Wave turbulence theory of discrete systems

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Plan

0. Problems of Kolmogorov´s theory and KAM theory.

1. General properties of wave interactions in systems with periodic boundary conditions (resonators), construction of discrete classes of interacting waves.

2. Example of transition from the resonator to an infinite domain (for Rossby waves).

3. Interconnections between discrete classes and KAM tori.

Problems appear in:

- resonators
- Faraday instability
- zonal flows in atmosphere and oceans
- "frozen turbulence" + power-law spectrum
- many other situations...

KAM theory - an attempt to improve Kolmogorov´s theory, at least is a simpler setting of **weak turbulence** (K.,A.,M. 1954-1963).

The basic mathematical fact used in KAM theory is **Thue theorem** (1909), giving low estimate for the distance between any algebraic number α of degree $n > 2$ and a rational number $p/q \in \mathbb{Q}$:

$$|\alpha - \frac{p}{q}| > \frac{c(\alpha)}{q^{\varepsilon+1+n/2}}, \quad \forall \varepsilon > 0$$

where $c(\alpha)$ is a constant depending on α and ε can be arbitrary small.

This fact allows to construct KAM tori (α from Thue theorem being a ratio of frequencies for interacting waves) and KAM theorem states then that **almost** all tori are preserved.

Almost means *in particular* that tori with rationally related frequencies are explicitly **excluded** from consideration.

Since the union of invariant tori has positive measure and $\mathbb{Q}$ has measure 0, this exclusion is supposed to be not very important.

Notice that due to KAM theory spectral space is decomposed into *disjoint invariant sets*, and though it contradicts Kolmogorov´s ergodicity but not very substantially as the size of the system tends to infinity (Arnold, 1964).

Resonators, i.e. systems with **discrete spectra** have qualitatively different properties (Kartashova 1990, 1994, 1998):

- (a) all interacting waves with wave vectors k_i satisfying condition
$$\sum_i \pm\omega(\vec{k}_i) = 0$$
are divided into disjoint discrete classes

- (b) interactions are local

- (c) number of interacting waves depends on the form of boundary conditions and for the great number of boundary conditions interactions are not possible

- (d) major part of the waves do not interact

- (e) these properties hold for some $0 < \varepsilon << 1$:
$$\sum_i \pm\omega(\vec{k}_i) = \varepsilon, \quad \sum_i \pm\vec{k}_i = 0, \quad k_{i,j} \in \mathbb{Z}$$

How to construct classes (a)?

Example: Let dispersion function is

$$\omega = c|\vec{k}|^{-1/2},\ \vec{k} = (m,n),\ m,n \in \mathbb{Z}$$

corresponding to planetary waves in a square domain, and equation for frequencies of interacting waves is

$$\pm\frac{1}{\sqrt{m_1^2+n_1^2}} \pm \frac{1}{\sqrt{m_2^2+n_2^2}} \pm \frac{1}{\sqrt{m_3^2+n_3^2}} = 0.$$

Necessary condition for the existence of integer solutions is following: **all frequencies have the same irrationality**, i.e. each $\omega_i = \omega(\vec{k}_i)$ can be presented as

$$\omega_i = a_i\sqrt{q},\ a_i \in \mathbb{N}\quad \forall i = 1,2,3$$

with different constants a_i and **the same square-free** q. Vectors with the same q form a discrete class Cl_q.

Since $\omega_i/\omega_j \in \mathbb{Q}\quad \forall i,j = 1,2,3$, **these classes describe the waves which are excluded from KAM theory**.

IMPORTANT REMARK:

(a) is necessary condition: **if** vectors k_i satisfy

$$\sum_i \pm\omega_i = 0,$$

then they belong to one class; this condition gives no information about the existence of a solution of the system

$$\sum_i \pm\omega_i = 0, \ \sum_i \pm\vec{k}_i = 0.$$

Thus, though each class contains infinite number of waves but only a small number of them do interact (due $\sum_i \pm\vec{k}_i = 0$).

(b) **locality of interactions**: there exists spectral domain of the *finite* radius R to which all the waves belong interacting with a given one (m, n)

spherical Rossby waves: $\omega = m/n(n+1) \ \Rightarrow$

$$R = 2(n^2 + n - 1)$$

Rossby waves in square basin: $\omega = 1/\sqrt{k} \ \Rightarrow$

$$R = k^2 + k, \ \ k = |\vec{k}|$$

Gravity-capillary waves: $\omega^2 = gk + ak^3 \ \Rightarrow$

$$R = \frac{16}{9}k^3$$

(e) **dependence on the form of resonator**:

Example:

"square" dispersion $\omega = \frac{1}{\sqrt{m^2+n^2}}$ in a square domain axa

transforms into

"rectangular" dispersion $\omega = \frac{1}{\sqrt{(am)^2+(bn)^2}}$ in a rectangular domain axb

$\Rightarrow$ solutions in a rectangular domain are **a subset** of solutions in a square domain, such that m is divisible by a and n is divisible by b.

Example: wave (4,6) takes part in resonant interactions in axa but not in ax2a

(d) **existence of non-interacting waves**:

spherical Rossby waves: $\omega = m/n(n+1) \;\Rightarrow$

$<60\%$ of all waves do interact; 80% of **all interacting waves** are parts of only one triad and less then 2% of waves take part in 4 triads or more, i.e. chains of the coupled triads soon break

Rossby waves in square basin: $\omega = 1/\sqrt{k} \;\Rightarrow$

$<80\%$ of all waves do interact, chains break

Capillary waves: $\omega^2 = k^3 \;\Rightarrow$ no 3-wave interactions

Numerical simulations have been done with BVE on a sphere in a form

$$\frac{\partial \triangle\psi}{\partial t} + 2\frac{\partial \psi}{\partial \lambda} + \varepsilon J(\psi, \triangle\psi) = 0.$$

Here ψ is the stream-function; variables t, ϕ and λ physically mean the time, the latitude ($-\pi/2 \leq \phi \leq \pi/2$) and the longitude ($0 \leq \lambda \leq 2\pi$) respectively; $0 < \varepsilon << 1$ is small parameter. The spherical Laplacian and Jacobian are given by formulae

$$\triangle\psi = \frac{\partial^2 \psi}{\partial \phi^2} + \frac{1}{\cos^2\phi}\frac{\partial^2 \psi}{\partial \lambda^2} - \tan\phi \frac{\partial \psi}{\partial \phi}$$

and

$$J(a, b) = \frac{1}{cos\phi}(\frac{\partial a}{\partial \lambda}\frac{\partial b}{\partial \phi} - \frac{\partial a}{\partial \phi}\frac{\partial b}{\partial \lambda})$$

respectively.

Example of transition from discrete to continuous spectra for BVE on a unit sphere.

A linear spherical wave in this case has form

$$\psi_{sphere} = AP_n^m(\sin\phi)\exp i[m\lambda + \frac{2m}{n(n+1)}t]$$

with $\omega = m/n(n+1)$ and $P_n^m(x)$ is the associated Legendre function of degree n and order m; and a linear plane wave is

$$\psi_{plane} = A\exp i(k_x x + k_y y + \omega)$$

with $\omega = k_x/(1+k_x^2+k_y^2)$. Regarding $m \sim n >> 1$ and using asymptotic approximation for Legendre functions, one can "convert" (not always but in a bounded latitudinal belt with the width $\sim n^{-1}$) one spherical wave into a linear combination of two plane waves

$$A\exp i(k_x(\varphi_0)x \pm k_y(\varphi_0)y + \omega),$$

where local wave numbers $k(\varphi_0)_x, k(\varphi_0)_y \in \mathbb{R}$ are functions of the initial spherical wave number m, n and of the so-called interaction latitude $\varphi_0 = \varphi_0(m_1, .., n_3)$ which is explicit function of all three interacting vectors. If $0 < \cos^2\varphi_0 < 1$, plane images of spherical waves interact as in classical β-plane approximation.

INTERESTING:

- Not all spherical waves have plane images (transition is **not always** possible)

- Plane wave system **keeps memory** about spherical interactions: coupling coefficient of the plane images of spherical waves is $\sim n^{3/2}$ and $\sim n^{7/6}$ otherwise.

Transition from a square domain to infinite β-plane gives even more substantial difference in magnitudes of coupling coefficients: $\sim n^2$ for plane images of the waves from square domain and $\sim n$ otherwise.

Conclusion: long-wave part of spectrum is dominated by a few exactly interacting waves with huge amplitudes while short-wave part of the spectrum consists of many approximately interacting waves with substantially smaller amplitudes.

Continuous WT (CWT) describes energetic behavior of a wave system for the whole spectrum leaving some **gaps in the spectrum** which are supposed to be not important in short-wave part.

Discrete WT (DWT) **fills the gaps** all over the spectrum.

In fact we have two layers of turbulence - CWT (layer I) and DWT (layer II), which are mutually complementary and should be regarded simultaneously $\Rightarrow$

A model of laminated turbulence:

Layer I: KAM tori and stochastic enough turbulence in the short-waves range with power-law spectra; direct/inverse energy cascades; wave-numbers range of energy pumping influences the results.

Layer II: a countable number of waves with big amplitudes all over the wave spectrum; some of the waves do not change their energies (non-interacting waves) and others do exchange energy within small independent groups; no energy cascades; results do not depend on the wave-numbers range of energy pumping.

According to apocryphal story, **Werner Heisel-berg** said on his deathbed:

"When I meet God, I am going to ask him two questions:

Why relativity?

And why turbulence?

I really believe he will have the answer for the first."

THE END

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