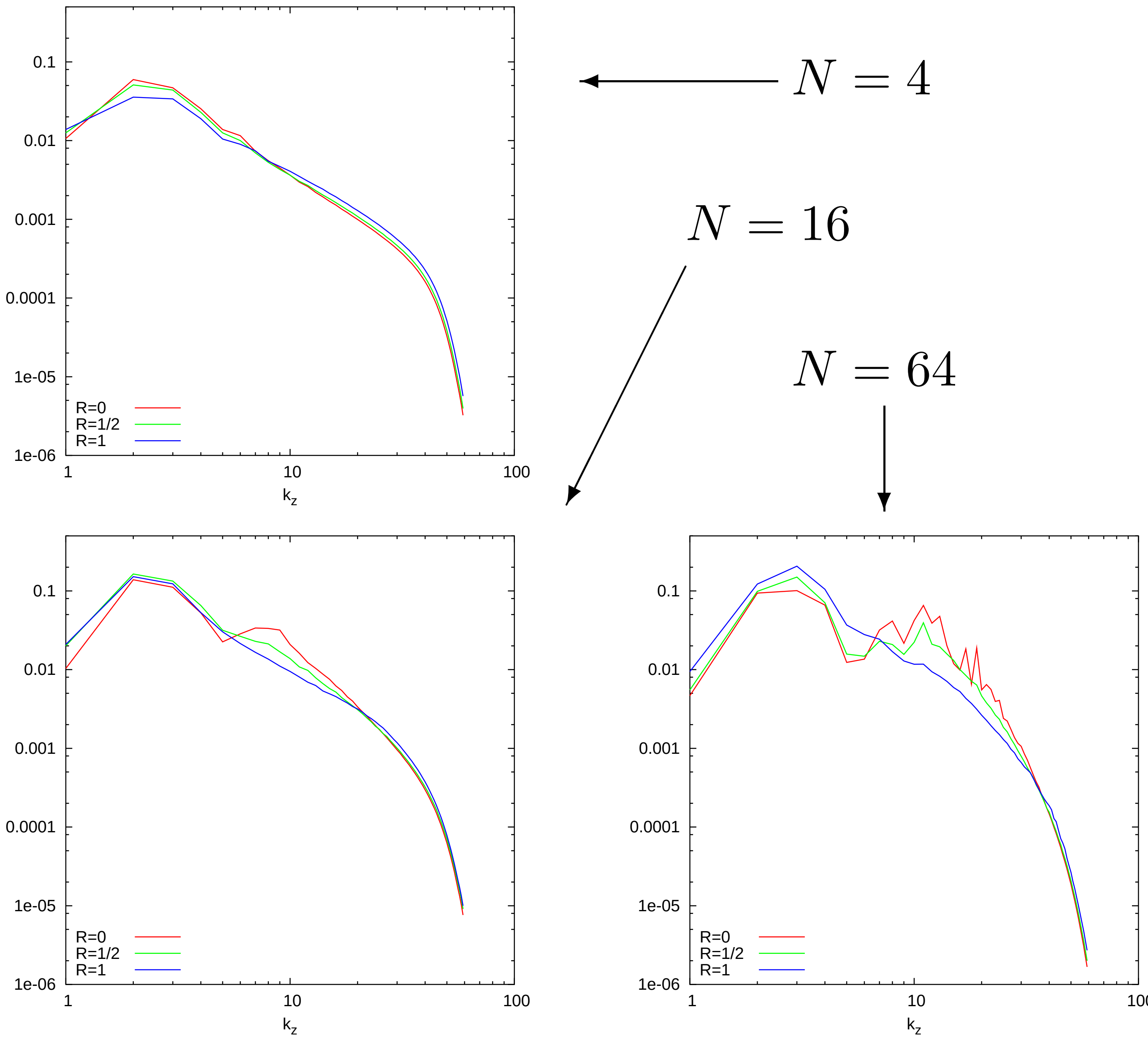


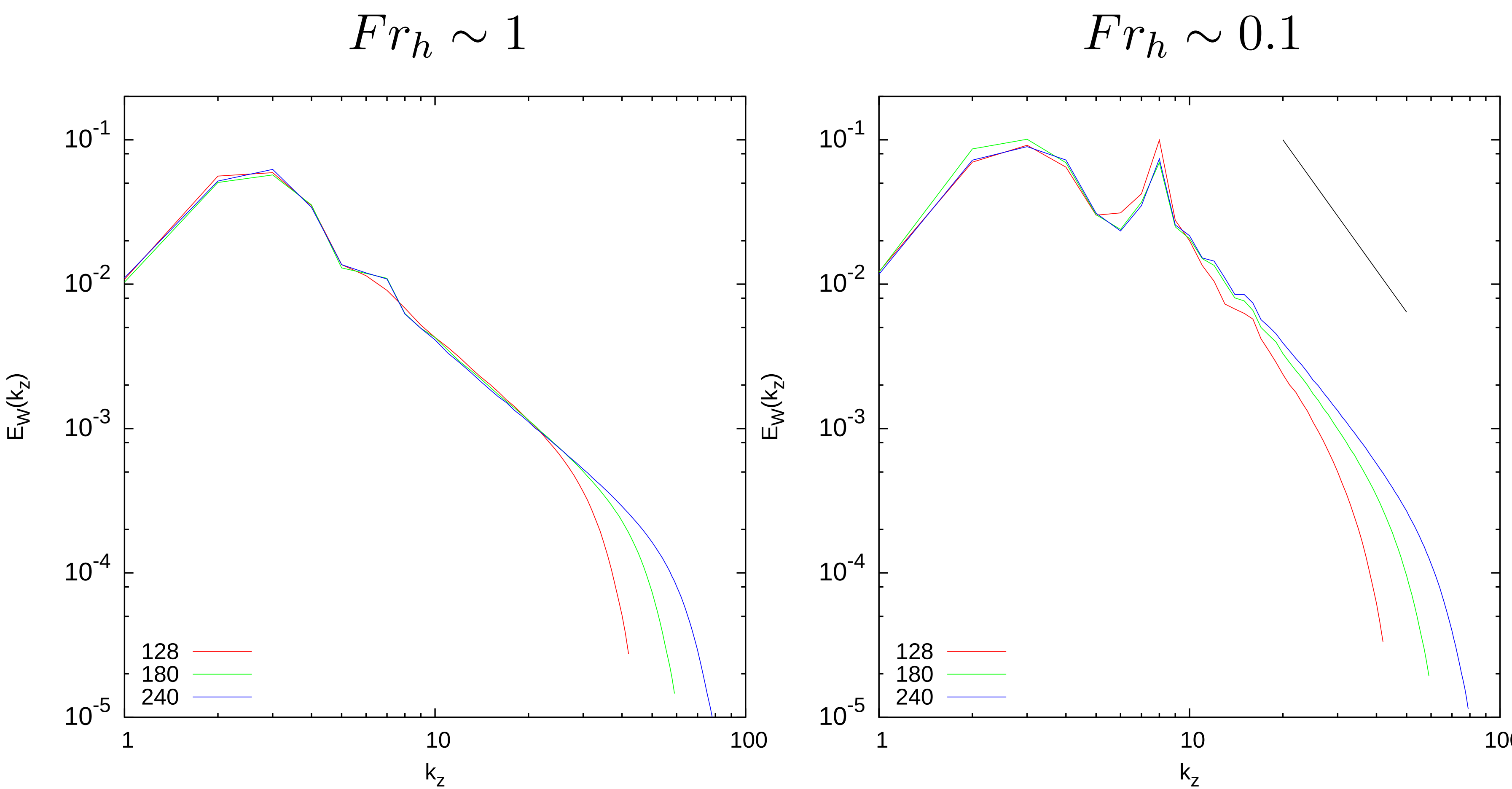
FIGURE 9. Vertical wavenumber spectra of wave energy, along with the hypothetical saturation spectrum $0.1\,N^2\,k_z^{-3}$, for (a) $N=4$, (b) $N=8$, (c) $N=16$ and (d) $N=32$ when $M=180$ and $R=0$.

2b. Wave forcing: Sensitivity to vortical modes



(res)

2b. Wave forcing: dependence on Re



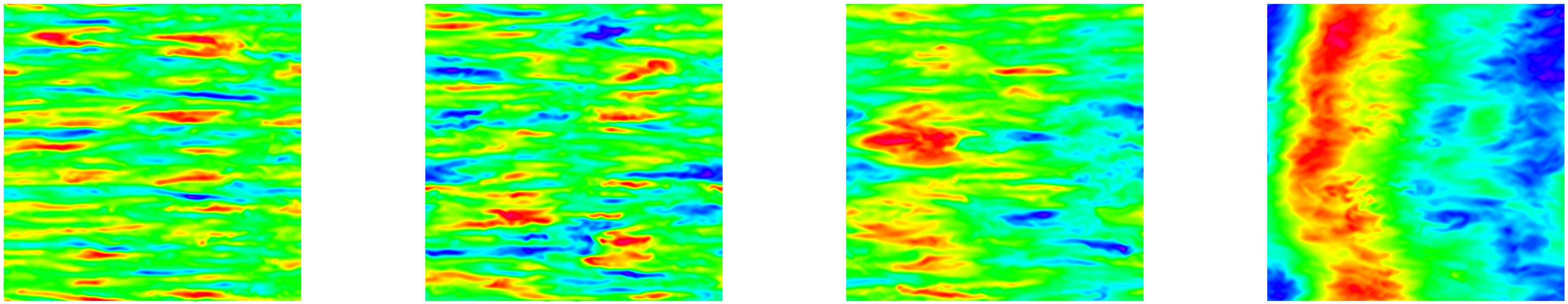
- Decreasing $Fr_h \rightarrow$ steeper spectra $\rightarrow$ nonlocal interactions.

Summary of Stratified Simulations

- Vortical motion:
  - Spectra $\sim k_h^{-5}$, k_z^0 when Fr_h, $Fr_z \ll 1$.
  - Simulations confirm $H \sim U/N$.

- Wave motion:
  - $N^2\, k_z^{-3}$ at smaller scales? Not really.
  - Spectra very bumpy when Fr_h, $Fr_z \ll 1$ (sensitive to forcing?)
  - Spectra steepen as Fr_h, $Fr_z \to 0$ BUT are sensitive to Re.

3. Strong stratification at various rotations



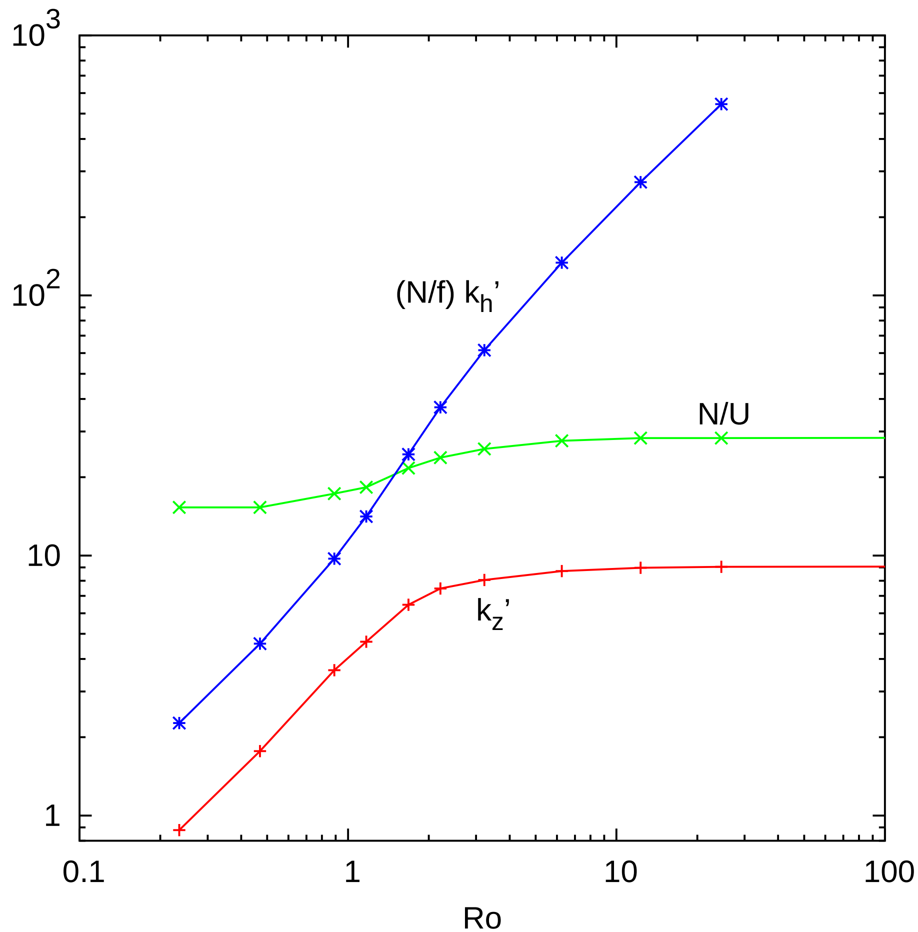
No rotation $\longrightarrow$ lots of rotation

This is the velocity component perpendicular to the screen.

The "adjustment from stratification-dominated $Bu \gg 1$ flows to the $Bu = O(1)$ regime occurs through gluing of horizontal pancake structures in the vertical by rotation" (Babin, Mahalov and Nicolænko 1998).

Burger no. $= Bu = Ro^2/F_r^2$

3. Strong stratification at various rotations

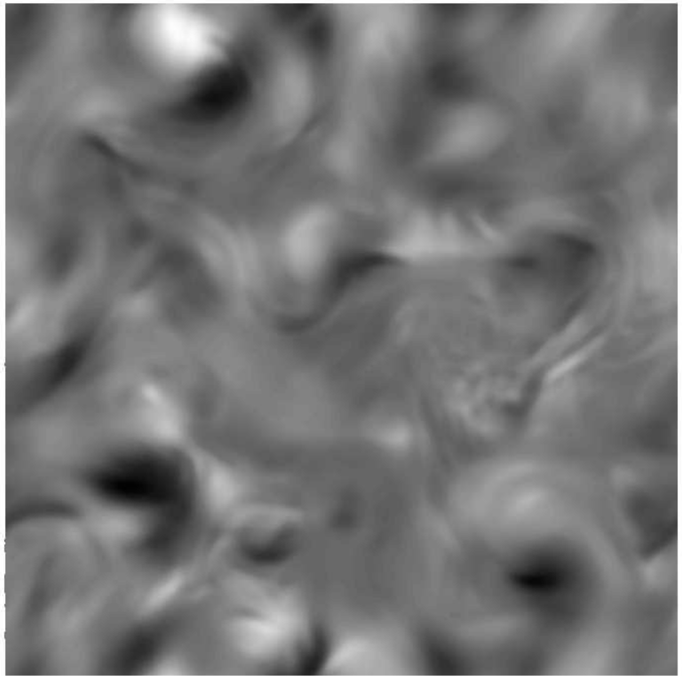


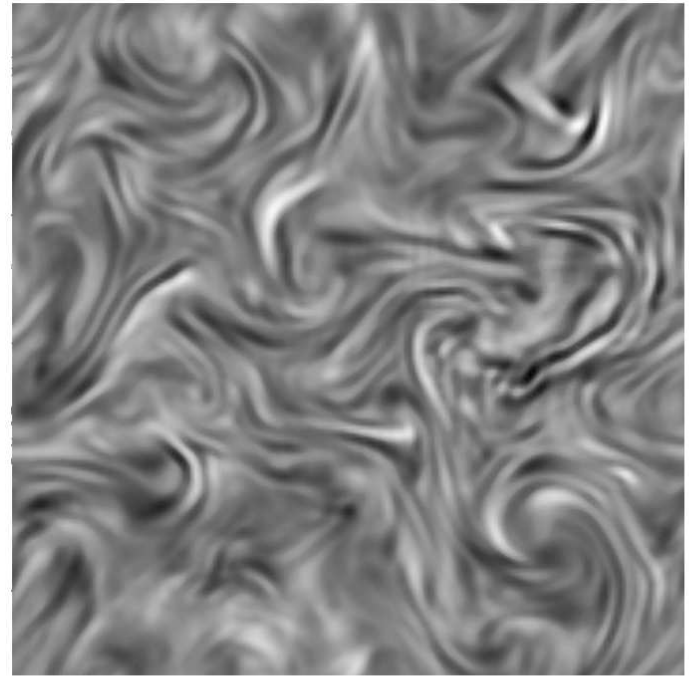
Stratified turbulence. Characteristic vertical wavenumber $= N/U$

Quasigeostrophic turbulence. Vertical wavenumber $= N/fL$ (Charney 1971)

This is how to set $\Delta z/\Delta x$ in the (sub)mesoscale.

3. Strong stratification at various rotations





Balance at Ro=0.09 ?

We have used the QG modes and the ω equation to diagnose vertical velocity (left).

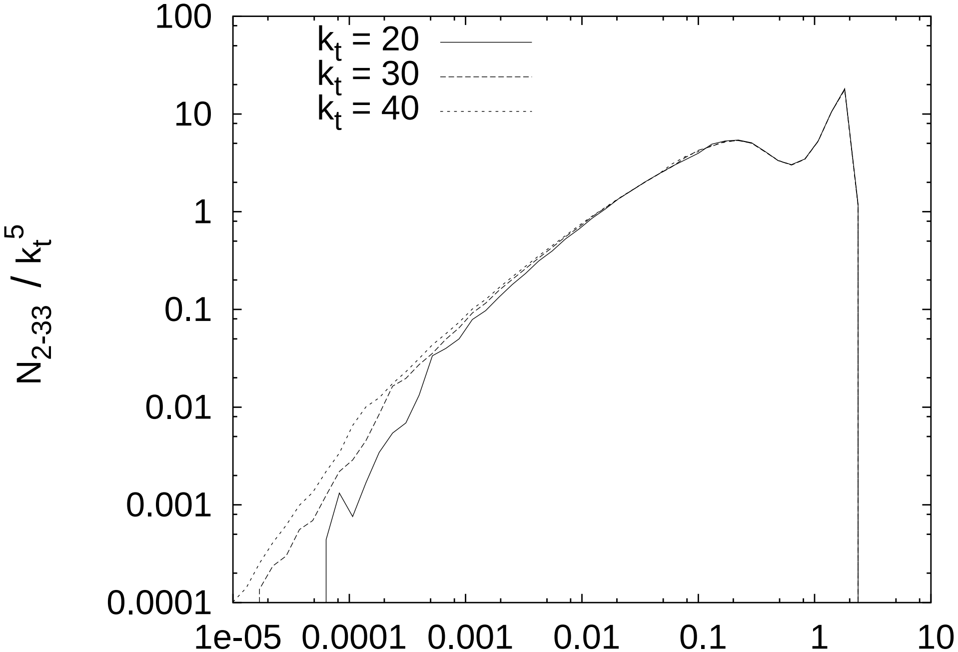
The real vertical velocity is on the right.

If balance exists, it isn't this simple.

4. Rotation only

Contradictory results: vorticity asymmetry, k_h^{-3} inverse cascades with forcing, two-dimensionalisation, etc.

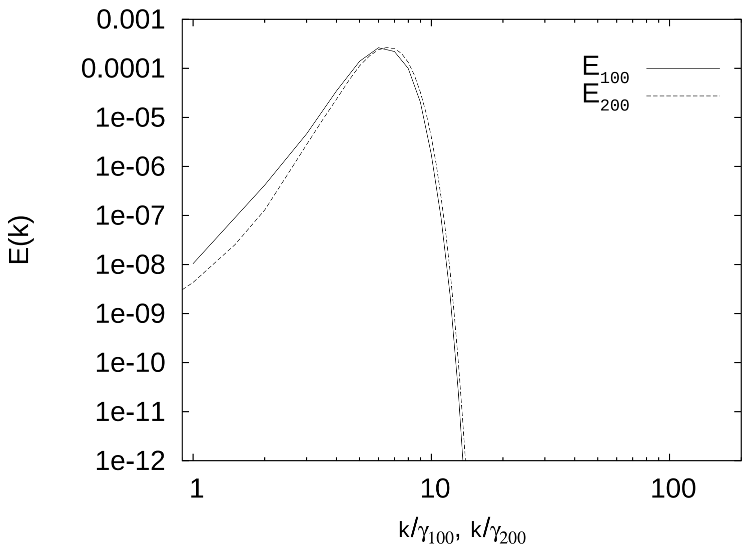
Smith & Lee (2005 JFM) near-resonant interactions must be well resolved.

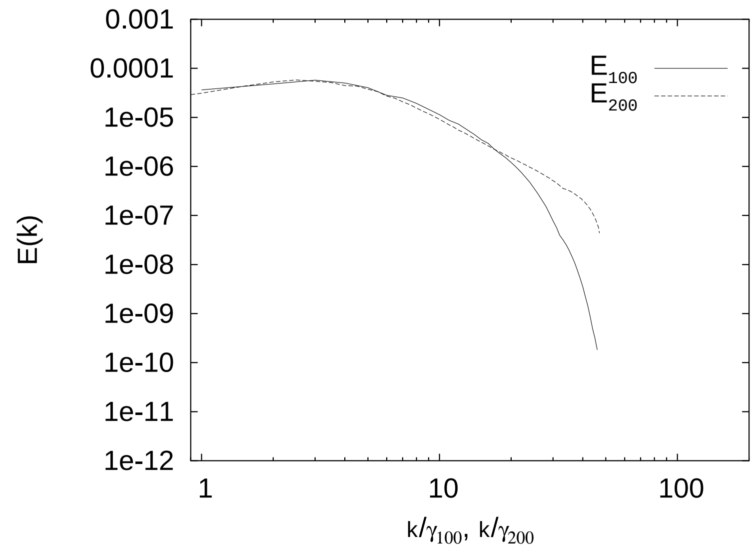


Histogram of $Ro \leq \frac{1}{f} | \pm \omega_{\mathbf{k}} \pm \omega_{\mathbf{p}} \pm \omega_{\mathbf{q}} | < Ro + dRo$ where $\mathbf{k} = \mathbf{p} + \mathbf{q}$.

4. Rotation only

We performed decay simulations with two different computational boxes and scaled initial conditions.



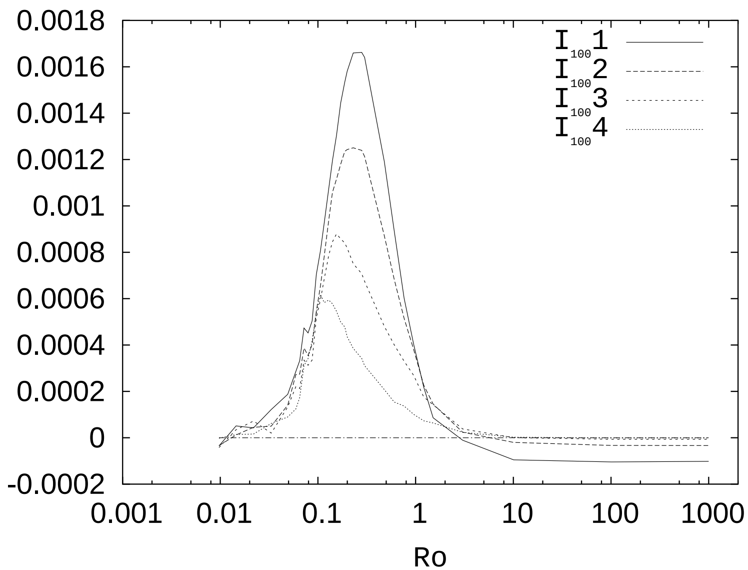


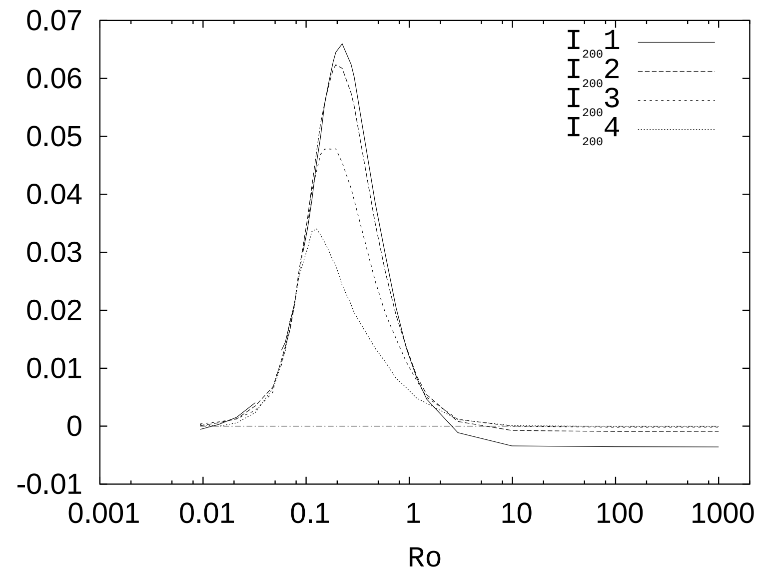
Preliminary isotropic Navier-Stokes simulation. Beginning (left) and at the time of max enstrophy (right).

Spectra at right describe initial conditions for rotating simulations

4. Rotation only

Below is the total energy transfer from 3D (wave) modes to 2D (vortex) modes vs Ro at various times.





Small computational box (left), large box (right)

Big max at intermediate Ro. (Not sensitive to box size.)
Consistent with asymptotic decoupling.

Conclusions

This stuff is pretty complicated