The Nastrom-Gage energy spectrum of the atmosphere

proposed theoretical explanations, and comparison with the predicted energy spectrum of superfluid turbulence

Eleftherios Gkioulekas, Ka-Kit Tung and Wendy Orlando

Department of Applied Mathematics, University of Washington

Publications

1. W.T. Welch, and K. K. Tung (1998), *J. Atmos. Sci.* **55**, 2833-2851.
2. K.K. Tung and W.T. Welch (2001), *J. Atmos. Sci.* **58**, 2009-2012.
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4. K.K. Tung and W.W. Orlando (2003b), Discrete and Continous Dynamical Systems B **3**, 145-162.
5. K.K. Tung (2004), *J. Atmos. Sci.*, **61**, 943-948.
6. E. Gkioulekas and K.K. Tung (2005), *Discrete and Continuous Dynamical Systems B*, **5**, 79-102
7. E. Gkioulekas and K.K. Tung (2005), *Discrete and Continuous Dynamical Systems B*, **5**, 103-124.
8. E. Gkioulekas and K.K. Tung, *J. Fluid Mech.*, submitted. [nlin.CD/0512005]
9. K.K. Tung and E. Gkioulekas, *J. Atmos. Sci.*, submitted. [nlin.CD/0507042]

KLB theory

Kraichnan, Leith, and Batchelor (KLB) proposed that in two-dimensional turbulence there is an upscale energy cascade and a downscale enstrophy cascade. The energy spectrum in the upscale energy range is

$$E(k) = C_{ir}\varepsilon^{2/3}k^{-5/3}, \tag{1}$$

and in the downscale enstrophy range is

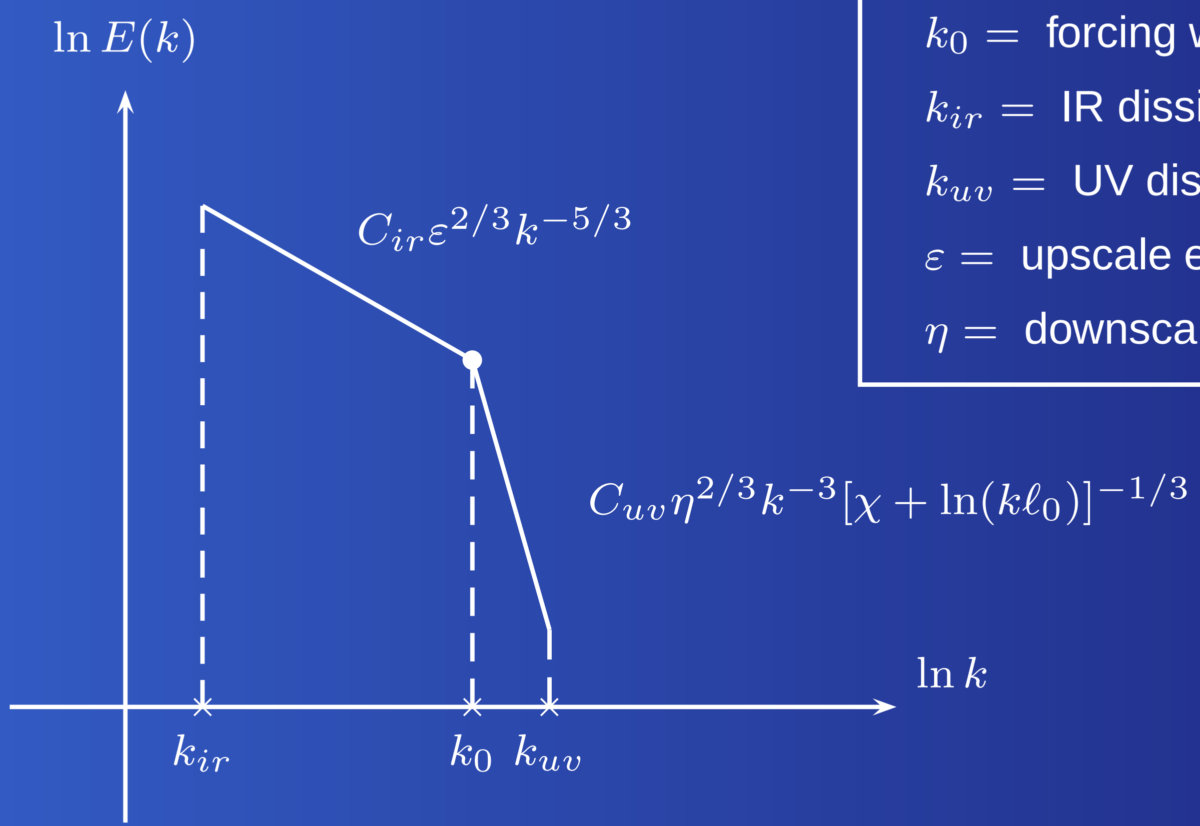
$$E(k) = C_{uv}\eta^{2/3}k^{-3}[\chi + \ln(k\ell_0)]^{-1/3}. \tag{2}$$

Falkovich and Lebedev (1994) predict that the vorticity ζ structure functions have logarithmic scaling given by

$$\langle[\zeta(\mathbf{r}_1) - \zeta(\mathbf{r}_2)]^n\rangle \sim [\eta\ln(\ell_0/r_{12})]^{2n/3}. \tag{3}$$

Confirmed using spectral reduction by Bowman, Shadwick and Morrison (1999).

KLB energy spectrum

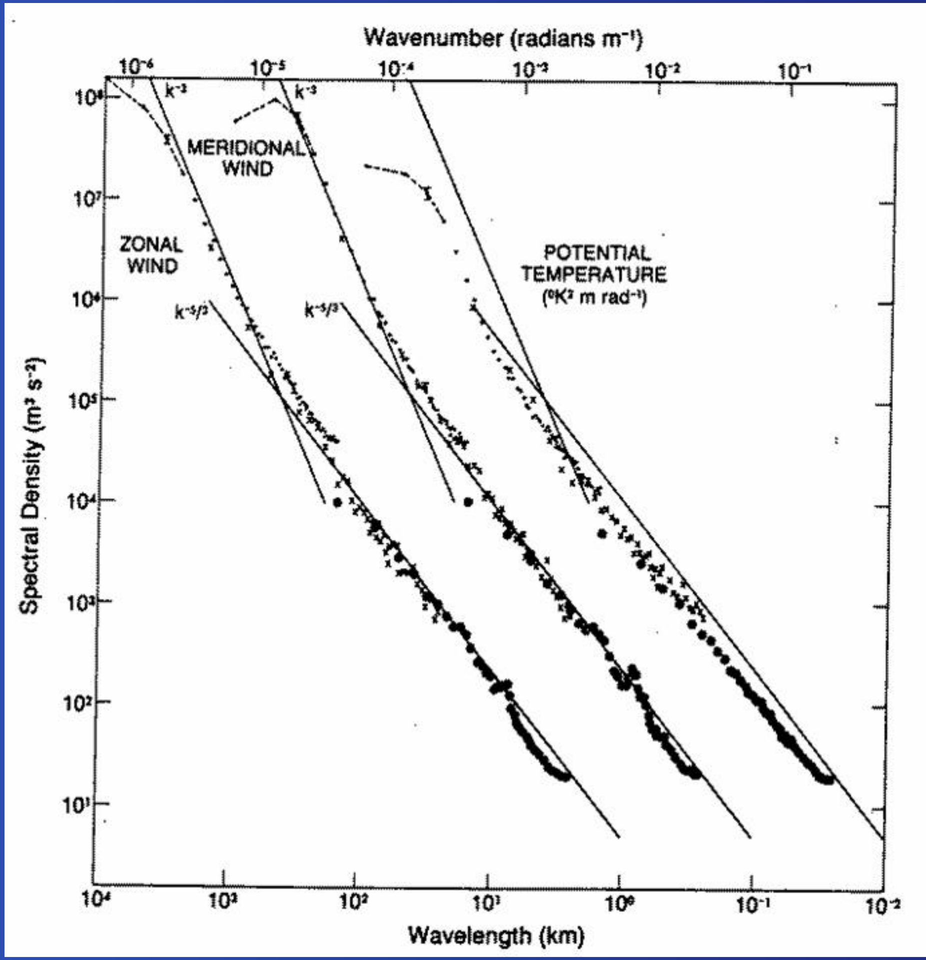


- $k_0 =$ forcing wavenumber
- $k_{ir} =$ IR dissipation wavenumber
- $k_{uv} =$ UV dissipation wavenumber
- $\varepsilon =$ upscale energy flux
- $\eta =$ downscale ensrropht flux

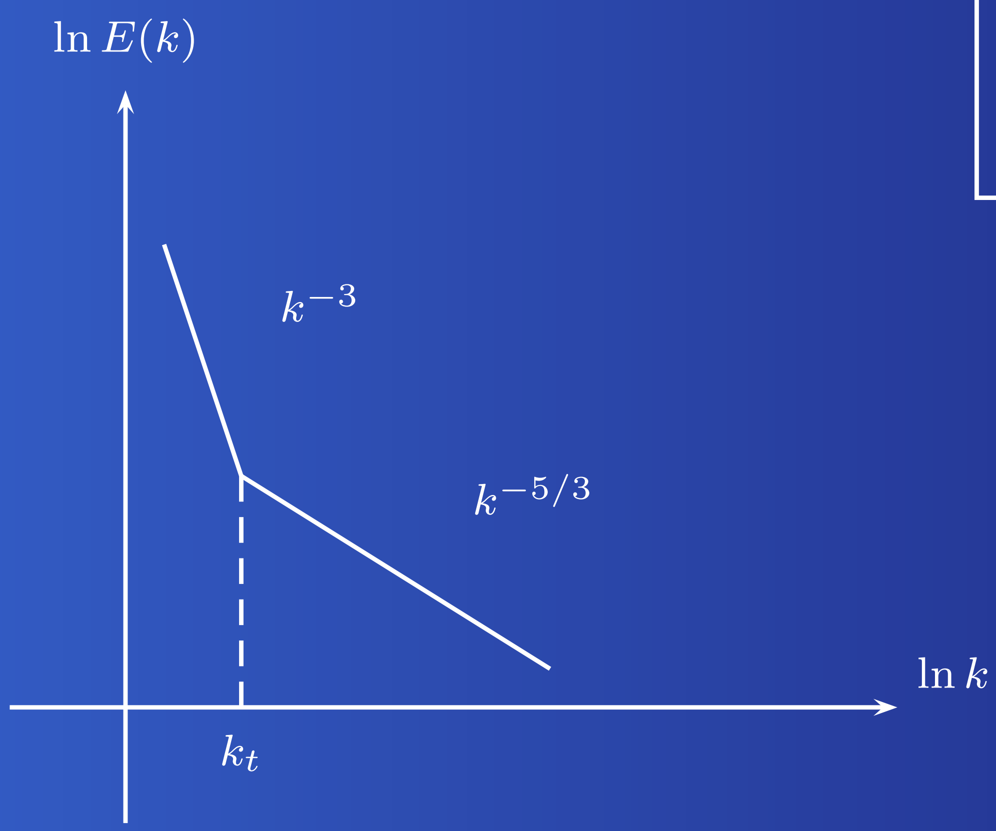
Motivation

- The study of two-dimensional turbulence was originally motivated by the hope that it would prove a useful model for atmospheric turbulence.
- This idea was later encouraged by Charney (1971) who claimed that quasi-geostrophic turbulence is isomorphic to two-dimensional turbulence.
- Early observations suggested that the energy spectrum of the atmosphere follows a k^{-3} power law behavior (see Tung and Orlando (2003) for review).
- Analysis of GASP measurements by Nastrom and Gage (1984) shows a transition to $k^{-5/3}$ scaling.

Nastrom-Gage spectrum



Nastrom-Gage spectrum schematic



$k^{-3} \longrightarrow$ 3000km – 800km

$k^{-5/3} \longrightarrow$ 600km– $\ll$ 1km

$k_t \approx$ 700km

The $k^{-5/3}$ part of NG spectrum

- The Nastrom-Gage energy spectrum was confirmed recently with MOSAIC program and GCM simulation
- The k^{-3} is interpretated as downscale enstrophy cascade.
- An explanation of the $k^{-5/3}$ in terms of internal gravity waves was ruled out by Gage and Nastrom (1986).
- Gage and Nastrom (1986) suggest a large-scale source of enstrophy (baroclinic instability) and a small scale source of energy that sends some energy upscale and downscale.
- Lilly (1989) theorized that energy source at small scales can be attributed to thunderstorms.
- The $k^{-5/3}$ portion of the spectrum, appears to be approximately the same whether it is in winter or summer, and whether the airplane flew over storms or not.

NLV spectrum I

- In superfluid turbulence there is a predicted energy spectrum by L'vov, Nazarenko and Volovik (2004) which transitions from k^{-3} to $k^{-5/3}$.
- Slope steepens at large scales due to effective linear damping.
- The energy spectrum for a 3D downscale energy cascade with Ekman damping is predicted as

$$E(k) = C\varepsilon^{-2/3}(k)k^{-5/3} \tag{4}$$

$$\frac{\partial \varepsilon(k)}{\partial k} = -\Gamma E(k) = -\Gamma C\varepsilon^{-2/3}(k)k^{-5/3} \tag{5}$$

which leads to the solution

$$E(k) = C[\varepsilon_0^{1/3} - \Gamma C k^{-2/3}]^2 k^{-5/3} \approx \begin{cases} C\varepsilon_0^{2/3}k^{-5/3}, & k \gg k_t \\ C^3\Gamma k^{-3}, & k \ll k_t \end{cases} \tag{6}$$

NLV spectrum II

- Is the k^{-3} slope in the Nastrom-Gage spectrum caused by Ekman damping?
- The predicted transition wavenumber k_t is given by

$$k_t = \sqrt{\frac{(\Gamma C)^3}{\varepsilon_0}} \tag{7}$$

- Using the energy flux ε_0 from Cho and Lindborg (2001) and the estimated Ekman damping Γ:

$$\varepsilon_0 \approx 6 \times 10^{-11}\ \text{km}^2\text{s}^{-3}, \quad \Gamma \approx (1/7)\ \text{days}^{-1}, \quad C \approx 1 \tag{8}$$

  we get an estimated transition scale

$$\ell_t = \sqrt{2\pi} k_t \approx 23000\ \text{km} \gg 700\ \text{km} \tag{9}$$

- The ε_0 estimate involves assumptions that need to be confirmed.

The double cascade theory. I

- Tung and Orlando (2003) conjectured that the observed atmospheric energy spectrum results from the downscale cascade of enstrophy and energy injected at the large scales by baroclinic instability and dissipated at the smallest length scales.
- If η_{uv} is the downscale enstrophy flux and ε_{uv} is the downscale energy flux, the transition from -3 slope to $-5/3$ slope occurs at the transition wavenumber k_t with order of magnitude estimated by

$$k_t \approx \sqrt{\eta_{uv}/\varepsilon_{uv}}. \tag{10}$$

The double cascade theory. II

- Recent measurements and data analysis by Cho and Lindborg (2001) have confirmed the existence of a *downscale* energy flux and estimate

$$\eta_{uv} \approx 2 \times 10^{-15}\text{s}^{-3} \quad (11)$$

$$\varepsilon_{uv} \approx 6 \times 10^{-11}\text{km}^2\text{s}^{-3} \quad (12)$$

- From these estimates we find the mean value of the transition scale

$$k_t = \sqrt{\eta_{uv}/\varepsilon_{uv}} \approx 0.57 \times 10^{-2}\text{km}^{-1} \Longrightarrow \lambda_t = 2\pi/k_t \approx 1 \times 10^3\text{km} \quad (13)$$

  which has the correct order of magnitude.

- Tung and Orlando (2003) have also demonstrated numerically that a two-layer quasi-geostrophic channel model with thermal forcing, Ekman damping, and hyperdiffusion can reproduce the atmospheric energy spectrum.

The two-layer model

The governing equations for the two-layer quasi-geostrophic model read:

$$\frac{\partial \zeta_1}{\partial t} + J(\psi_1, \zeta_1) = d_1 + f_1, \quad \frac{\partial \zeta_2}{\partial t} + J(\psi_2, \zeta_2) = d_2 + f_2, \tag{14}$$

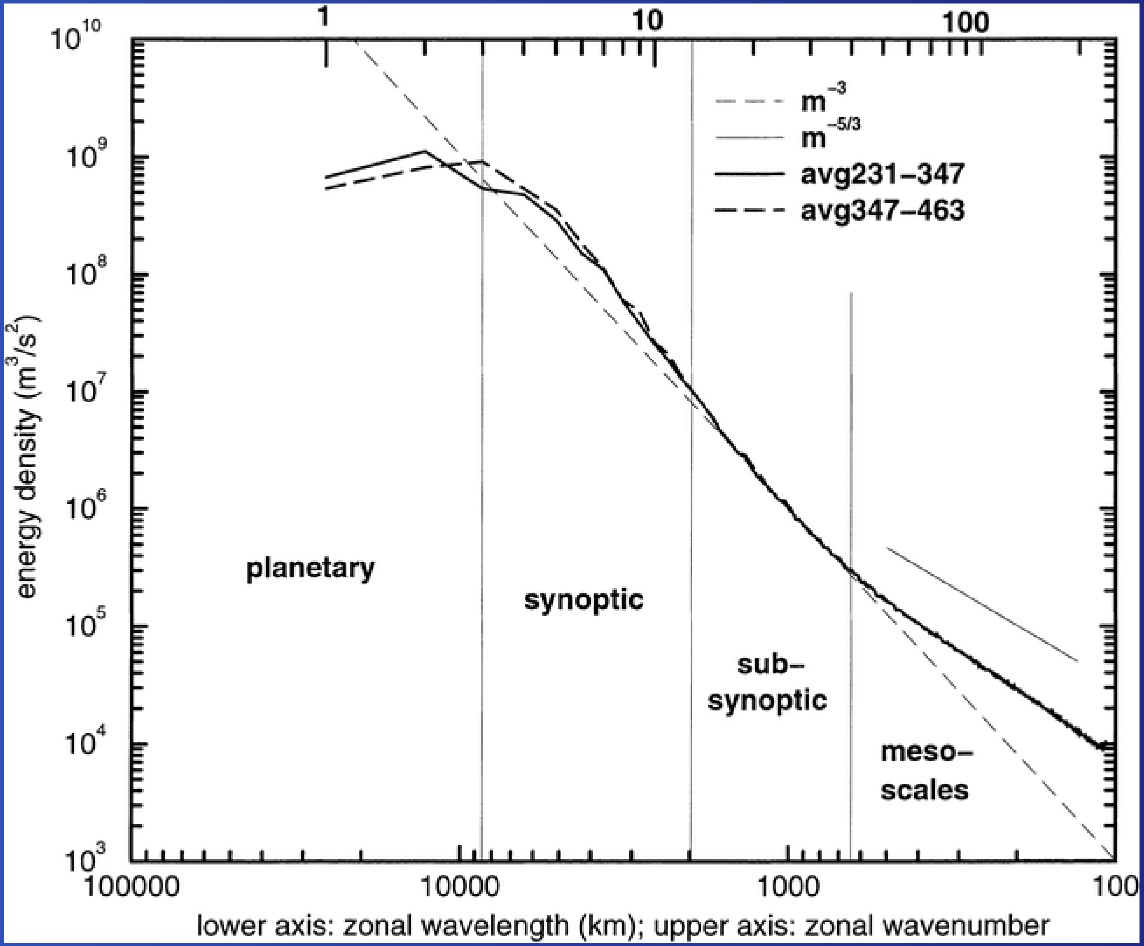
where ζ_1 is the potential vorticity of the top layer and ζ_2 the potential vorticity of the bottom layer. The relationship between the vorticities ζ_1 and ζ_2 and the streamfunctions ψ_1 and ψ_2 reads:

$$\zeta_1 = \Delta\psi_1 - \frac{k_R^2}{2}(\psi_1 - \psi_2), \quad \zeta_2 = \Delta\psi_2 + \frac{k_R^2}{2}(\psi_1 - \psi_2), \tag{15}$$

Here, $k_R \equiv (2\sqrt{2}f)/(hN)$ is the Rossby radius of deformation wavenumber, and

$$f_1 = \frac{k_R^2}{2f}Q, \quad f_2 = -\frac{k_R^2}{2f}Q, \quad d_1 = \nu(-\Delta)^{\kappa+1}\psi_1, \quad d_2 = \nu(-\Delta)^{\kappa+1}\psi_2 - \nu_E\Delta\psi_2 \tag{16}$$

Tung and Orlando spectrum



Debate

- Smith (2004) debated the theory of Tung and Orlando (2003) by arguing that the downscale energy cascade can never have enough flux to move the transition wavenumber k_t into the inertial range.
- Smith (2004) uses two-dimensional Navier-Stokes for his argument. Tung (2004) replies that the two-layer model is a different dynamical system than the two-dimensional Navier-Stokes equations
- Debate clarified further in a series of papers by Gkioulekas and Tung:
  1. K.K. Tung (2004), *J. Atmos. Sci.*, **61**, 943-948.
  2. E. Gkioulekas and K.K. Tung (2005), *Discrete and Continuous Dynamical Systems B*, **5**, 79-102
  3. E. Gkioulekas and K.K. Tung (2005), *Discrete and Continuous Dynamical Systems B*, **5**, 103-124.
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Superposition principle. I

Gkioulekas and Tung (2005) have shown that a leading downscale enstrophy cascade *and a subleading downscale energy cascade* contribute linearly to the total energy spectrum:

$$E(k) = E_{uv}^{(\varepsilon)}(k) + E_{uv}^{(\eta)}(k) + E_{uv}^{(p)}(k), \; \forall k\ell_0 \gg 1, \tag{17}$$

where $E_{uv}^{(\varepsilon)}(k)$, $E_{uv}^{(\eta)}(k)$ are the contributions of the downscale energy and enstrophy cascade, given by

$$\begin{aligned} E_{uv}^{(\varepsilon)}(k) &= a_{uv}\varepsilon_{uv}^{2/3} k^{-5/3} \mathcal{D}_{uv}^{(\varepsilon)}(k\ell_{uv}^{(\varepsilon)}) \\ E_{uv}^{(\eta)}(k) &= b_{uv}\eta_{uv}^{2/3} k^{-3}[\chi + \ln(k\ell_0)]^{-1/3} \mathcal{D}_{uv}^{(\eta)}(k\ell_{uv}^{(\eta)}), \end{aligned} \tag{18}$$

Thus, in the inertial range where the effect of forcing and dissipation can be ignored, the energy spectrum will take the simple form

$$E(k) \approx a_{uv}\varepsilon_{uv}^{2/3} k^{-5/3} + b_{uv}\eta_{uv}^{2/3} k^{-3}[\chi + \ln(k\ell_0)]^{-1/3}. \tag{19}$$

Superposition principle. II

- Justification based on work by L'vov and Procaccia as follows:
- Define the fully unfused correlation tensors for velocity u_α and vorticity ζ:

$$F_n^{\alpha_1\alpha_2\ldots\alpha_n}(\{\mathbf{x}_k,\mathbf{x'}_k\}_{k=1}^n,t) = \left\langle \left[\prod_{k=1}^n u_{\alpha_k}(\mathbf{x},t) - u_{\alpha_k}(\mathbf{x}',t)\right]\right\rangle , \qquad (20)$$

$$V_n(\{\mathbf{x}_k,\mathbf{x'}_k\}_{k=1}^n,t) = \left\langle \left[\prod_{k=1}^n \zeta(\mathbf{x}_k,t) - \zeta(\mathbf{x'}_k,t)\right]\right\rangle \qquad (21)$$

- The relation between F_n and V_n is $V_n = \mathcal{T}_n F_n$ or:

$$V_n(\{\mathbf{x}_k,\mathbf{x'}_k\}_n,t) = \prod_{k=1}^n [\varepsilon_{\alpha_k\beta_k}(\partial_{\alpha_k,\mathbf{x}_k} + \partial_{\alpha_k,\mathbf{x'}_k})]F_n^{\alpha_1\cdots\alpha_n}(\{\mathbf{x}_k,\mathbf{x'}_k\}_n,t) \qquad (22)$$

Superposition principle. III

- F_n and V_n satisfy the balance equations:

$$\frac{\partial F_n}{\partial t} + \mathcal{O}_n F_{n+1} + I_n = \mathcal{D}_n F_n + Q_n \tag{23}$$

$$\frac{\partial V_n}{\partial t} + \mathcal{T}_n \mathcal{O}_n F_{n+1} + \mathcal{I}_n = \mathcal{D}_n V_n + \mathcal{Q}_n \tag{24}$$

  Here $Q_n, \mathcal{Q}_n$ are forcing terms and $I_n, \mathcal{I}_n$ are sweeping terms, $\mathcal{O}_n$ local interactions, and $\mathcal{D}_n$ the dissipation operator.

- Belinicher, L'vov and Procaccia (1998) argue that in 3D turbulence, the scaling of the downscale energy cascade originates from the solvability condition on the homogeneous equation

$$\mathcal{O}_n F_{n+1} = 0 \tag{25}$$

  This argument predicts multifractal scaling.

- Use Feynman mind-trick: “The same equations have the same solutions”

Superposition principle. IV

- The enstrophy cascade solution demands that the vorticity statistics be stationary. True in 2D. Not true in 3D.
- In two-dimensional turbulence, homogeneous solutions originate from

$$\mathcal{O}_n F_{n+1} = 0 \Longrightarrow \text{1 solution: energy cascade} \quad (26)$$

$$\mathcal{T}_n \mathcal{O}_n F_{n+1} = 0 \Longrightarrow \text{2 solutions: energy and enstrophy cascade} \quad (27)$$

- The balance equations essentially have two homogeneous solutions (energy/enstrophy cascade) and a particular solution (coherent structures) which is caused by Q_n and I_n.

Superposition principle. V

- The realistic solutions for each cascade include a dissipation range. These solutions originate from the modified equation

$$\mathcal{T}_n \mathcal{O}_n F_{n+1} - \mathcal{T}_n \mathcal{D}_n F_n = 0. \tag{28}$$

- The dissipative terms $\mathcal{D}_n$ modify the linear operator $\mathcal{O}_n$ and in doing so *modify the homogeneous solutions* responsible both for the leading and subleading cascades both downscale and upscale. The modification amounts to truncating the inertial range with the dissipation range.
- The location of the dissipation scale corresponding to one of the homogeneous solutions present is independent of the energy or enstrophy flux corresponding to the other homogeneous solutions.
- Thus, the dissipation scales can be estimated with dimensional analysis.

KLB limit

- The dissipation scale $\ell_{uv}^{(\eta)}$ of the downscale enstrophy cascade and the dissipation scale $\ell_{uv}^{(\varepsilon)}$ of the downscale energy cascade are given by

$$\ell_{uv}^{(\eta)} = \ell_0 \left[\frac{\mathcal{R}_{uv}^{(\eta)}}{\mathcal{R}_{0,uv}^{(\eta)}} \right]^{-1/(2\kappa)} = \left[\frac{1}{\mathcal{R}_{0,uv}^{(\eta)}} \frac{\eta_{uv}^{1/3}}{\nu} \right]^{-1/(2\kappa)} \quad (29)$$

$$\ell_{uv}^{(\varepsilon)} = \ell_0 \left[\frac{\mathcal{R}_{uv}^{(\varepsilon)}}{\mathcal{R}_{0,uv}^{(\varepsilon)}} \right]^{3/(2-6\kappa)} = \left[\frac{1}{\mathcal{R}_{0,uv}^{(\varepsilon)}} \frac{\varepsilon_{uv}^{1/3}}{\nu} \right]^{3/(2-6\kappa)} \quad (30)$$

- In the KLB limit $\mathcal{R}_{uv}^{(\eta)} \to +\infty$ the dissipation scale of the subleading downscale energy cascade is given asymptotically by

$$\ell_{uv}^{(\varepsilon)} \approx \ell_{uv}^{(\eta)} \left(\frac{\mathcal{R}_{0,uv}^{(\varepsilon)}}{\mathcal{R}_{0,uv}^{(\eta)}} \right)^{3/(6\kappa-2)} , \quad \text{and } k_t \lambda_{uv} \to 1 \quad (31)$$

Danilov Inequality. I

- Why is the downscale energy cascade hidden in two-dimensional turbulence?
- In two-dimensional turbulence, the energy flux $\Pi_E(k)$ and the enstrophy flux $\Pi_G(k)$ are constrained by

$$k^2\Pi_E(k) - \Pi_G(k) < 0, \tag{32}$$

  for all wavenumbers k outside of the forcing range.
- The transition wavenumber k_t where the break from k^{-3} to $k^{-5/3}$ should occur, approaches the dissipation scale from the dissipation range, thus a transition cannot be seen visually in two-dimensional turbulence.
- Thus, the contribution of the downscale energy cascade to the energy spectrum is overwhelmed by the contribution of the downscale enstrophy cascade.

Danilov Inequality. II

- Recall that in the two-layer model, the dissipation terms read:

$$d_1 = \nu(-\Delta)^{\kappa+1}\psi_1, \tag{33}$$

$$d_2 = \nu(-\Delta)^{\kappa+1}\psi_2 - \nu_E\Delta\psi_2 \tag{34}$$

- We have shown that it is the asymmetric presence of Ekman damping on the bottom layer but not the top layer which causes the violation of the Danilov inequality in the two-layer model and moves the transition wavenumber k_t into the inertial range.
- A necessary (but not sufficient) condition to *violate* Danilov's inequality is

$$\nu_E > 2\nu k_{\max}^{2p}\left(\frac{k_{\max}}{k_R}\right)^2 \tag{35}$$

- Many open questions remain.

Open question: Helicity cascade

- It has been claimed that in the Nastrom-Gage spectrum we have a transition from $k^{-7/3}$ (downscale helicity cascade), instead of k^{-3}, to $k^{-5/3}$. References advocating this position include:
  1. A.Bershadskii, E.Kit, and A. Tsinober, *Proc. R. Soc. Lond. A* **441** (1993), 147–155.
  2. S.S. Moiseev and O.G. Chkhetiani, *JETP* **83** (1996), 192–198.
  3. H. Branover, A. Eidelman, E. Golbraikh, and S. Moiseev, *Turbulence and structures: chaos, fluctuations, and helical self-organization in nature and the laboratory*, Academic Press, San Diego, 1999.
- Cho and Lindborg (2001) showed that $S_3(r) \sim r^3$ (diagonal components) in the polar stratosphere data, which supports an enstrophy cascade. There is also an unexplained robust r^2 contribution to the off-diagonal components in the stratosphere from 10 km to 1,000 km in scale.
- Open question. Need transition scale calculation to confirm or deny.

Conclusion

- The $k^{-5/3}$ portion of the Nastrom-Gage spectrum is a downscale energy cascade.
- The k^{-3} interpretation for small wavenumbers could be wrong.
- The $k^{-3} \rightarrow k^{-5/3}$ interpretation can be accounted for with a two-layer model
- The dynamics of the two-layer model are interesting and not well-understood.