Turbulence and Coherent structures in Bose Gases

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Plan of presentation

- Strongly nonequilibrium BEC formation in a weakly interacting Bose gas (with Boris Svistunov)

  weak turbulence → strong turbulence → superfluid turbulence

- BEC formation in a mixture of two Bose gases (with Chen Yin)

  increased rate of condensation $N_2 \approx \frac{1}{4} N_1$
  mixture of "drift" and "cascade" scenarios

- Complete families of solitary waves in condensate systems

Strongly nonequilibrated Bose-Einstein condensation

[NGB & Svistunov, Phys. Rev. A **66** 013603 (2002)]

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\frac{\nabla^2\psi}{2m} + U|\psi|^2\psi$$

NLS gives an accurate microscopic description of the formation of a BEC from a strongly degenerate gas of weakly interacting bosons
[Levich and Yakhot, JPA (1978); Kagan and Svistunov, PRL (1997)]

Kinetic description of Weak turbulence regime
[Zakharov *et al* (1985); Svistunov (1991); Kagan et al (1992); Semikoz & Tkachev (1997); Josserand & Pomeau (2001); NGB & Svistunov (2002); Nazarenko & Zakharov (2005)]

$$\psi(\mathbf{r}, t=0) = \sum_{\mathbf{k}} a_{\mathbf{k}} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}),$$

KE – equation on "occupation numbers" $n_{\mathbf{k}} = |a_{\mathbf{k}}|^2$ averaged over ensemble of states.

Criterion of Weak Turbulence regime:
frequency of rotation of phases of $a_{\mathbf{k}} \gg$ energy of nonlinear interactions

$$k_*^2/m \gg U n_{k_*} k_*^3$$

NLS in Fourier components

$$i\dot{a}_1 = \frac{k_1^2}{2m}a_1 + U\sum_{234} a_2^* a_3 a_4 \,\delta_{\mathbf{k}_1;\mathbf{k}_3+\mathbf{k}_4-\mathbf{k}_2}$$

$$\dot{n}_1 = \frac{\partial}{\partial t}a_1^* a_1 = 2UIm\sum_{234} a_1^* a_2^* a_3 a_4 \,\delta_{\mathbf{k}_1;\mathbf{k}_3+\mathbf{k}_4-\mathbf{k}_2}$$

Take ensemble average $A_{1234} = \langle a_1^* a_2^* a_3 a_4 \rangle$

$$\dot{A}_{1234} = i\Delta\epsilon A_{1234} + i2U(n_2 n_3 n_4 + n_1 n_3 n_4 - n_1 n_2 n_3 - n_1 n_2 n_4)$$

where $\Delta\epsilon = \epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4$, $\epsilon_i = k_i^2/2m$

Kinetic equation becomes

$$\dot{n}_1 = \frac{4\pi U^2}{(2\pi)^6}\int d\mathbf{k}_2 d\mathbf{k}_3 \delta(\Delta\epsilon)(n_2 n_3 n_4 + n_1 n_3 n_4 - n_1 n_2 n_3 - n_1 n_2 n_4)$$

where $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$.

Characteristic kinetic time τ_{kin}: $\tau_{kin}^{-1} \sim m^3 U^2 \epsilon_*^2 n_{\epsilon_*}^2$

Self-similar cascade solution

Kinetic equation of the form $\dot{n}_\epsilon \sim \epsilon^2 n_\epsilon^3$
Conservation of the number of particles $N \propto \int \epsilon^{1/2} n_\epsilon d\epsilon$
Conservation of energy $E \propto \int \epsilon^{3/2} n_\epsilon d\epsilon$.

First possibility: drift of the particle distribution towards lower energies
Conservation of particles: $n_\epsilon \propto \epsilon_0^{-3/2}(t) f(\epsilon/\epsilon_0)$
From KE: $\epsilon_0(t) \propto t$ – *not possible!*

Second possibility: particle cascade
Divergent integral of conservation of particles
Existence of t^* at which $\epsilon = 0$

$$n_\epsilon \propto \epsilon_0(t)^{-\alpha} f(\epsilon/\epsilon_0)$$

$$f(x) \propto 1/x^\alpha \qquad x \to \infty$$

To find $\epsilon_0(t)$:
$f(x) = x^{-\alpha}(1 + cx^{-\sigma})$ as $x \to \infty$
but $n_\epsilon(t) = n_\epsilon(t^*) + \dot{n}_\epsilon(t^*)(t - t^*) + \cdots, \quad \epsilon \gg \epsilon_0(t)$
Consistency with KE $\epsilon_0(t) \propto (t^* - t)^{1/\sigma}$ where $\sigma = 2(\alpha - 1)$; $1 < \alpha < \frac{3}{2}$
$\alpha \approx 1.24$ [Semikoz and Tkatchev PRD, 1997]

Strongly Non-equilibrium Bose-Einstein Condensation

Weak turbulence regime:

$$n_\epsilon = A\epsilon_0(t)^{-\alpha} f(\epsilon/\epsilon_0), \qquad t \le t^*$$

$$\epsilon_0(t) = B(t^* - t)^{1/2(\alpha-1)}$$

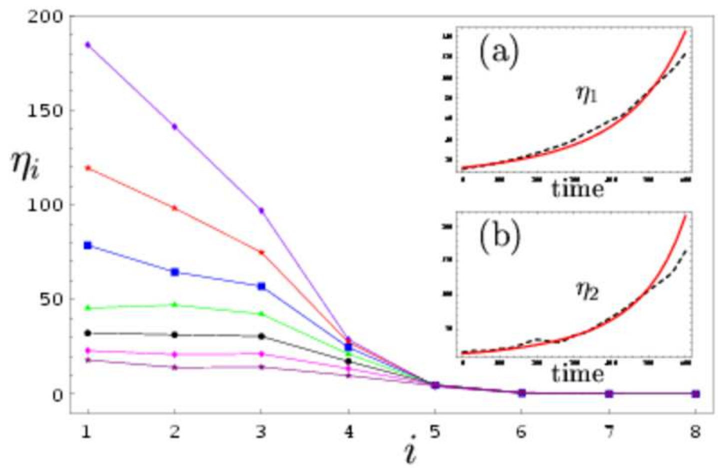
with $m^3 U^2 A^2 \approx \pi^3 \hbar^7 B^{2(\alpha-1)}$.

Direct Numerical Simulations of NLS: $\psi(\mathbf{r}, t = 0) = \sum_{\mathbf{k}} a_{\mathbf{k}} \exp(\mathrm{i}\mathbf{k} \cdot \mathbf{r})$

$$a_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}} n_0 f(\epsilon/\epsilon_0)}\, \exp[i\phi_{\mathbf{k}}]$$

where $\xi_{\mathbf{k}}$ and $\phi_{\mathbf{k}}$ are random numbers.
Grid size $N = 256$, $n_0 = 15$, $\epsilon_0 = 1/18$.
Period of amplitude oscillation $t_p = 2\pi/\epsilon_0 = 113$.
Number of periods before the blow-up is $P = t^*/t_p \approx 8$.

$\eta_i(t) = \sum^{(\text{shell } i)} n_{\mathbf{k}}(t)/M_i$, where M_i is the number of harmonics in the i-th shell



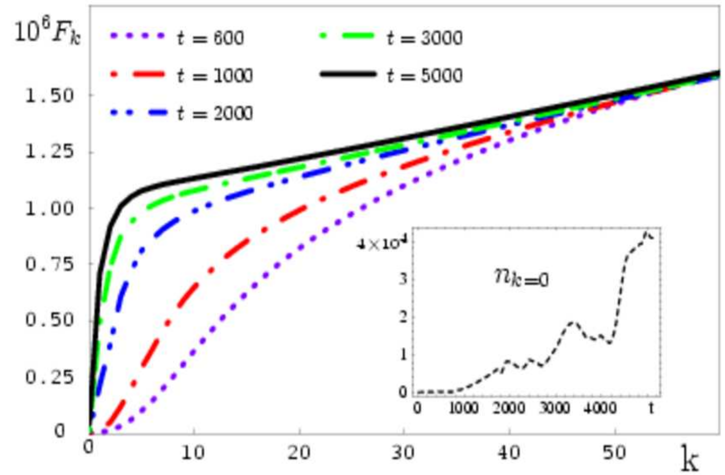
At $t_0 < t^*$ the self-similarity of solution breaks down $\Rightarrow$
Regime of STRONG TURBULENCE with formation of QUASICONDENSATE with vortex tangle.
From the dimensional analysis and numerics the characteristic time, t_0, and the characteristic wave vector, k_0, at the beginning of the strong turbulence regime are given by the relations

$$t_* - t_0 \sim 40[\hbar^{2\alpha+5}/m^3U^2A^2]^{1/(2\alpha-1)} ,$$

$$k_0 \sim 40[AU(m/\hbar)^{\alpha+1}]^{1/(2\alpha-1)} .$$

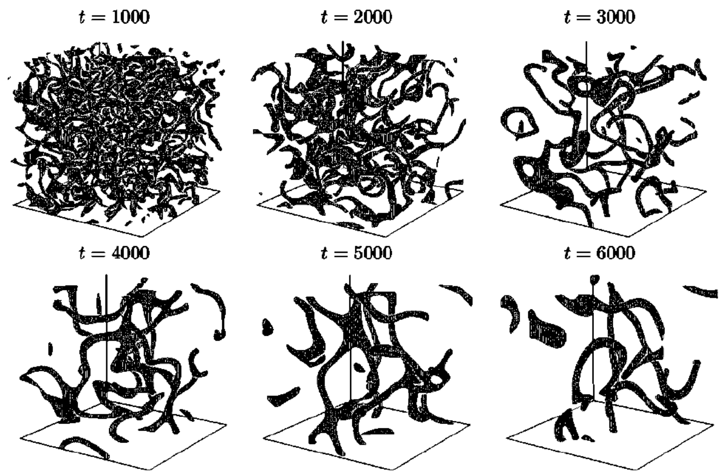
Separation between lines is of the order of their cores $\sim k_0^{-1}$.

Evolution of the integral distribution of particles $F_k = \sum_{k' \leq k} n_{\mathbf{k}'}$:

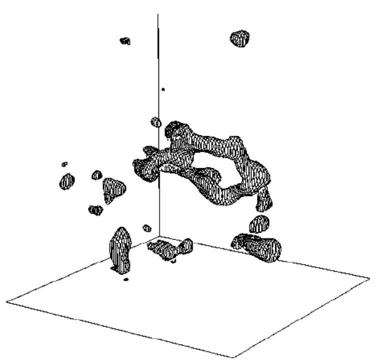


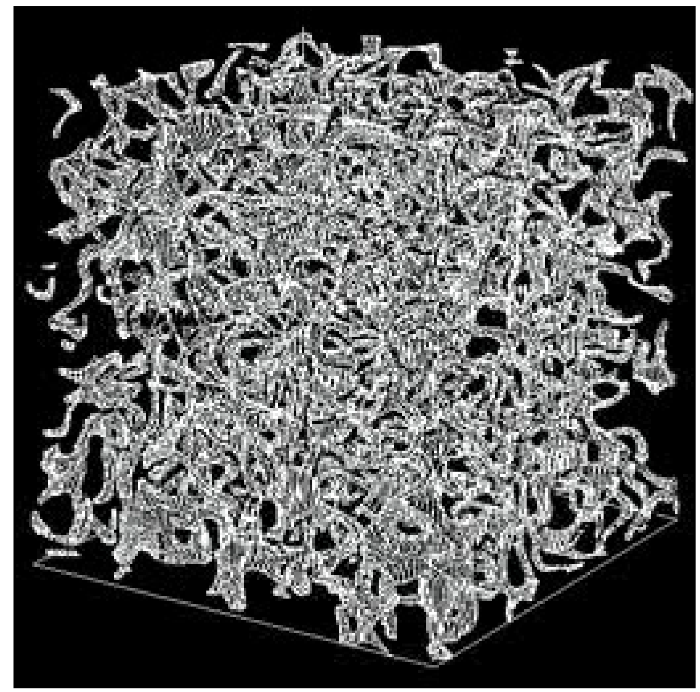
Evolution of topological defects

High-frequency wave suppression



Averaging in time





SUPERFLUID TURBULENCE: Evolution of the vortex tangle

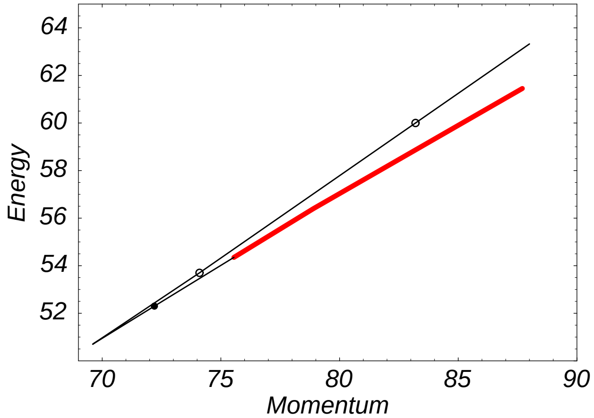
$\tau_{st} \sim R^2/\ln(R/a)$; R is the interline spacing, a is the vortex core size.

Superfluid turbulence decay: Nore *et al* (1997); Kobayashi & Tsubota (2005)

Solitary waves in BEC

Cylindrical coordinates (s, θ, z): $2\mathrm{i}U\frac{\partial\psi}{\partial z} = \nabla^2\psi + (1 - |\psi|^2)\psi.$

Family of solitary waves [Jones & Roberts, J.Phys A (1982)]



$$p = \frac{1}{2i}\int \nabla\psi(\psi^* - 1) - \nabla\psi^*(\psi - 1)\, dV$$

$$\mathcal{E} = \frac{1}{2}\int |\nabla\psi|^2 + \frac{1}{2}(1 - |\psi|^2)^2\, dV.$$

Generalized Pade approximations with the correct asymptotic behaviour [NGB, J. Phys. A, **37**, 1617 (2004)]

Stability: Lower branch is linearly stable. Upper branch is linearly unstable to axisymmetric infinitesimal perturbations, but the growth rates are small. Spectrum σ^2 is real and changes sign at the cusp.
[NGB & Roberts, J.Phys. A: **37**, 11333 (2004)]

Coupled Gross-Pitaevskii system

Simultaneous trapping and cooling of atoms in distinct spin or hyperfine levels ^{87}Rb (JILA, NIST) or of different atomic species ^{41}K-^{87}Rb (LENS)
Wave functions ψ_1 and ψ_2
Number of particles $N_1 = \int |\psi_1|^2 \, dV$ and $N_2 = \int |\psi_2|^2 \, dV$

$$
\begin{aligned}
i\hbar \frac{\partial \psi_1}{\partial t} &= \left[-\frac{\hbar^2}{2m_1} \nabla^2 + V_{11} |\psi_1|^2 + V_{12} |\psi_2|^2 \right] \psi_1, \\
i\hbar \frac{\partial \psi_2}{\partial t} &= \left[-\frac{\hbar^2}{2m_2} \nabla^2 + V_{12} |\psi_1|^2 + V_{22} |\psi_2|^2 \right] \psi_2,
\end{aligned}
$$

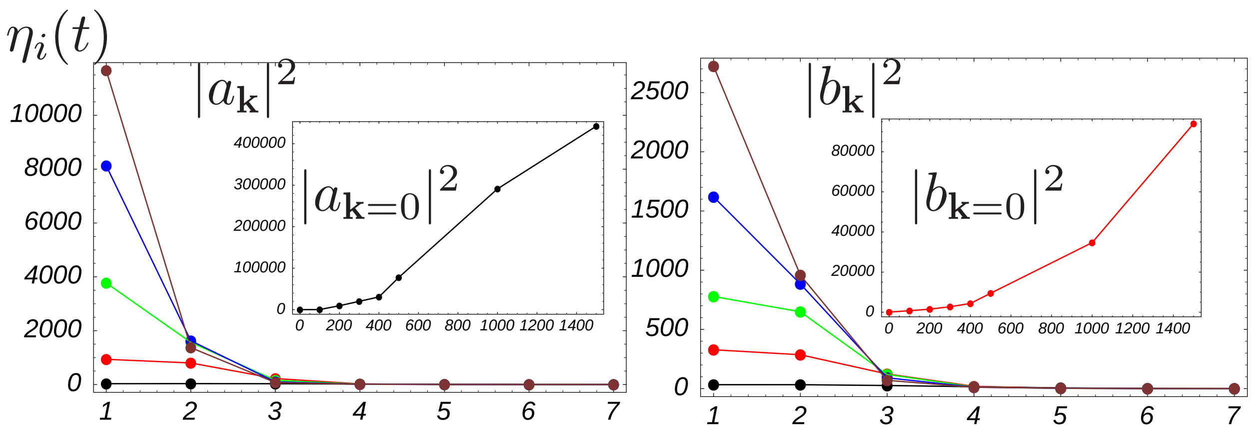
m_i is the mass of the atom of the ith condensate; coupling constants $V_{ij} = 2\pi\hbar^2 a_{ij}/m_{ij}$;
a_{ij} are scattering lengths; $m_{ij} = m_i m_j/(m_i + m_j)$ is the reduced mass.
Chemical potentials $\mu_1 = V_{11} n_1 + V_{12} n_2$, $\mu_2 = V_{12} n_1 + V_{22} n_2$, where $n_i = |\psi_{i\infty}|^2$
Dispersion relation $(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) = \omega_{12}^4$,
where $\omega_i^2(k) = c_i^2 k^2 + \hbar^2 k^4 / 4m_i^2$ with sound velocity $c_i^2 = n_i V_{ii}/m_i$ and $\omega_{12}^2 = c_{12}^2 k^2$
where $c_{12}^2 = n_1 n_2 V_{12}^2 / m_1 m_2$.
Acoustic branches are $\omega_\pm \approx c_\pm k$ with $2c_\pm^2 = c_1^2 + c_2^2 \pm \sqrt{(c_1^2 - c_2^2)^2 + 4c_{12}^4}$.
Dynamical stability $V_{11} V_{22} > V_{12}^2$

Condensation in two-component Bose gases

$$m_1 = m_2, \quad V_{11} = V_{22}$$

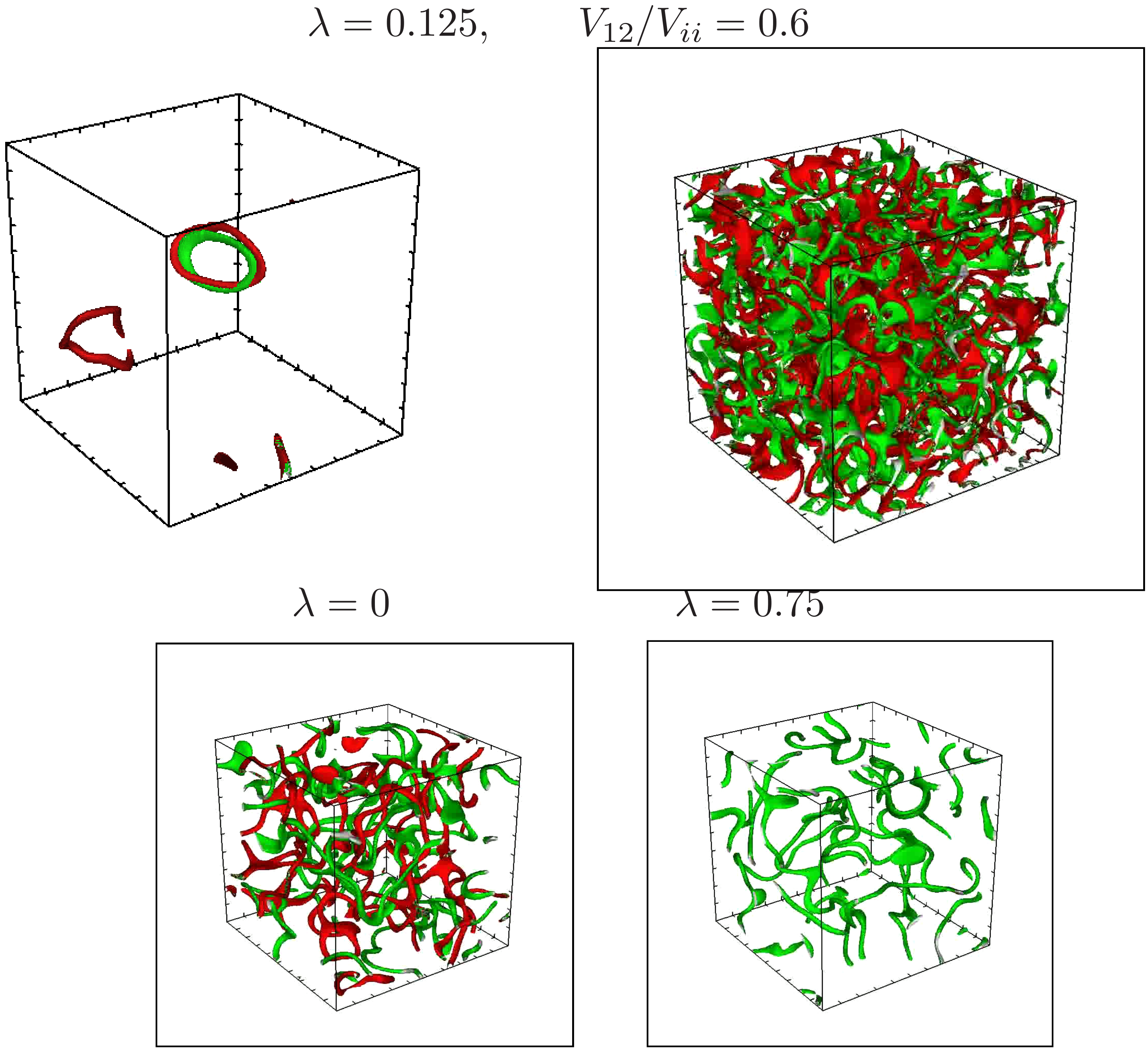
$$\psi_1(\mathbf{r}, t=0) = \sum_{\mathbf{k}} a_{\mathbf{k}} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}), \qquad \psi_2(\mathbf{r}, t=0) = \sum_{\mathbf{k}} b_{\mathbf{k}} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}),$$

Two component Bose gas with $\lambda = \frac{N_1 - N_2}{N_1} = 0.125$ and $V_{12}/V_{ii} = 0.6$



Shells: $i - 1 \le \log_2(k/2\pi) < i$

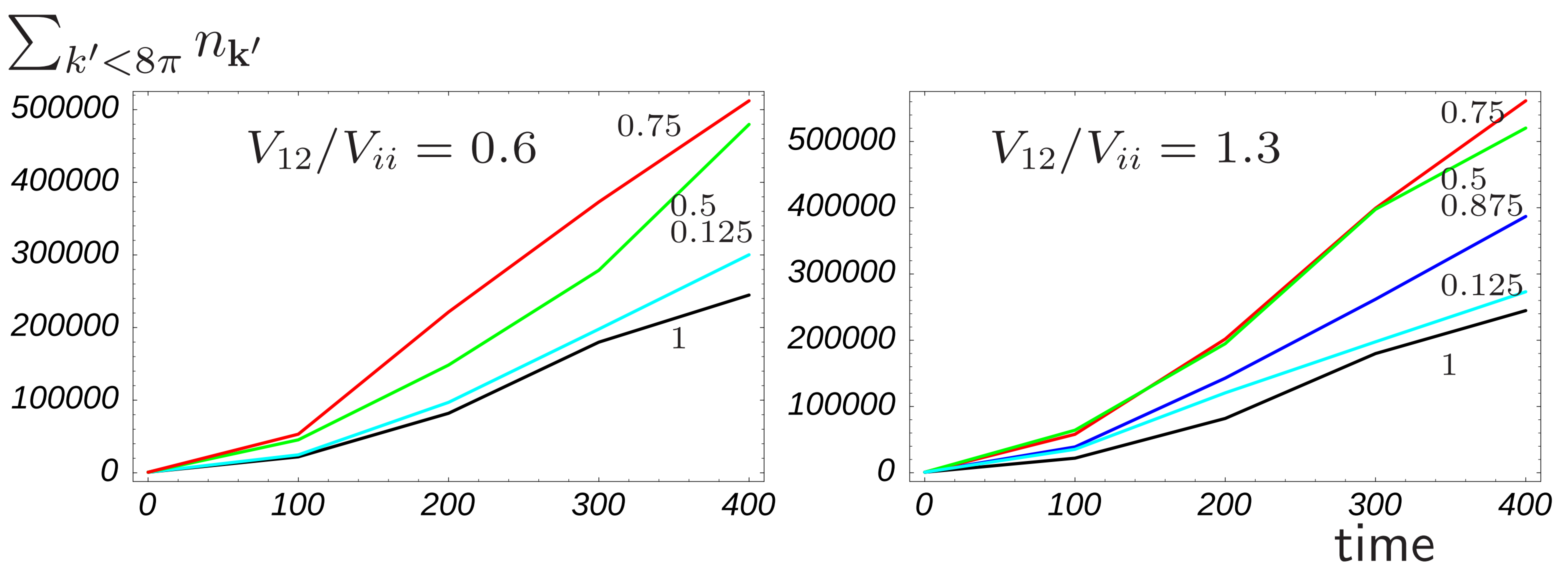
$\eta_i(t) = \sum^{(\text{shell } i)} n_{\mathbf{k}}(t)/M_i$, where M_i is the number of harmonics in the i-th shell


$\lambda = 0.125, \qquad V_{12}/V_{ii} = 0.6$
$\lambda = 0$
$\lambda = 0.75$

Surprises, questions etc.

[1] Faster condensation for ψ_1 depending on $\lambda = (N_1 - N_2)/N_1 \geq 0$. Fastest for $\lambda = 3/4$.

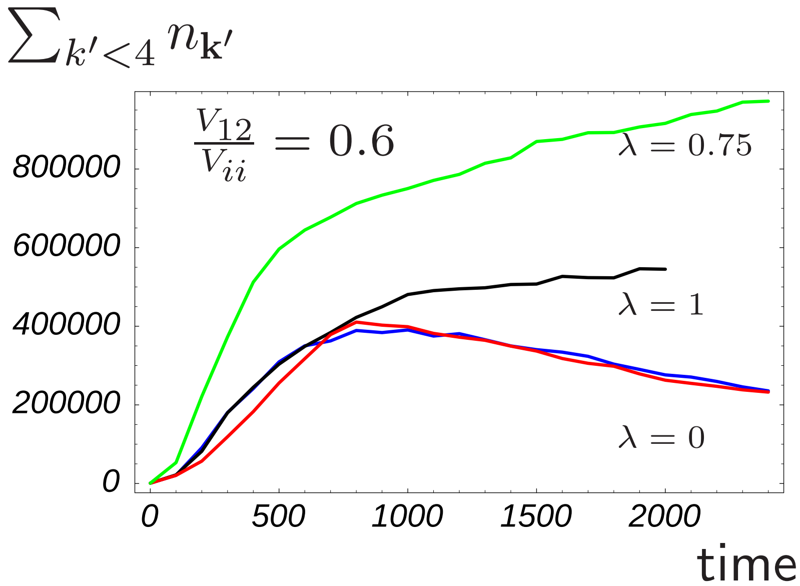
Weak Turbulence regime:



[2] ψ_2 does not condense for $\lambda > 1/4$
ψ_2 is stochastic field of zero mean

[3] Structures created during the condensation?

[4] Condensation for $\lambda < 0.25$?



KE for $n_{\mathbf{k}} = |a_{\mathbf{k}}|^2$ and $l_{\mathbf{k}} = |b_{\mathbf{k}}|^2$

$$\dot{n}_1 = \frac{\partial}{\partial t} a_1^* a_1 = 2Im \sum_{234} \left(U_{11} a_1^* a_2^* a_3 a_4 + U_{12} a_1^2 b_2^* b_3 a_4 \right) \delta_{\mathbf{k}_1;\mathbf{k}_3+\mathbf{k}_4-\mathbf{k}_2}$$

$$\dot{l}_1 = \frac{\partial}{\partial t} b_1^* b_1 = 2Im \sum_{234} \left(U_{12} b_1^* a_2^* a_3 b_4 + U_{22} b_1^2 b_2^* b_3 b_4 \right) \delta_{\mathbf{k}_1;\mathbf{k}_3+\mathbf{k}_4-\mathbf{k}_2}$$

After averaging

$$\dot{n}_\epsilon \sim \epsilon^2 [V_{11}^2\, n_\epsilon^3, \qquad V_{12}^2\, n_\epsilon^2\, l_\epsilon, \qquad V_{12}^2\, n_\epsilon\, l_\epsilon^2]$$
$$\dot{l}_\epsilon \sim \epsilon^2 [V_{22}^2\, l_\epsilon^3, \qquad V_{12}^2\, l_\epsilon^2\, n_\epsilon, \qquad V_{12}^2\, l_\epsilon\, n_\epsilon^2]$$

Characteristic kinetic time τ_{kin}: $\tau_{kin}^{-1} \sim m^3 \epsilon_*^2 [V_{11}^2 n_{\epsilon_*}^2 + \frac{1}{2} V_{12}^2 n_{\epsilon_*} l_{\epsilon_*}]$

[1] Initially drift scenario becomes possible: eg. $\dot{n}_\epsilon \sim V_{12}^2 \, \epsilon^2 \, n_\epsilon^2$
Conservation of particles: $n_\epsilon \propto \epsilon_0^{-3/2} f(\epsilon/\epsilon_0)$

From KE: $\epsilon_0(t) \propto t^{-1/(2-3/2(p-1))}$ where $p = 2$ and

$$f(x) \rightarrow x^{-1}, \qquad x \rightarrow \infty$$

[2] KE describes the evolution to thermodynamical equilibrium

$$n_\epsilon^{eq} = \frac{T}{\epsilon + \mu_1}, \quad l_\epsilon^{eq} = \frac{T}{\epsilon + \mu_2}$$

with ultraviolet cut-off k_c

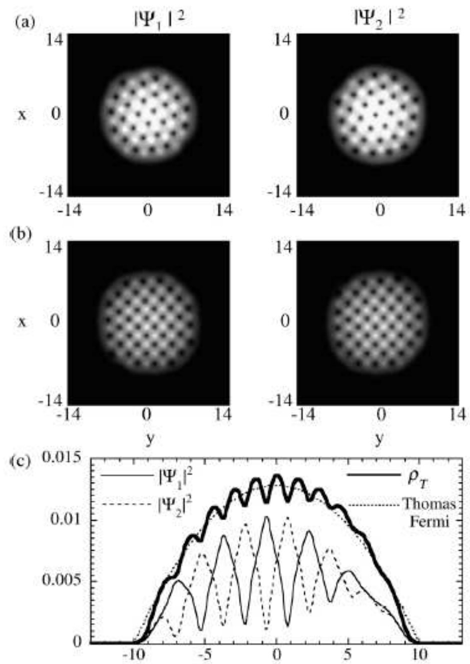
Assume $N_1 > N_2$: conservation of particles $\mu_1 \rightarrow 0$ for $T \neq 0 \Rightarrow \mu_2 \neq 0$

ψ_2 is stochastic field of zero mean, finite correlation length $1/\sqrt{\mu}$.

Solitary waves: previous work

Rotating condensates

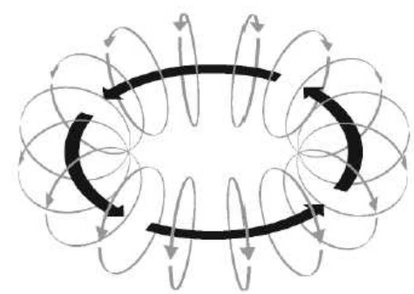
[Kita et al 2002,
Mueller and Ho 2002,
Kasamatsu et al 2003,
Mueller 2004]

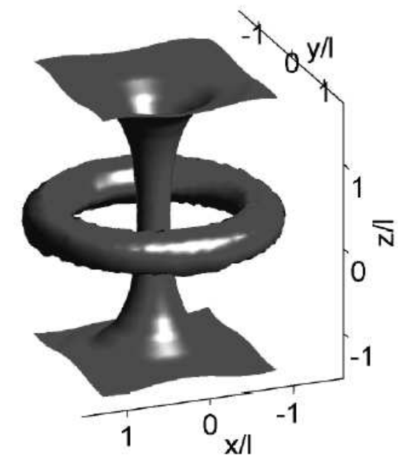


$V_{11}V_{22} > V_{12}^2$

Skyrmions (vortons)

[Khawaja and Stoof 2001,
Ruostekoski and Anglin 2001,
Savage and Roustekoski 2003,
Battye et al 2002]





$V_{11}V_{22} < V_{12}^2$

Governing equations

$$
\begin{aligned}
2\mathrm{i}U\frac{\partial \psi_1}{\partial z} &= \nabla^2\psi_1 + (1 - |\psi_1|^2 - \alpha_1|\psi_2|^2)\psi_1 \\
2\mathrm{i}U\frac{\partial \psi_2}{\partial z} &= \gamma\nabla^2\psi_2 + (1 - \alpha_1|\psi_1|^2 - \frac{\alpha_1}{\alpha_2}|\psi_2|^2 - \Lambda^2)\psi_2, \\
&\psi_1 \to \psi_{1\infty}, \quad \psi_2 \to \psi_{2\infty}, \quad \mathrm{as} \quad |\mathbf{x}| \to \infty.
\end{aligned}
$$

where $\alpha_i = V_{12}/V_{ii}$, $\gamma = m_1/m_2$ and $\Lambda^2 = (\mu_1 - \mu_2)/\mu_1$.
Dimensionless units:

$$
\mathbf{x} \to \frac{\hbar}{(2m_1\mu_1)^{1/2}}\mathbf{x}, \qquad t \to \frac{\hbar}{2\mu_1}t, \qquad \psi_i \to \sqrt{\frac{\mu_1}{V_{11}n_i}}\psi_i.
$$

Cylindrical coordinates (s, θ, z).
Introduce stretched variables $z' = z$ and $s' = s\sqrt{1 - 2U^2}$
Map infinite domain onto the box $(0, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ by
$\widehat{z} = \tan^{-1}(Dz')$ and $\widehat{s} = \tan^{-1}(Ds')$.
Transformed equations are expressed in second-order finite difference form with 100^2 grid points. Newton-Raphson iteration procedure using banded matrix linear solver based on bi-conjugate gradient stabilised iterative method with preconditioning.

Energy and Impulse

The momentum (or impulse) of the $i-$th component

$$\mathbf{p}_i = \frac{1}{2i}\int [(\psi_i^* - \psi_{i\infty})\nabla\psi_i - (\psi_i - \psi_{i\infty})\nabla\psi_i^*]\, dV.$$

Form the energy, $\mathcal{E}$: *energy of the system with a solitary wave – energy of an undisturbed system of the same mass*

$$\mathcal{E} = \frac{1}{2}\int \left\{|\nabla\psi_1|^2 + \gamma|\nabla\psi_2|^2 + \frac{1}{2}(\psi_{1\infty}^2 - |\psi_1|^2)^2 + \frac{\alpha_1}{2\alpha_2}(\psi_{2\infty}^2 - |\psi_2|^2)^2\right\} dV$$

$$+ \frac{\alpha_1}{2}\int \prod_{i=1}^{2}(\psi_{i\infty}^2 - |\psi_i|^2)\, dV.$$

Perform the variation $\psi_i \rightarrow \psi_i + \delta\psi_i$

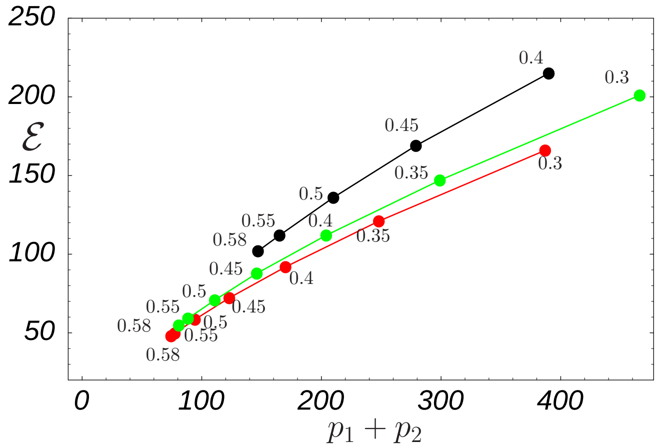
Discard surface integrals that vanish provided $\delta\psi_i \rightarrow 0$ for $|\mathbf{x}| \rightarrow \infty$:

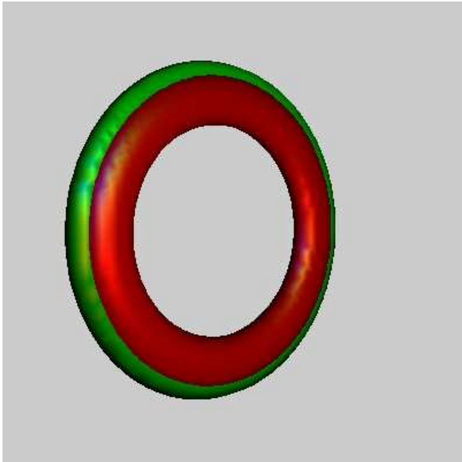
$$U = \partial\mathcal{E}/\partial(p_1 + p_2)$$

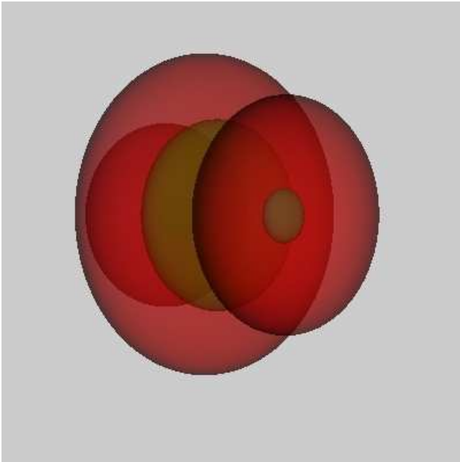
Solitary wave complexes [NGB, Phys. Rev. Lett., 94, 120401 (2005)]

Vortex Ring – Vortex Ring (VR-VR); Vortex Ring – Rarefaction Pulse (VR-RP);
Vortex Ring – "Slaved Wave" (VR-SW); "Slaved Wave"– Rarefaction pulse (SW-RP)

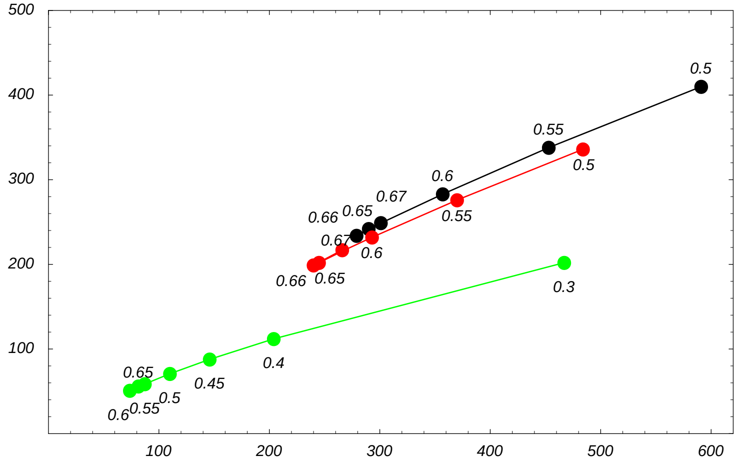
$m_1 = m_2, \quad \Lambda = 0.1, \quad \alpha = 0.1$

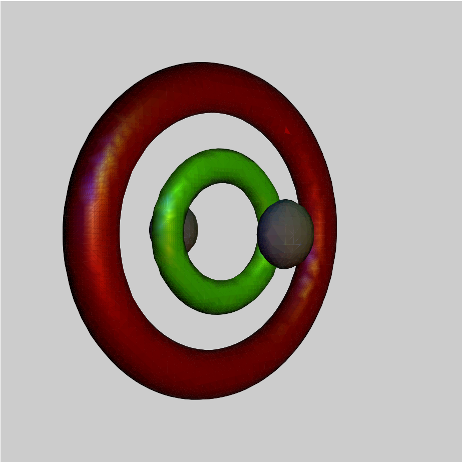




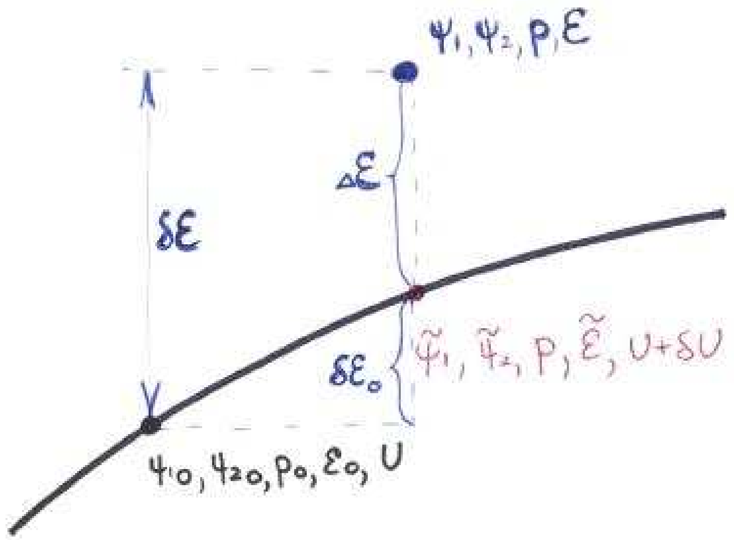


$2m_1 = m_2$





Stability



$\psi_i = \psi_{0i}(as, bz)$
a and b are constants.
The critical choice is $a \neq 1, b = 1$.
Define $\mathbf{p} \equiv \mathbf{p}_1 + \mathbf{p}_2$
$\mathbf{p} = a^{1-D}\mathbf{p}_0$

$$\mathcal{E} = \tfrac{1}{2}a^{3-D}\int\sum|\nabla_H\psi_i|^2\,dV + a^{1-D}\left(\tfrac{1}{2}\int\sum\left|\frac{\partial\psi_i}{\partial z}\right|^2 dV + \tfrac{1}{4}\int\sum(\psi_{\infty i}^2 - |\psi_i|^2)^2\,dV + \tfrac{\alpha}{2}\int\prod(\psi_{\infty i}^2 - |\psi_i|^2)\,dV\right).$$

Using the integral properties $\mathcal{E} = a^{1-D}\left[\mathcal{E}_0 + \tfrac{1}{2}(a^2-1)(D-1)(\mathcal{E}_0 - Up_0)\right]$.

To second order in $a' = a - 1$:

$$p - p_0 = \frac{\partial p_0}{\partial U}\delta U + \frac{1}{2}\frac{\partial^2 p_0}{\partial U^2}(\delta U)^2$$
$$p - p_0 = (1-D)p_0 a' - \tfrac{1}{2}D(1-D)p_0 a'^2$$

Determine $\delta U \Longrightarrow$ Expand $\delta\mathcal{E}_0 \Longrightarrow$ Compare with $\delta\mathcal{E}$ implied to order a'^2:

$$\Delta\mathcal{E} = \delta\mathcal{E} - \delta\mathcal{E}_0 = -\frac{1}{2}(D-1)[(D-3)(\mathcal{E} - Up) + (D-1)p^2\partial U/\partial p]a'^2.$$

$\mathcal{E} > Up$ and in 2D, $\frac{\partial U}{\partial p} < 0$, therefore, $\Delta\mathcal{E} > 0$
In 3D, $\frac{\partial U}{\partial p} < 0$ on the lower branch, $\Delta\mathcal{E} > 0$; $\frac{\partial U}{\partial p} > 0$ on the upper branch, $\Delta\mathcal{E} < 0$

Vortons, springs, etc.

Simplest tractable microscopic model in proper universality class of cosmological systems

Solitary waves moving along the vortex line.

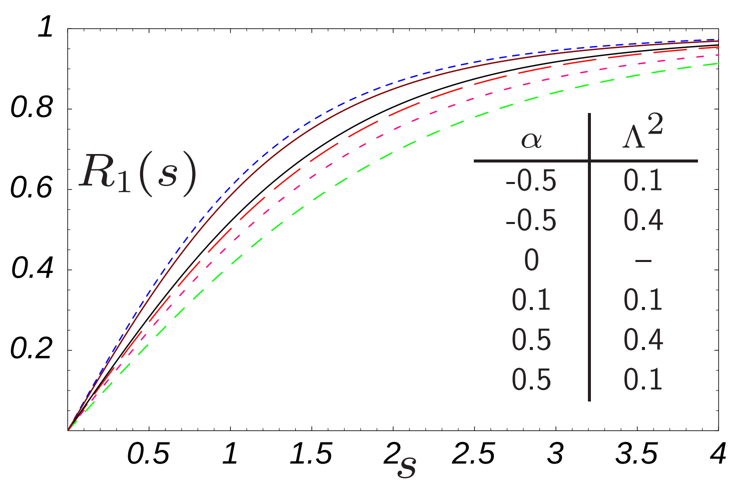
Ansatz $\psi_1 = (R_1(s) + \chi_1(s,z))\exp(i\theta)$, $\psi_2 = R_2(s) + \chi_2(s,z)$, where

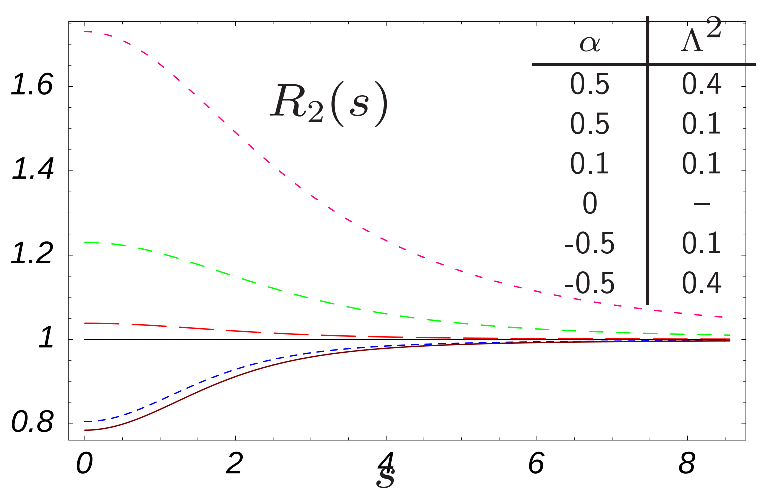
$$R_1'' + \frac{R_1'}{r} - \frac{R_1}{r^2} + (1 - R_1^2 - \alpha R_2^2)R_1 = 0,$$

$$R_2'' + \frac{R_2'}{r} + (1 - \alpha R_1^2 - R_2^2 - \Lambda^2)R_2 = 0.$$

$R_1 \sim \psi_{1\infty} - \frac{1}{2\psi_{1\infty}s^2}$, $\qquad R_2 \sim \psi_{2\infty} \pm K_0(2\psi_{2\infty}^2 s) \sim \psi_{2\infty} \pm \exp(-2\psi_{2\infty}^2 s)$

$(\psi_{2\infty}^2 = (1-\alpha-\Lambda^2)/(1-\alpha^2), \psi_{1\infty}^2 = 1 - \alpha\psi_{2\infty}^2)$



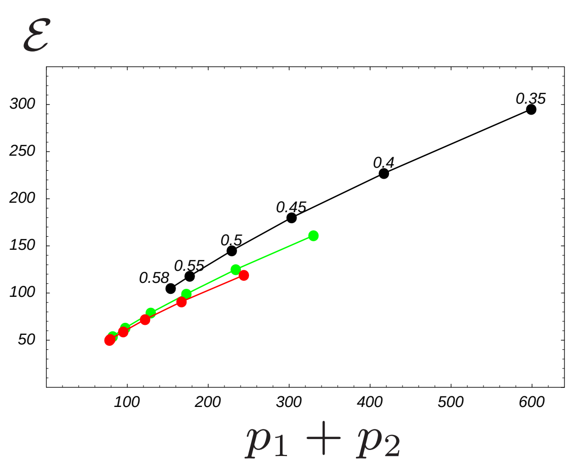


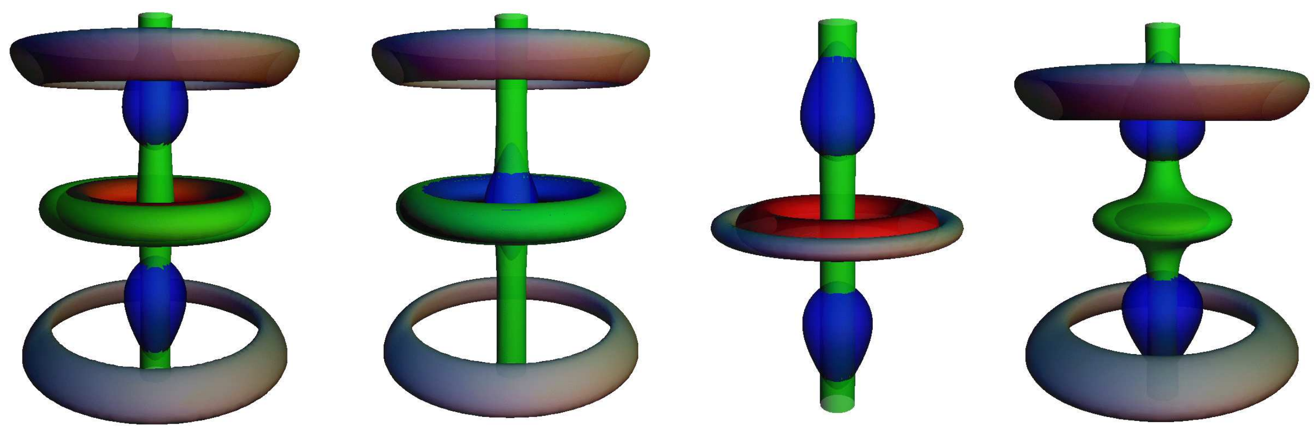
$$2\mathrm{i}U\frac{\partial\chi_1}{\partial z}=\frac{1}{s}\frac{\partial}{\partial s}\left[s\frac{\partial\chi_1}{\partial s}\right]+\frac{\partial^2\chi_1}{\partial z^2}-\frac{\chi_1}{s^2}$$

$$+(1-|R_1+\chi_1|^2-\alpha|R_2+\chi_2|^2)(R_1+\chi_1)-(1-R_1^2-\alpha R_2^2)R_1.$$

$$2\mathrm{i}U\frac{\partial\chi_2}{\partial z}=\frac{1}{s}\frac{\partial}{\partial s}\left[s\frac{\partial\chi_2}{\partial s}\right]+\frac{\partial^2\chi_2}{\partial z^2}$$

$$+(1-\alpha|R_1+\chi_1|^2-|R_2+\chi_2|^2-\Lambda^2)(R_2+\chi_2)-(1-\alpha R_1^2-R_2^2-\Lambda^2)R_2.$$





$\rho_1=\frac{1}{5}\psi_{1\infty}^2$ $\rho_2=\frac{1}{5}\psi_{2\infty}^2$ $\rho_2>\psi_{2\infty}^2$ $\rho_1>\psi_{1\infty}^2$

Conclusions:

- scenario of strongly non-equilibrium Bose-Einstein condensation in uniform weakly interacting Bose gas:
  weak turbulence $\longrightarrow$ strong turbulence $\longrightarrow$ superfluid turbulence
  (1) details of weak turbulence regime: $t^*, \quad t_0, \quad k_0$
  (2) formation of short-range order; bimodal particle distribution
- BEC formation in a mixture of two Bose gases
  (1) kinetic theory
  (2) mixture of "drift" and "cascade" scenarios
  (3) possibility of **increased rate of condensation for** ψ_1: $\lambda_{max} \approx 3/4$
  (4) no condensation for ψ_2 if $\lambda > 1/4$
- Coherent structures in condensate systems:
  (1) vortex rings in one-component and "nonlocal" condensates;
  (2) vortex rings of various radii in each of the components;
  (3) a vortex ring in one component coupled to a rarefaction solitary wave of the other component;
  (4) two coupled rarefaction waves;
  (5) either a vortex ring or a rarefaction pulse coupled to a localised disturbance of a very low momentum;
  (6) all of the above moving along the vortex line (vortons, springs, etc.);