Exercise session I

Exercise 1. Ito and Stratonovich stochastic ODE's.

Consider the stochastic ODE for the Lagrangian trajectory with noise in a smooth velocity field:

$$dR = v(t,R)\,dt + \sqrt{2\varkappa}\,d\beta(t)$$

where $\beta(t)$ is the d-dimensional Brownian motion and $d\beta(t) = \eta(t)\,dt$ where $\eta(t)$ is a (d-dim) white noise: a Gaussian process with mean $\overline{\eta^i(t)} = 0$ and covariance

$$\overline{\eta^i(t)\,\eta^j(t')} = \delta^{ij}\,\delta(t-t').$$

Ito calculus: $\quad d\beta^i(t)\,d\beta^j(t) = \delta^{ij}\,dt$

$$df(R) = (\nabla_i f)(R)\,dR + \varkappa(\nabla^2 f)(R)$$ ← Ito term

which is equivalent to the integral equation

$$f(R(t)) - f(R(0)) = \int_0^t (\nabla_i f)(R(s))\,v(s,R(s))\,ds$$

$$+ \sqrt{2\varkappa}\underbrace{\int_0^t (\nabla_i f)(R(s))\,d\beta^i(s)}_{I} + \varkappa\int_0^t (\nabla^2 f)(R(s))\,ds$$

I – Ito stochastic integral. By definition,

$$I = \lim_{0<t_0<t_1<\ldots<t_n<t} \sum_{m=0}^{n} (\nabla_i f)(R(t_m))\big(\beta(t_{m+1}) - \beta(t_m)\big)$$

In particular the expectation $\langle I \rangle = 0$!

Stratonovich gave another prescription for the stochastic integral:

$$S \equiv \int_0^t (\nabla_i f)(R(s)) \circ d\beta(s) =$$

$$= \lim_{0<t_0<t_1<\dots<t_n<t} \sum_{m=0}^{n} \frac{1}{2}\left[(\nabla_i f)(R(t_m)) + (\nabla_i f)(R(t_{m+1}))\right]\left(\beta^i(t_{m+1}) - \beta^i(t_m)\right)$$

$$= I + \lim_{0<t_0<\dots<t_n<t} \sum_{m=0}^{n} \frac{1}{2}\left[\nabla_i f(R(t_{m+1})) - \nabla_i f(R(t_m))\right]\left(\beta^i(t_{m+1}) - \beta^i(t_m)\right)$$

$$= I + \lim_{0<t_0<\dots<t} \sum_{m=0}^{n} \frac{1}{2} (\nabla_i \nabla_j f)(R(t_m))\Big[v^j(t, R(t_m))(t_{m+1} - t_m) + \sqrt{2\kappa}\left(\beta^j(t_{m+1}) - \beta^j(t_m)\right)\Big]\left(\beta^i(t_{m+1}) - \beta^i(t_m)\right)$$

Essentially by the law of large numbers the limit is that of the mean value and one gets

$$S = I + \frac{1}{2}\sqrt{2\kappa} \int_0^t (\nabla^2 f)(R(s))\, ds$$

Hence

$$f(R(t)) - f(R(0)) = \int_0^t (\nabla_i f)(R(s))\, v(s, R(s))\, ds + \sqrt{2\kappa} \int_0^t (\nabla_i f)(R(s)) \circ d\beta^i(s)$$

or

$$df(R) = (\nabla_i f)(R) \circ dR^i$$

Simplification of Stratonovich: no Ito term in the chain rule

Complication of Stratonovich: mean does not vanish.

We shall stick to the Ito convention

Exercise 2. Show that if $R(t;t_0,r_0)$ solves the stochastic ODE

$$dR = v(t,R) + \sqrt{2\varkappa}\, d\beta(t)$$

with $R(t_0;t_0,r_0) = r_0$ then, denoting by the overbar the average over β,

$$n(t,r) = \int \overline{\delta(r - R(t;t_0,r_0))}\, n(t_0,r_0)\, dr_0$$

solves the transport equation

$$\partial_t n + \nabla\cdot(n v) - \varkappa \nabla^2 n = 0$$

Solution. As in the case without the noise, we consider

$$\int f(r)\partial_t n(t,r)\, dr = \frac{d}{dt}\int f(r)\, n(t,r)\, dr$$

$$= \frac{d}{dt}\int f(r)\, \overline{\delta(r - R(t;t_0,r_0))}\, n(t_0,r_0)\, dr_0\, dr$$

$$= \frac{d}{dt}\int \overline{f(R(t;t_0,r_0))}\, n(t_0,r_0)\, dr_0$$

$$= \int \overline{\left[(\nabla_i f)(R(t;t_0,r_0))\, v^i(t;t_0,r_0) + \varkappa (\nabla^2 f)(R(t;t_0,r_0))\right]}\, n(t_0,r_0)\, dr_0$$

$$= \int \left[(\nabla_i f)(r)\, v^i(t,r) + \varkappa \nabla^2 f(r)\right] \overline{\delta(r - R(t;t_0,r_0))}\, n(t_0,r_0)\, dr\, dr_0$$

$$= \int \left[(\nabla_i f)(r)\, v^i(t,r) + \varkappa \nabla^2 f(r) \right] u(t,r)\, dr$$

$$= \int f(r) \left[\nabla_i \left(u(t,r)\, v^i(t,r) \right) + \varkappa \nabla^2 u(t,r) \right] dr$$

$\square$

<u>Exercise 3</u> Show that if $R(s;t,r)$ solves

$$dR = v(s,R)\, ds + d\beta(s)$$

with the final condition $R(t;t,r) = r$ then

$$\theta(t,r) = \int \overline{\delta(r_0 - R(t_0;t,r))}\, \theta(t_0,r_0)\, dr_0$$

$$= \overline{\theta(t_0, R(t_0;t,r))}$$

solves the transport equation

$$\partial_t \theta + (v\cdot\nabla)\theta - \varkappa \nabla^2 \theta = 0$$

<u>Solution</u>. Suppose that $\theta(t,r)$ is the solution of the last equation. To show that for all t_0

$$\theta(t,r) = \overline{\theta(t_0;R(t_0;t,r))}$$

it is enough to show the derivative over t_0 of the right hand side vanishes. By the Ito calculus,

$$\frac{d}{dt_0} \overline{\theta(t_0;R(t_0;t,r))} = \overline{\partial_{t_0} \theta(t_0;R(t_0;t,r))}$$

why minus?

$$+ \overline{(\nabla_i \theta)(t_0,R(t_0;t,r))\, v^i(t_0,R(t_0;t,r))} - \varkappa\, \overline{\nabla^2 \theta(t_0;R(t_0;t,r))} = 0$$

$\square$

Exercise 4. Show that if $R(t;t_0,r_0)$ is as in Exercise 2 and $W^i{}_j(t;t_0,r_0) = \dfrac{\partial R^i(t;t_0,r_0)}{\partial r_0^j}$ then for the vector field

$$B(t,r) = \int \delta(r - R(t;t_0,r_0))\, W(t;t_0,r_0)\, B(t_0,r_0)\, dr_0$$

solves the transport equation

$$\partial_t B + (v\cdot\nabla)B + (\nabla\cdot v)B - (B\cdot\nabla)v - \varkappa\nabla^2 B = 0.$$

Solution. For a vector-valued function $f(r)$,

$$\int f(r)\cdot\partial_t B(t,r)\, dr = \frac{d}{dt}\int f(r)\cdot B(t,r)\, dr$$

$$= \frac{d}{dt}\int f(R(t;t_0,r_0))\cdot W(t;t_0,r_0)\, B(t_0,r_0)\, dr_0$$

$$= \int \left[(\nabla_i f)(R(t;t_0,r_0))\, v^i(t,R(t;t_0,r_0)) + \varkappa(\nabla^2 f)(R(t;t_0,r_0))\right] \cdot W(t,t_0,r_0)\, B(t_0,r_0)\, dr_0$$

$$+ \int f(R(t;t_0,r_0))\cdot(\nabla v)^t(t,R(t;t_0,r_0))\, W(t;t_0,r_0)\, B(t_0,r_0)\, dr_0$$

$$= \int\left[(\nabla_i f)(r)\, v^i(t,r) + \varkappa(\nabla^2 f)(r) + (\nabla v^i(t,r))\, f_i(r)\right]\cdot B(t,r)\, dr$$

$$= \int f(r)\left[-\nabla_i\left(v^i(t,r)\, B(t,r)\right) + \nabla^2 B(t,r) + (B(t,r)\cdot\nabla)\, v(t,r)\right] dr$$

□

Exercise session II

Exercise 1. Show that the trajectory equations

$$dR = v(t,R)\,dt$$

for the white noise in time v are the same with Ito and Stratonovich convention.

Solution. The corresponding integral equation is

$$R(t) = r_0 + \int_0^t v(s,R(s))\,ds$$

With the Ito convention

$$\int_0^t v(s,R(s))\,ds = \lim_{0<t_0<\dots<t_n<t} \sum_n \int_{t_n}^{t_{n+1}} v(s,R(t_n))\,ds \equiv I$$

With the Stratonovich one

$$\int_0^t v(s,R(s))\,ds = \lim_{0<\dots<t} \sum_n \int_{t_n}^{t_{n+1}} \tfrac{1}{2}\left[v(s,R(t_n)) + v(s,R(t_{n+1}))\right]ds$$

$$\equiv S$$

$$S - I = \lim \sum_n \int_{t_n}^{t_{n+1}} \tfrac{1}{2}\left[v(s,R(t_{n+1})) - v(s,R(t_n))\right]ds$$

$$= \lim \sum_n \int_{t_n}^{t_{n+1}} \tfrac{1}{2}\left(R^i(t_{n+1}) - R^i(t_n)\right) \nabla_i v(s,R(t_n))\,ds$$

$$= \lim \sum_m \int_{t_m}^{t_{m+1}} \frac{1}{2} \int_{t_m}^{t_{m+1}} v^i(s', R(t_m))\, ds'\, \nabla_i v(s, R(t_m))\, ds$$

$$= \frac{1}{2}\int_0^t \partial_i D^i(0)\, ds = \frac{1}{2}\partial_i D^i(0)\, t = 0 \quad \text{if } \partial_i D^i(0) = 0$$

(e.g. in the isotropic case). □

<u>Exercise 2</u> a) Show that for the equation for the infinitesimal separation of the Lagrangian trajectories

$$d\,\delta R = (\delta R \cdot \nabla) v(t, R)\, dt$$

where $R(t)$ solves

$$dR = v(t, R)\, dt$$

the Ito and Stratonovich conventions coincide

b). Show that for the equation
$$d\,\delta R = S(t)\,\delta R\, dt$$
where $S(t)$ is the matrix-valued white white noise with the covariance

$$\langle S^i{}_k(t)\, S^j{}_l(t') \rangle = -\delta(t-t')\, \nabla_k \nabla_l D^{ij}(0) \equiv \delta(t-t')\, C^{ij}_{kl}$$

the do not, in general.

<u>Solution</u> a). With the Ito convention,

$$I \equiv \int_0^t (\delta R(s) \cdot \nabla) v(s, R(s))\, ds = \lim \sum_m \int_{t_m}^{t_{m+1}} (\delta R(t_m) \cdot \nabla) v(s, R(t_m))\, ds$$

and with the Stratonovich one:

$$S \equiv \int_0^t (\delta R(s)\cdot\nabla)\, v(s,R(s))\, ds$$

$$= \lim \sum_m \int_{t_m}^{t_{m+1}} \tfrac{1}{2}\Big[(\delta R(t_m)\cdot\nabla)\, v(s,R(t_m))\, ds + (\delta R(t_{m+1})\cdot\nabla)\, v(s,R(t_{m+1}))\, ds\Big]$$

$$S - I = \tfrac{1}{2} \lim \sum_m \int_{t_m}^{t_{m+1}} \Big[\delta R^i(t_{m+1})\, \nabla_i v(s,R(t_{m+1}))\, ds - \delta R^i(t_m)\, \nabla_i v(s,R(t_m))\, ds\Big]$$

$$= \tfrac{1}{2} \lim \sum_m \int_{t_m}^{t_{m+1}} \Big[(\delta R^i(t_{m+1}) - \delta R^i(t_m))\, \nabla_i v(s,R(t_m))\, ds + \delta R^i(t_m)\, (R^j(t_{m+1}) - R^j(t_m))\, \nabla_j\nabla_i v(s,R(t_m))\, ds\Big]$$

$$= \tfrac{1}{2} \lim \sum_m \int_{t_m}^{t_{m+1}} \Big[\int_{t_m}^{t_{m+1}} \delta R^j(t_m)(\nabla_j v^i)(s',R(t_m))\, ds'\, \nabla_i v(s,R(t_m))\, ds + \delta R^i(t_m) \int_{t_m}^{t_{m+1}} v^j(s',R(t_m))\, \nabla_j\nabla_i v(s,R(t_m))\, ds\Big]$$

$$= \tfrac{1}{2} \int_0^t \Big[\delta R^j(s)\big(-\nabla_j\nabla_i D^{i\cdot}(R(s))\big)\, ds + \delta R^i(s)\, \nabla_j\nabla_i D^{j\cdot}(R(s))\, ds\Big]$$

$$= 0.$$

□

Solution b). With the Ito convention,

$$\tilde{I} \equiv \int_0^t S(s)\,\delta R(s)\,ds = \lim \sum_n \int_{t_n}^{t_{n+1}} S(s)\,\delta R(t_n)\,ds$$

and with the Stratonovich one:

$$\tilde{S} \equiv \int_0^t S(s)\,\delta R(s)\,ds = \lim \sum_n \int_{t_n}^{t_{n+1}} S(s)\,\tfrac{1}{2}\left[\delta R(t_n)+\delta R(t_{n+1})\right]ds$$

Hence

$$\tilde{S}-\tilde{I} = \lim \sum_n \int_{t_n}^{t_{n+1}} S(s)\,\tfrac{1}{2}\left[\delta R(t_{n+1})-\delta R(t_n)\right]ds$$

$$= \lim \sum_n \int_{t_n}^{t_{n+1}} S^{\cdot}{}_{j}(s)\,\tfrac{1}{2}\left[\int_{t_n}^{t_{n+1}} S^{j}{}_{l}(s')\,\delta R^{l}(t_n)\,ds'\right]ds$$

$$= \tfrac{1}{2}\int_0^t C^{\cdot j}_{jl}\,\delta R^{l}(s)$$

But $C^{ij}_{jl} = -\nabla_j \nabla_l D^{ij}(0)$ does not vanish in the compressible Kraichnan ensemble. For $C^{ij}_{kl} = \beta(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k) + \gamma\,\delta^{ij}\delta_{kl}$, $C^{ij}_{jl} = \left[(d+1)\beta+\gamma\right]\delta^i_l$

$$= (2\beta + d\gamma)\,\wp\,.$$

Note that the difference between a) and b) comes from the $R(t)$ dependence in $\nabla_j v^i(t, R(t))$.

□

<u>Exercise 3</u> Find the PDF of $|\delta R(t)| \equiv \Delta(t)$ in the isotropic Kraichnan ensemble.

<u>Solution</u>. With the Ito calculus

$$d(\delta R)^2 = 2\,\delta R \cdot S(t)\,\delta R\,dt + C^{ii}_{kl}\,\delta R^k \delta R^l\,dt$$

Hence

$$\frac{d}{dt}\langle f((\delta R)^2)\rangle = \langle C^{ii}_{kl}\,\delta R^k \delta R^l\; f'((\delta R)^2)\rangle$$
$$+ \langle 2\,C^{ij}_{kl}\,\delta R^i \delta R^k \delta R^j \delta R^l\; f''((\delta R)^2)\rangle$$

For $\quad C^{ij}_{kl} = \beta\left(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k\right) + \gamma\,\delta^{ij}\delta_{kl}\,,$

we get

$$\frac{d}{dt}\langle f((\delta R)^2)\rangle = \langle (2\beta + d\gamma)(\delta R)^2\, f'(\delta R^2)\rangle$$
$$+ \langle (4\beta + 2\gamma)(\delta R)^4\, f''(\delta R^2)\rangle.$$

Or, since $\quad \frac{d}{d\Delta^2} = \frac{1}{2\Delta}\frac{\partial}{\partial\Delta}\,,$

$$\frac{d}{dt}\langle f(\Delta)\rangle = \frac{1}{2}(2\beta + d\gamma)\langle \Delta\,\partial_\Delta f(\Delta)\rangle$$
$$+ \frac{1}{2}(2\beta+\gamma)\langle \Delta^3\,\partial_\Delta \frac{1}{\Delta}\partial_\Delta f(\Delta)\rangle$$
$$= \left\langle \left[\frac{2\beta+\gamma}{2}\left(\frac{\partial}{\partial \ln\Delta}\right)^2 + \left(-\beta + \frac{(d-2)}{2}\gamma\right)\frac{\partial}{\partial \ln\Delta}\right] f(\Delta)\right\rangle$$

Setting $s(t) \equiv \ln \Delta(t)$, we get

$$\frac{d}{dt}\langle f(s(t))\rangle = \langle (A\partial_s^2 + B\partial_s) f(s(t))\rangle$$

for $A = \frac{2\beta+\gamma}{2}$, $B = -\beta + \frac{d-2}{2}\gamma$.

Since $\langle f(s(t))\rangle = \int f(s)\langle \delta(s-s(t))\rangle ds$, we infer that

$$\langle \delta(s - s(t;s_0))\rangle = e^{t(A\partial_s^2 + B\partial_s)}(s_0, s).$$

Since

$$A\partial_s^2 + B\partial_s = e^{-\frac{B}{2A}s}\left(A\partial_s^2 - \frac{B^2}{4A}\right)e^{\frac{B}{2A}s},$$

$$e^{t(A\partial_s^2 + B\partial_s)}(s_0, s) = e^{-\frac{B}{2A}s_0}\, e^{t\left(A\partial_s^2 - \frac{B^2}{4A}\right)}(s_0,s)\, e^{\frac{B}{2A}s}$$

$$= \frac{1}{\sqrt{4\pi A t}}\, e^{-\frac{(s-s_0)^2}{4At} - \frac{B^2 t}{4A} + \frac{B}{2A}(s-s_0)}$$

$$= \frac{1}{\sqrt{4\pi A t}}\, e^{-\frac{(s-s_0-Bt)^2}{4At}}$$

Hence

$$\langle \delta(\Delta - \Delta(t))\rangle = \left\langle \frac{1}{\Delta}\delta(\ln\Delta - \ln\Delta(t))\right\rangle$$

$$= \frac{1}{\sqrt{4\pi A t}}\, e^{-\left(\ln\frac{\Delta}{\Delta_0} - Bt\right)^2/4At}\, \frac{1}{\Delta}.$$

and we get the log-normal distribution with $B = -\beta + \frac{d-2}{2}\gamma$ equal to the Lyap. exp. λ_1. $\square$

Exercise 4. Show that

$$\lim_{\Delta_0 \to 0} \langle \delta(\Delta - \Delta(t;\Delta_0)) \rangle = \delta(\Delta)$$

signaling the deterministic behavior of the Lagrangian trajectories in the Batchelor regime.

Solution. For a test function f,

$$\int f(\Delta) \langle \delta(\Delta - \Delta(t;\Delta_0)) \rangle \, d\Delta$$

$$= \int f(e^s) \, e^{-(s-s_0-Bt)^2/4At} \frac{1}{\sqrt{4\pi A t}} \, ds$$

$$= \int f(e^{s+s_0+Bt}) \, e^{-s^2/4At} \frac{1}{\sqrt{4\pi A t}} \, ds$$

$$= \int f(\Delta_0 e^{s+Bt}) \, e^{-\frac{s^2}{4At}} \frac{1}{\sqrt{4\pi A t}} \, ds$$

$$\xrightarrow[\Delta_0 \to 0]{} f(0) \, .$$

$\square$

Exercise session III

Exercise 1. Prove that if $W(t)$ satisfies the Ito eq.

$$dW = S(t)W\,dt$$

with $\langle S(t)\rangle = 0$, $\langle S^i_k(t) S^j_l(t')\rangle = \delta(t-t')\, C^{ij}_{kl}$

and

$$C^{ij}_{kl} = \beta(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k) + \gamma\, \delta^{ij}\delta_{kl}$$

then for $W = O'\begin{pmatrix} e^{\rho_1} & \\ & e^{\rho_d}\end{pmatrix} O$

$$\frac{d}{dt}\langle \tilde{f}(\vec{\rho})\rangle = \langle (\mathcal{L}\tilde{f})(\vec{\rho})\rangle$$

for

$$\mathcal{L} = \frac{\beta+\gamma}{2}\left(\sum_i \frac{\partial^2}{\partial \rho_i^2} + \sum_{i\neq j} \coth(\rho_i - \rho_j)\frac{\partial}{\partial \rho_i}\right) + \frac{\beta}{2}\left(\sum_i \frac{\partial}{\partial \rho_i}\right)^2 - \frac{(d+1)\beta+\gamma}{2}\sum_i \frac{\partial}{\partial \rho_i}$$

Solution. By Ito calculus,

$$\frac{d}{dt}\langle f(W)\rangle = \left\langle \frac{1}{2} C^{ij}_{kl} W^k_m W^l_n \frac{\partial}{\partial W^i_m}\frac{\partial}{\partial W^j_n} f(W)\right\rangle$$

$$= \frac{1}{2}\left\langle \left(\beta W^i_m W^j_n + \beta W^j_m W^i_n + \gamma\delta^{ij} W^k_m W^k_n\right)\frac{\partial}{\partial W^i_m}\frac{\partial}{\partial W^j_n} f(W)\right\rangle$$

$$= \frac{1}{2}\Big\langle \Big(\beta W^i_m \frac{\partial}{\partial W^i_m} W^j_n \frac{\partial}{\partial W^j_n} - \beta W^i_m \frac{\partial}{\partial W^i_m} + \beta W^j_m \frac{\partial}{\partial W^i_m} W^i_n \frac{\partial}{\partial W^j_n}$$

$$- d\beta W^i_m \frac{\partial}{\partial W^i_m} + \gamma W^k_m \frac{\partial}{\partial W^i_m} W^k_n \frac{\partial}{\partial W^i_n} - \gamma W^i_m \frac{\partial}{\partial W^i_m}\Big) f(W)\Big\rangle$$

Let us introduce the action of the generators of the $gl(N)$ Lie algebra

$$(E_i^j f)(W) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(e^{-\varepsilon E_i^j} W) \qquad \text{for } E_i^j = \begin{array}{c} \quad j \\ \begin{pmatrix} & & \\ & 1 & \\ & & \end{pmatrix} i \end{array}$$

or $(E_i^j)^n{}_m = \delta_i^n \delta_m^j$

$$(E_i^j f)(W) = -(E_i^j W)^n{}_m \frac{\partial}{\partial W^n{}_m} f(W) = -\delta^n_i \delta^j_r W^r{}_m \frac{\partial}{\partial W^n{}_m} f(W)$$

$$= -W^j{}_m \frac{\partial}{\partial W^i{}_m} f(W)$$

and the generator of dilations $D = -\sum_i E_i^i$

$$D f(W) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(e^{\varepsilon} W) = W^i{}_m \frac{\partial}{\partial W^i{}_m} f(W) .$$

We get

$$\frac{d}{dt}\langle f(W)\rangle = \frac{1}{2}\Big\langle\Big(\beta D^2 - \beta D + \beta E_i^j E_j^i$$

$$- d\beta D + \gamma (E_i^k)^2 - \gamma D\Big) f(W)\Big\rangle$$

If $f(W) = \tilde{f}(\vec{\varrho})$ then $D f(W) = \sum_i \frac{\partial}{\partial \varrho_i} \tilde{f}(\vec{\varrho})$

Moreover

$$(J_{ik} f)(W) \equiv (E_i^k - E_k^i) f(W) = 0$$

since J_{ik} generates the left rotations.

Hence

$$\frac{d}{dt}\langle f(W)\rangle = \frac{\beta+\gamma}{2} E_i^j E_j^i + \frac{\beta}{2} D^2 - \frac{(d+1)\beta + \gamma}{2} D ,$$

The D terms reproduce correctly the the $\sum_i \frac{\partial}{\partial \rho_i}$ contributions to L. Since for $W = \begin{pmatrix} e^{\rho_1} & & \\ & \ddots & \\ & & e^{\rho_d} \end{pmatrix}$,

$$(E_i{}^i f)(W) = -\frac{\partial}{\partial \rho^i} \tilde{f}(\vec{\rho})$$

the sum

$$\frac{\beta+\gamma}{2} \sum_i (E_i{}^i)^2 f(W) = \frac{\beta+\gamma}{2} \sum_i \left(\frac{\partial}{\partial \rho^i}\right)^2 \tilde{f}(\vec{\rho}).$$

We are left with the calculation of

$$\frac{\beta+\gamma}{2} \sum_{i \neq j} E_i{}^j E_j{}^i f(W)$$

Since $E_i{}^j E_j{}^i$ act only on 2×2 submatrix composed of $W^k{}_m$ with $n, m = i$ or j, it is enough to do the calculation of

$$(E_1{}^2 E_2{}^1 + E_2{}^1 E_1{}^2) f(W)$$

in 2 dimensions.

$$(E_1{}^2 E_2{}^1 f)\begin{pmatrix} e^{\rho_1} & 0 \\ 0 & e^{\rho_2} \end{pmatrix} = \frac{d}{d\varepsilon_1} \frac{d}{d\varepsilon_2}\Big|_0 f\left(e^{-\varepsilon_2 E_2^1} e^{-\varepsilon_1 E_1^2} \begin{pmatrix} e^{\rho_1} & 0 \\ 0 & e^{\rho_2} \end{pmatrix}\right)$$

$$= \frac{d}{d\varepsilon_1} \frac{d}{d\varepsilon_2}\Big|_0 f(W_{12}) \quad \text{with } W_{12} = \begin{pmatrix} 1 & 0 \\ -\varepsilon_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\varepsilon_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\rho_1} & 0 \\ 0 & e^{\rho_2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\rho_1} & -\varepsilon_1 e^{\rho_2} \\ -\varepsilon_2 e^{\rho_1} & (1+\varepsilon_1 \varepsilon_2) e^{\rho_2} \end{pmatrix}$$

Since f depends only on the stretching exponents of W_{12}, we have to find them keeping only the terms linear in $\varepsilon_1, \varepsilon_2$ and $\varepsilon_1\varepsilon_2$.

Since
$$W_{12}W_{12}^T = \begin{pmatrix} e^{2\varrho_1} & , -\varepsilon_2 e^{2\varrho_1} - \varepsilon_1 e^{2\varrho_2} \\ -\varepsilon_2 e^{2\varrho_1} - \varepsilon_1 e^{2\varrho_2}, & (1+2\varepsilon_1\varepsilon_2)e^{2\varrho_2} \end{pmatrix} + \dots$$

the eigenvalues of $W_{12}W_{12}^T$ satisfy the equation
$$\lambda^2 - \lambda\left(e^{2\varrho_1} + e^{2\varrho_2} + 2\varepsilon_1\varepsilon_2 e^{2\varrho_2}\right) + e^{2\varrho_1}e^{2\varrho_2} + \dots = 0$$

or

$$\lambda = \frac{1}{2}\left[e^{2\varrho_1} + e^{2\varrho_2} + 2\varepsilon_1\varepsilon_2 e^{2\varrho_2} \pm \sqrt{(e^{2\varrho_1} - e^{2\varrho_2})^2 + 4\varepsilon_1\varepsilon_2(e^{2\varrho_1} + e^{2\varrho_2})e^{2\varrho_2}}\right]$$

$$= \frac{1}{2}\left[e^{2\varrho_1} + e^{2\varrho_2} + 2\varepsilon_1\varepsilon_2 e^{2\varrho_2} \pm \left(e^{2\varrho_1} - e^{2\varrho_2} + 2\varepsilon_1\varepsilon_2 \frac{e^{2\varrho_1} + e^{2\varrho_2}}{e^{2\varrho_1} - e^{2\varrho_2}} e^{2\varrho_2}\right) + \dots\right]$$

$$= \begin{cases} e^{2\varrho_1}\left(1 + 2\varepsilon_1\varepsilon_2 \frac{e^{2\varrho_2}}{e^{2\varrho_1} - e^{2\varrho_2}} + \dots\right) \\ e^{2\varrho_2}\left(1 - 2\varepsilon_1\varepsilon_2 \frac{e^{2\varrho_2}}{e^{2\varrho_1} - e^{2\varrho_2}} + \dots\right) \end{cases}$$

and the modified exponents are

$$\begin{cases} \varrho_1 + \varepsilon_1\varepsilon_2 \frac{e^{2\varrho_2}}{e^{2\varrho_1} - e^{2\varrho_2}} + \dots \\ \varrho_2 - \varepsilon_1\varepsilon_2 \frac{e^{2\varrho_2}}{e^{2\varrho_1} - e^{2\varrho_2}} + \dots \end{cases}$$

Thus

$$E_1{}^2 E_2{}^1 \tilde{f}(\vec{\varrho}) = \frac{e^{2\varrho_2}}{e^{2\varrho_1} - e^{2\varrho_2}}\left(\frac{\partial}{\partial\varrho_1} - \frac{\partial}{\partial\varrho_2}\right)\tilde{f}(\vec{\varrho})$$

Similarly,

$$\left(E_2{}^1 E_1{}^2 f\right)\begin{pmatrix} e^{\varrho_1} & 0 \\ 0 & e^{\varrho_2} \end{pmatrix} = \frac{d}{d\varepsilon_1}\frac{d}{d\varepsilon_2}\Big|_0 f\left(e^{-\varepsilon_2 E_1{}^2} e^{-\varepsilon_1 E_2{}^1} \begin{pmatrix} e^{\varrho_1} & 0 \\ 0 & e^{\varrho_2} \end{pmatrix}\right)$$

$$= \frac{d}{d\varepsilon_1}\frac{d}{d\varepsilon_2}\Big|_0 f(W_{21}) \quad \text{with} \quad W_{21} = \begin{pmatrix} 1 & -\varepsilon_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\varepsilon_1 & 0 \end{pmatrix}\begin{pmatrix} e^{\varrho_1} & 0 \\ 0 & e^{\varrho_2} \end{pmatrix}$$

$$= \begin{pmatrix} (1+\varepsilon_1\varepsilon_2)e^{\varrho_1} & -\varepsilon_2 e^{\varrho_2} \\ -\varepsilon_1 e^{\varrho_1} & e^{\varrho_2} \end{pmatrix}$$

$$W_{21} W_{21}^T = \begin{pmatrix} (1+2\varepsilon_1\varepsilon_2)e^{2\varrho_1} & -\varepsilon_1 e^{2\varrho_1} - \varepsilon_2 e^{2\varrho_2} \\ -\varepsilon_1 e^{2\varrho_1} - \varepsilon_2 e^{2\varrho_2} & e^{2\varrho_2} \end{pmatrix}$$

and for the eigenvalues we get

$$\lambda = \frac{1}{2}\left[e^{2\varrho_1} + e^{2\varrho_2} + 2\varepsilon_1\varepsilon_2 e^{2\varrho_1} \pm \sqrt{\left(e^{2\varrho_1} - e^{2\varrho_2}\right)^2 + 4\varepsilon_1\varepsilon_2 e^{2\varrho_1}\left(e^{2\varrho_1} + e^{2\varrho_2}\right) + \ldots}\right]$$

$$= \frac{1}{2}\left[e^{2\varrho_1} + e^{2\varrho_2} + 2\varepsilon_1\varepsilon_2 e^{2\varrho_1} \pm \left(e^{2\varrho_1} - e^{2\varrho_2} + 2\varepsilon_1\varepsilon_2 e^{2\varrho_1} \frac{e^{2\varrho_1} + e^{2\varrho_2}}{e^{2\varrho_1} - e^{2\varrho_2}}\right) + \ldots\right]$$

$$= \begin{cases} e^{2\varrho_1}\left(1 + 2\varepsilon_1\varepsilon_2 \dfrac{e^{2\varrho_1}}{e^{2\varrho_1} - e^{2\varrho_2}} + \ldots\right) \\ e^{2\varrho_2}\left(1 - 2\varepsilon_1\varepsilon_2 \dfrac{e^{2\varrho_1}}{e^{2\varrho_1} - e^{2\varrho_2}} + \ldots\right) \end{cases}$$

leading to the modified exponents

$$\varrho_1 + \varepsilon_1\varepsilon_2 \frac{e^{2\varrho_1}}{e^{2\varrho_1} - e^{2\varrho_2}} + \ldots$$

$$\varrho_2 - \varepsilon_1\varepsilon_2 \frac{e^{2\varrho_1}}{e^{2\varrho_1} - e^{2\varrho_2}} + \ldots$$

and the result

$$E_2{}^1 E_1{}^2 \tilde{f}(\vec{\rho}) = \frac{e^{2\rho_1}}{e^{2\rho_1} - e^{2\rho_2}} \left(\frac{\partial}{\partial \rho_1} - \frac{\partial}{\partial \rho_2} \right) \tilde{f}(\vec{\rho})$$

Altogether, we get

$$\sum_{i<j} (E_i{}^j E_j{}^i + E_j{}^i E_i{}^j) \tilde{f}(\vec{\rho})$$

$$= \sum_{i<j} \frac{e^{2\rho_i} + e^{2\rho_j}}{e^{2\rho_i} - e^{2\rho_j}} \left(\frac{\partial}{\partial \rho_i} - \frac{\partial}{\partial \rho_j} \right) \tilde{f}(\vec{\rho})$$

$$= \sum_{i \neq j} \coth(\rho_i - \rho_2) \frac{\partial}{\partial \rho_i} \tilde{f}(\vec{\rho}) .$$

This ends the proof

$\square$