Constant Flux States in Models of Wave Turbulence

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Example 1: “Weak” wave turbulence on a lake
Example 2: “Strong” wave turbulence on the ocean
What is Wave Turbulence?

- Ensemble of interacting dispersive waves.
- Energy injected at some characteristic frequency, $\omega_f$.
- Waves are damped at very large and / or very small scales. Scale separation.
- Energy transferred by interaction between waves.
- Concept of *inertial range*

In the limit of large inertial ranges, system often becomes scale invariant and exhibit a power law spectrum: $E(k) \sim k^{-x}$. 
Hamiltonian Model

Hamiltonian evolution for the complex Fourier amplitudes, $a\vec{k}$, $a^*\vec{k}$:

$$\frac{\partial a\vec{k}}{\partial t} = i\frac{\delta H}{\delta a^*\vec{k}} + f\vec{k} - \gamma\vec{k}a\vec{k}$$

- Wave-vector, $\vec{k}$ is a $d$-dimensional vector.
- $H$ has a (linear) kinetic and (nonlinear) potential term

$$H = T + U = \int t(k)dk + \int u(k)dk$$

where $t(k) = \omega\vec{k}a\vec{k}a^*\vec{k}$.

- Elementary solutions: $a(x, t) = \int dk \, a_k e^{i(k \cdot x - \omega_k t)}$.
- Forcing and dissipation functions are separated in $\vec{k}$-space.
Structure of the interaction

3 wave systems:

\[ u(k_1) = \int V_{k_1 k_2 k_3} \left( a_{k_1} a_{k_2} a_{k_3}^* + a_{k_1}^* a_{k_2}^* a_{k_3} \right) \delta(k_1 - k_2 - k_3) \, dk_2 \, dk_3 \]

4 wave systems:

\[ u(k_1) = \int T_{k_1 k_2 k_3 k_4} \left( a_{k_1} a_{k_2} a_{k_3}^* a_{k_4}^* + a_{k_1}^* a_{k_2}^* a_{k_3} a_{k_4} \right) \delta(k_1 + k_2 - k_3 - k_4) \, dk_2 \, dk_3 \, dk_4 \]
Conservation Laws and Scale Invariance

Scale invariance: Many interesting cases possess scale invariance:

- Dispersion relation: \( \omega_{hk} = h^\alpha \omega_k \).
- Nonlinear interactions:
  \[
  V_{hk_1hk_2hk_3} = h^\beta V_{k_1k_2k_3} \quad \text{3-wave}
  \]
  \[
  T_{hk_1hk_2hk_3hk_4} = h^\beta T_{k_1k_2k_3k_4} \quad \text{4-wave}
  \]

The parameters \( \alpha, \beta \) determine the scaling properties of the system.

Conservation laws:

- Energy, \( H \).
- Momentum: \( \vec{P} = \int \vec{k} a_k a_k^* d\vec{k} \). (but often, \( \vec{P} = 0 \))
- Wave-action: \( N = \int a_k^* a_k^* d\vec{k} \). (4-wave only)
Hamilton’s equations give an equation governing for the spectrum. For the 3-wave case:

\[
\frac{\partial n(k_1)}{\partial t} = 2 \int V_{k_1 k_2 k_3} \text{Im} \langle a_{k_1}^* a_{k_2} a_{k_3} \rangle \delta(k_1 - k_2 - k_3) dk_2 dk_3 \\
- 2 \int V_{k_2 k_3 k_1} \text{Im} \langle a_{k_2}^* a_{k_3} a_{k_1} \rangle \delta(k_2 - k_3 - k_1) dk_2 dk_3 \\
- 2 \int V_{k_3 k_1 k_2} \text{Im} \langle a_{k_3}^* a_{k_1} a_{k_2} \rangle \delta(k_3 - k_1 - k_2) dk_2 dk_3
\]

Similarly the 4-wave case ...but now what? Closure problem.
The weak turbulence limit

In weak turbulence, potential energy is small compared to the kinetic energy.

\[ \frac{u(k)}{t(k)} = \epsilon \ll 1 \]

- \( \epsilon \) provides a small parameter to do perturbation theory.
- Nonlinearity becomes localised on resonant triads or quartets. Resonances transfer energy over long timescales \((1/\epsilon^2)\)
- Roughly:

  \[ \text{Im} \langle a_{k_1}^* a_{k_2} a_{k_3} \rangle \sim V_{k_1 k_2 k_3} (n_2 n_3 - n_1 n_2 - n_1 n_3) \delta(\omega_1 - \omega_2 - \omega_3) \delta(k_1 - k_2 - k_3) \]

- Lowest order of perturbation theory gives a wave kinetic theory for the slow evolution of \( n(k) \).
The Wave Kinetic Equation

Isotropic 3-wave kinetic equation in frequency space is:

\[ \frac{\partial n_1}{\partial t} = \epsilon^2 \int U_{\omega_1\omega_2\omega_3} (n_2n_3 - n_1n_2 - n_1n_3) \delta(\omega_1 - \omega_2 - \omega_3) d\omega_2 d\omega_3 
- \epsilon^2 \int U_{\omega_2\omega_3\omega_1} (n_3n_1 - n_2n_3 - n_2n_1) \delta(\omega_2 - \omega_3 - \omega_1) d\omega_2 d\omega_3 
- \epsilon^2 \int U_{\omega_3\omega_1\omega_2} (n_1n_2 - n_3n_1 - n_3n_2) \delta(\omega_3 - \omega_2 - \omega_1) d\omega_2 d\omega_3 \]

4-wave kinetic equation in frequency space:

\[ \frac{\partial n_1}{\partial t} = \epsilon^2 \int U_{\omega_1\omega_2\omega_3\omega_4} (n_2n_3n_4 + n_1n_3n_4 - n_1n_2n_4 - n_1n_2n_3) \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) d\omega_2 d\omega_3 d\omega_4 \]
Kolmogorov-Zakharov Solutions of the Kinetic Equation

Stationary scaling solutions, \( n(k) = c k^{-x} \), can be found exactly:

\[
(\omega_2, \omega_3) \rightarrow \left( \frac{\omega_1 \omega'_2}{\omega'_3}, \frac{\omega_2^2}{\omega'_3} \right), \quad (\omega_2, \omega_3) \rightarrow \left( \frac{\omega_1^2}{\omega'_2}, \frac{\omega_1 \omega'_3}{\omega'_2} \right).
\]

3W case: kinetic energy cascade:

\[
n(k) = c_3^{(P)} \sqrt{P} k^{-\beta-d}
\]

4W case: Two cascades - kinetic energy and wave action:

\[
n(k) = c_4^{(P)} P^{1/3} k^{-(2\beta+3d)/3}
n(k) = c_4^{(Q)} Q^{1/3} k^{-(2\beta-3d+\alpha)/3}
\]
What about the strong turbulence case?

- In the strong turbulence case, potential energy may be as large or larger than the kinetic energy.
- No perturbative expansion.
- However, conservation laws still hold but it is total energy rather than kinetic energy which is conserved. For 4W case, wave action remains conserved.
- More like hydrodynamic turbulence case.
- Conservation laws should still provide constraints on the inertial range statistics, even for strong turbulence in the spirit of Kolmogorov’s 4/5 Law.
Consider 4-wave wave-action cascade. In the inertial range:

\[
\frac{\partial N_{k_1}}{\partial t} \equiv \frac{\partial Q_{k_1}}{\partial k_1} = \int \prod_{i=2,3,4} (k_i^{d-1} dk_i) \left[ L_{1,2,3,4} + L_{2,1,3,4} - L_{3,4,1,2} - L_{4,3,1,2} \right]
\]

where

\[
N_{k_1} = \int n(k_1) k_1^{d-1} d\Omega_1, \quad L_{1,2,3,4} = T_{k_1,k_2,k_3,k_4} K_{k_1,k_2,k_3,k_4}
\]

\[
K_{k_1,k_2,k_3,k_4} = \int \prod_{i=1,2,3,4} d\Omega_i \text{Im} \langle a_{k_1}^* a_{k_2}^* a_{k_3} a_{k_4} \rangle \delta(k_1 + k_2 - k_3 - k_4)
\]

This is true whether nonlinearity is weak or strong.
Constant Flux Relation for Wave Action

Assume $K_{k_1,k_2,k_3,k_4}$ is a scaling function of degree, $h$, with the obvious symmetries.

$$K_{ak_1,ak_2,ak_3,ak_4} = a^h K_{k_1,k_2,k_3,k_4}$$

One can still apply the Zakharov Transformation even without the $\omega$ delta functions, to obtain:

$$\frac{\partial N_{k_1}}{\partial t} = \int \prod_{i=2,3,4} (k_i^{d-1} dk_i) L_{1,2,3,4} [k_1^y + k_2^y - k_3^y - k_4^y]$$

where $y = -h - \beta - 4d$. RHS clearly vanishes for $h = \beta + 4d$, corresponding to a constant flux of $n(k)$ in the inertial range.

Weak limit:

$$K_{k_1,k_2,k_3,k_4} \sim T(k) n^3 \delta(k) \delta(\omega) \sim k^{\beta-3*(2\beta+3d-\alpha)/3+d-\alpha} \sim k^{\beta-4d}$$
Exact conservation law for total energy

Problem: if the nonlinearity is strong then it is the total energy density, \( h(k) = t(k) + u(k) \), rather than just the kinetic energy density whose flux should be constant in the stationary state. The following statement is exact:

\[
\frac{\partial h_{k_1}}{\partial t} \equiv \frac{\partial J_{k_1}}{\partial k_1} = 4 \int \prod_{i=2,3} (k_i^{d-1} dk_i) [L_{1,2,3} - L_{2,1,3}]
\]

\[
h_{k_1} = \int (t(k_1) + u(k_1)) k_1^{d-1} d\Omega_1, \quad L_{1,2,3} = V_{k_1,k_2,k_3} K_{k_1,k_2,k_3}
\]

\[
K_{k_1,k_2,k_3} = \int \prod_{i=1,2,3} d\Omega_i \text{Re} \langle a_{k_1}^* \dot{a}_{k_2}^* a_{k_3} \rangle \delta(k_1 - k_2 - k_3)
\]
Constant Flux Relation for Energy

Again, if $K_{k_1,k_2,k_3}$ is a scaling function of degree, $h$, the Zakharov transformation can be applied to give

$$\frac{\partial J_{k_1}}{\partial k_1} = 4 \int \prod_{i=2,3} (k_i^d - 1 \, dk_i) L_{1,2,3} \left[ 1 - \left( \frac{k_1}{k_2} \right)^{3d + \beta + h} \right]$$

Clearly we have a constant flux when $h = -\beta - 3d$. This is different from previously. The CFR correlation function $\text{Re} \langle a_{k_1}^* \dot{a}_{k_2}^* a_{k_3} \rangle$ contains time derivatives. Corresponding result for 4-wave energy cascade is:

$$\int \prod_{i=1,2,3,4} d\Omega_i \text{Re} \langle a_{k_1}^* \dot{a}_{k_2}^* a_{k_3} a_{k_4} \rangle \delta(k_1 + k_2 - k_3 - k_4) \sim k^{-\beta - 4d}$$
The Assumption of Locality

In deriving the CFR scaling, we implicitly assumed convergence of the exact collision integral when the flux-carrying correlation function takes the scaling form. For example, for the 3-wave case, we can deduce that

\[ K_{k_1,k_2,k_3} = (k_2 k_3)^{\frac{h}{2}} \Phi \left( \frac{k_2}{k_3} \right) \]

We require that the scaling function, \( \Phi \) decays fast enough at 0 and \( \infty \) to ensure convergence. This is to say that the transfer of energy is local. Our arguments at present do not tell anything about \( \Phi \).

For weak turbulence, locality can be checked a posteriori. For hydrodynamic turbulence no such problem exists (the entire argument is local in \( x \).) Here, it is an assumption.
The MMT Model

The MMT model is the simplest real wave turbulence model:

\[ H = \int \omega_{k_1} \phi_{k_1} \phi_{k_1}^* \, dk_1 + \int T_{k_1 k_2 k_3 k_4} \phi_{k_1}^* \phi_{k_2}^* \phi_{k_3} \phi_{k_4} \delta(k_1 + k_2 - k_3 - k_4) \, dk_{1,2,3,4} \]

with \( \omega_k = k^\alpha \) and \( T_{k_1 k_2 k_3 k_4} = (k_1 k_2 k_3 k_4)^\beta 4 \).

- Original model is one dimensional but it can clearly be written down in higher dimensions too.
- Does not generally exhibit weak turbulence due to resonance sparsity in 1-d.
- Exhibits direct and inverse cascades.
- A good candidate to check CFR?
A Tentative Numerical Result for the MMT Model

Snapshot of $\psi(x, t)$

Energy density ratio, $t(k)/u(k)$.

Measured following contribution to the flux:

$$\langle \dot{a}_k a_{-k-2} \overline{a}_{k-1} \overline{a}_{-k-1} \rangle$$

$$\alpha = 0.25, \quad \beta = 0.25$$

4-point function.
A “Shell” Model of 3-wave Turbulence

What is the simplest possible system which should obey a non-trivial CFR?

- Discrete waves (0-dimensional)
- 3-wave interactions
- Positive wave vectors only.
- “Local” interactions only: $k + k \rightarrow 2k$. So only have powers of 2: $k_n = 2^n$.

These simplifications lead one to write down the following minimal Hamiltonian:

$$H = \sum_n k_n^\alpha a_n a_n^* + k_{n-1}^\beta \left(a_n^* a_{n-1}^2 + a_n a_{n-1} a_{n-1}^* \right)$$

Really a system of oscillators, strongly coupled in a special way.
Momentum CFR for the “Shell” Model

Equation of motion:

\[ \frac{\partial a_n}{\partial t} = i\omega_n a_n + igk_{n-1}^\beta a_{n-1}^2 + 2igk_{n}^\beta a_{n-1} \bar{a}_n \]

- Minimal model conserves \( H \) and momentum \( P = \sum_n k_n a_n a^*_n \).
- Momentum cascade is inverse and energy cascade is direct.
- Force and damp different modes to generate “turbulence”.
- Model is exactly solvable for the stationary states. Momentum CFR:

\[ \text{Im} \langle a_{n+1} a^{*2}_n \rangle = -\frac{J_P}{4} k_n^{-\beta-1} \]

where \( J_P \) is the momentum flux.
Numerical Confirmation of Momentum CFR for the “Shell” Model

Numerical integration of the minimal model with forcing and damping confirm the CFR prediction for momentum:

Time evolution for $\beta = 0$ \( \text{Im} \langle a_{n+1} a_n^* a_n^2 \rangle \) for different $\beta$.

Corresponding energy cascade is still a work in progress.
Conclusions and Ongoing Work

Conclusions:

- Conservation laws in wave turbulence can be used to fix the scaling of the flux-carrying correlation function in the inertial range, even in the case of strong wave turbulence.

- Energy conservation in wave turbulence, (assuming that our results are borne out by numerics), is of a different character to hydrodynamic turbulence. It links correlation functions of different order.

Current work:

- Verify energy cascade in the minimal model.

- Observe CFR scaling for more realistic models, especially MMT.

- Physical predictions in real space?