Scaling of space–time modes with Reynolds number in two-dimensional turbulence

Nicholas Kevlahan

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Collaborators

- Jahrul Alam
  McMaster University (PhD student)
- Oleg Vasilyev
  University of Colorado at Boulder
## Outline

- **Introduction**
- **Adaptive wavelet numerical simulation**
- **Results**
- **Conclusions**
Intermittency and turbulence

- The active regions of turbulence are distributed inhomogeneously in space and time.
- The active proportion of the flow is believed to decrease with Reynolds number.
- This intermittency is a fundamental property of turbulence.
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- Friz & Robinson (2001) proved this conjecture for stationary periodic 2D turbulence.
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- Assuming homogeneity, the spatial computational complexity of turbulence scales like $Re^{9/4}$ (or $Re^1$ in 2D).
- Similarly, space–time computational complexity scales like $Re^3$ (or $Re^{3/2}$ in 2D).
- Yakhot & Sreenivasan recently claimed it is even worse: $Re^4$.
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- However, these estimates **ignore** intermittency.
Questions

- What is the actual scaling of spatial degrees of freedom with Reynolds number, $Re^\beta$?
- What is the actual scaling of space-time degrees of freedom with Reynolds number, $Re^\alpha$?
- Is turbulence more intermittent in space or time?
- What is the fractal dimension of the active regions of the flow? (Assuming the $\beta$–model.)
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Numerical estimation of space-time modes

- Use a simultaneous space–time adaptive wavelet solver.
- Take the number of active space–time wavelet modes as an upper bound on the number of space–time degrees of freedom.
- Consider periodic, unforced, 2D turbulence.
- Perform a sequence of simulations for $1260 \leq Re \leq 40400$. 
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- High rate of data compression (e.g. jpeg2 2000 image compression).
- Fast $O(N)$ transform.
- Fast signal de-noising (optimal for additive Gaussian noise).
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Wavelet multiresolution analysis of $L^2(\mathbb{R})$

A sequence of approximation subspaces

$$M = \{ V^j \subset L^2(\mathbb{R}) \mid j \in J \}$$
s.t.

- $V^j \subset V^{j+1}$ (subspaces are nested).
- $\bigcup_{j \in J} V^j$ is dense in $L^2(\mathbb{R})$.
- Each $V^j$ has a Riesz basis of scaling functions $\{ \phi^j_k \mid k \in K^j \}$. 
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Wavelets $\psi^j_k$ span the complement space $W^j$, where $V^{j+1} = V^j \oplus W^j$, i.e. wavelet coefficients give the detail.
Nested collocation wavelet grids

Scaling functions are constructed from interpolating polynomials of degree \(2N - 1\) on nested grids:

\[ \mathcal{G}^j = \left\{ x_k^j \in \Omega : x_k^j = x_{2k}^{j+1}, \ k \in K^j \right\} \]

Collocation: each scaling function and wavelet is associated to a unique grid point.
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$$u(x) = \sum_{k \in K^J} u(x^J_k) \phi^J_k(x) = \sum_{k \in K^0} u(x^0_k) \phi^0_k(x) + \sum_{j=0}^{J-1} \sum_{k \in \mathcal{L}^j} d^j_k \psi^j_k(x)$$
Wavelet compression

\[ u(x) = \sum_{k \in K^0} u(x_k^0) \phi_k^0(x) + \sum_{j=0}^{+\infty} \sum_{k \in L^j} d_k^j \psi_k^j(x) \]

**Function** \( u(x) \)

**Wavelet locations** \( x_k^j \)
Wavelet compression

\[ u_{\ge}(x) = \sum_{k \in \mathcal{K}^0} u(x_k^0) \phi_k^0(x) + \sum_{j=0}^{J-1} \sum_{k \in \mathcal{L}^j} d_{k^j}^j \psi_{k^j}(x) \]

\[ \sum_{k \in \mathcal{L}^j} |d_{k^j}^j| \ge \epsilon \]

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\[ \| u(x) - u_{\geq}(x) \|_2 = O(\epsilon) \]
\[ \mathcal{N} = O(\epsilon^{-1/2N}) \]
\[ \| u(x) - u_{\geq}(x) \|_2 = O(\mathcal{N}^{-2N}) \]

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Space–time adaptive wavelet turbulence calculation

Advantages
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- Global error control in time.
- Local time step.
- Potentially optimal complexity for highly intermittent problems.
- Number of grid points is an approximation to the number of space–time degrees of freedom in the flow.
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Numerical method: pseudo BVP in space–time domain

- Add dynamic pseudo boundary condition for long time boundary.
- Use adaptive wavelet multilevel solver with V-cycles for BVP.
- FAS approximation to cope with nonlinear equations.
- Iterate until residual satisfies $L_2$ norm tolerance.
- Split space–time domain in time direction into manageable slices.
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1D+t example: Burgers equation

\[ \frac{\partial u}{\partial t} + (U + u)\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, \quad x \in (-1, 1), \quad t > 0 \]

- **Steepening shock:** \( U = 0, \ u(x, 0) = -\sin(\pi x), \ u(\pm 1, t) = 0. \)
- **Moving shock:** \( U = 1, \ u(x, 0) = -\tanh((x + 1/2)/(2\nu)), \ u(\pm \infty, t) = \mp 1. \)
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Scaling of space–time modes with Reynolds number
Burgers equation: steepening shock

Solution

Grid

Adapted grid

Scaling of space–time modes with Reynolds number
Burgers equation: moving shock

Solution and adapted grid for the Burgers equation at different times.
Burgers equation: time integration error

Global error in time

Comparison with time marching

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Scaling of space–time modes with Reynolds number
2D decaying turbulence simulations

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<tr>
<th>Run</th>
<th>Re</th>
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<th>$\Delta x$</th>
<th>$\lambda$</th>
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<tbody>
<tr>
<td>I</td>
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**Table:** Parameters for space–time turbulence simulations.
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Comparison simulations were also done using a standard pseudo-spectral code, and time marching adaptive wavelet simulations were done to estimate the number of spatial degrees of freedom.
$Re = 40 \, 400$ simulation, $t = [0, 400]$
$Re = 40\ 400$ simulation, $t = [21, 128]$
Vorticity field at $Re = 40400$

7895 wavelet modes  
263,169 Fourier modes  
Energy spectrum
Vorticity at $t = 126$

$Re = 1260$

$Re = 2530$

$Re = 5050$

$Re = 10100$

$Re = 20200$

$Re = 40400$
Adaptive wavelet grids at $Re = 40\,400$

(a) $t \in [0, 2.1]$  
(b) $t \in [123.8, 126.0]$  
(c) Spatial grid only at $t = 126.0$
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Note the strong time intermittency of the solution: the smallest time step is strongly localized in space.
Scaling of modes with Reynolds number

Space–time

Space only
Scaling of modes with Reynolds number

Note that intermittency reduces the number of modes significantly compared with the usual computational estimates.
The $\beta$-model for two-dimensional turbulence implies that the spatial modes should scale like $N \sim \text{Re}^{\frac{3D_F}{D_F+4}}$. 
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- **Spatial fractal dimension** is $D_F \approx 1.2$
The $\beta$-model for two-dimensional turbulence implies that the spatial modes should scale like $N \sim \text{Re}^{\frac{3D_F}{D_F+4}}$.

- Spatial fractal dimension is $D_F \approx 1.2$
- A simple extension gives a temporal fractal dimension $D_F \approx 0.3$
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\[ N \sim \text{Re}^{\frac{3D_F}{D_F+4}}. \]

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Assumes that the active proportion of the flow decreases like lengthscale to the power $D - D_F$. 
Conclusions

- Spatial modes scale like $Re^{0.7}$
- Space–time modes scale like $Re^{0.9}$
- Spatial fractal dimension of active regions is 1.2
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This is the first quantitative estimate of the Reynolds number dependence of the space–time intermittency of turbulence