Exercises for the Warwick course on "Soluble models of turbulent transport"

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July 7, 2006

Exercise session I

Problem I.1. To recall the basic formalism of the stochastic ODE's, consider the equation for the noisy Lagrangian trajectory $\boldsymbol{R}(t)$ in a given smooth d-dimensional velocity field:

$$d\boldsymbol{R} \;=\; \boldsymbol{v}(t,\boldsymbol{r})\,dt \;+\; \sqrt{2\kappa}\,d\boldsymbol{\beta}(t) \tag{1.1}$$

where $\boldsymbol{\beta}(t)$ is the d dimensional Brownian motion. $\boldsymbol{\beta}(t) = \int^{t} \boldsymbol{\eta}(s)\,ds$ where $\boldsymbol{\eta}(t)$ is the white noise: a Gaussian (generalized) process with mean zero and covariance

$$\overline{\eta^i(t)\,\eta^j(t')} \;=\; \delta^{ij}\,\delta(t-t')\,. \tag{1.2}$$

Eq. (1.1) does not require a choice of convention (why?) but for any regular function f, the composed process $f(\boldsymbol{R}(t))$ satisfies the Ito stochastic ODE

$$d\,f(\boldsymbol{R}) \;=\; (\nabla_i f)(\boldsymbol{R})\,[v^i(t,\boldsymbol{R})\,dt \;+\; \sqrt{2\kappa}\,\beta^i(t)] \;+\; \kappa\,(\boldsymbol{\nabla}^2 f)(\boldsymbol{R})\,dt \tag{1.3}$$

or the Stratonovich one without the last (Ito) term:

$$df(\boldsymbol{R}) \;=\; (\nabla_i f)(\boldsymbol{R})\,[v^i(t,\boldsymbol{R})\,dt \;+\; \sqrt{2\kappa}\circ d\beta^i(t)]\,. \tag{1.4}$$

Problem I.2. Show that if $\boldsymbol{R}(t;t_0,\boldsymbol{r}_0)$ solves the stochastic ODE (1.1) with the condition $\boldsymbol{R}(t_0;t_0,\boldsymbol{r}_0)=\boldsymbol{r}_0$ then

$$n(t,\boldsymbol{r}) \;=\; \int \overline{\delta(\boldsymbol{r}-\boldsymbol{R}(t;t_0,\boldsymbol{r}_0))}\;n(t_0,\boldsymbol{r}_0)\,d\boldsymbol{r}_0\,, \tag{1.5}$$

where the overline denotes the average over the white noise $\boldsymbol{\eta}$, solves the density transport equation:

$$\partial_t n \;+\; \boldsymbol{\nabla}\cdot(n\boldsymbol{v}) \;-\; \kappa\boldsymbol{\nabla}^2 n \;=\; 0\,. \tag{1.6}$$

Problem I.3. With the same notations, show that

$$\theta(t,\boldsymbol{r}) \;=\; \int \overline{\delta(\boldsymbol{r}_0-\boldsymbol{R}(t_0;t,\boldsymbol{r}))}\;\theta(t_0,\boldsymbol{r}_0)\,d\boldsymbol{r}_0 \;=\; \overline{\theta(t_0,\boldsymbol{R}(t_0;t;\boldsymbol{r}))} \tag{1.7}$$

satisfies the scalar transport equation:

$$\partial_t\theta + \boldsymbol{v}\cdot\boldsymbol{\nabla}\theta - \kappa\boldsymbol{\nabla}^2\theta = 0\,. \tag{1.8}$$

Problem I.4. For $W^i_{\ j}(t;t_0,\boldsymbol{r}_0) = \frac{\partial}{\partial r_0^j}R^i(t;t_0,\boldsymbol{r}_0)$, show that

$$\boldsymbol{B}(t,\boldsymbol{r}) = \int \overline{\delta(\boldsymbol{r}-\boldsymbol{R}(t;t_0,\boldsymbol{r}_0))\,W(t;t_0\boldsymbol{r}_0)}\;\boldsymbol{B}(t_0,\boldsymbol{r}_0)\,d\boldsymbol{r}_0 \tag{1.9}$$

solves equation for the magnetic field transport:

$$\partial_t\boldsymbol{B} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{\nabla}\cdot\boldsymbol{v})\boldsymbol{B} - (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \kappa\boldsymbol{\nabla}^2\boldsymbol{B} = 0\,. \tag{1.10}$$

Exercise session II

Problem II.1. Show that the stochastic ODE for the Lagrangian trajectories

$$d\boldsymbol{R} = \boldsymbol{v}(t,\boldsymbol{R})\,dt \tag{2.1}$$

in the Kraichnan model (white noise in time) velocities $\boldsymbol{v}$ is the same with the Ito and Stratonovich conventions.

Problem II.2a. Show that the stochastic ODE for the infinitesimal separation of the Lagrangian trajectories

$$d\,\delta\boldsymbol{R} = (\delta\boldsymbol{R}\cdot\boldsymbol{\nabla})\boldsymbol{v}(t,\boldsymbol{R}(t))\,dt \tag{2.2}$$

where $\boldsymbol{R}(t)$ solves Eq. (2.1), is the same with the Ito and Stratonovich convention.

II.2b. Show that for the stochastic equation

$$d\,\delta\boldsymbol{R} = S(t)\,\delta\boldsymbol{R}\,dt \tag{2.3}$$

for the matrix-valued white noise $S(t)$ with the covariance

$$\left\langle S^i_{\ k}(t)\,S^j_{\ l}(t')\right\rangle = -\,\delta(t-t')\,\nabla_k\nabla_l D^{ij}(\mathbf{0}) \equiv \delta(t-t')\,C^{ij}_{kl} \tag{2.4}$$

the Ito and Stratonowich conventions do not coincide, in general.

Problem II.3. Find the probability distribution function $\left\langle\delta(\Delta-\Delta(t;\Delta_0))\right\rangle$ of the length of the infinitesimal separation $|\delta\boldsymbol{R}(t)|\equiv\Delta(t)$ in the isotropic Kraichnan model with

$$C^{ij}_{kl} = \beta(\delta^i_k\,\delta^j_l + \delta^i_l\,\delta^j_k) + \gamma\,\delta^{ij}\,\delta_{kl}\,. \tag{2.5}$$

Extract from the result the value of the top Lyapunov exponent.

Problem II.4. Show that

$$\lim_{\Delta_0\to 0}\ \Big\langle\delta(\Delta-\Delta(t;\Delta_0))\Big\rangle\ =\ \delta(\Delta)\,,\tag{2.6}$$

where Δ_0 is the time zero value of $\Delta(t)$.

Exercise session III

Problem III.1. Prove that if $W(t)$ satisfies the multiplicative Ito stochastic ODE

$$dW\ =\ S(t)\,dt\tag{3.1}$$

where $S(t)$ is the matrix-valued white noise with the covariance given by Eqs. (2.4) and (2.5) above then

$$\frac{d}{dt}\Big\langle f(W)\Big\rangle\ =\ \Big\langle(\mathcal{L}f)(W)\Big\rangle\tag{3.2}$$

where

$$\mathcal{L}\ =\ \frac{\beta}{2}\sum_{ij}\mathcal{E}_i^{\ j}\mathcal{E}_j^{\ i}\ +\ \frac{\gamma}{2}\sum_{ij}(\mathcal{E}_i^{\ j})^2\ +\ \frac{\beta}{2}\mathcal{D}^2\ -\ \frac{(d+1)\beta+\gamma}{2}\mathcal{D}\tag{3.3}$$

where $\mathcal{E}_i^{\ j}$ are the generators of the left action of the Lie algebra $gl(d)$:

$$(\mathcal{E}_i^{\ j}f)(W)\ =\ \frac{d}{d\epsilon}\Big|_0\,f(\mathrm{e}^{-\epsilon E_i^j}W)\tag{3.4}$$

for $E_i^{\ j}$ denoting the matrices with 1 on the intersection of the i-th raw and j-th column and zeros elsewhere and where CD is the generator of dilations:

$$(\mathcal{D}f)(W)\ =\ \frac{d}{d\epsilon}\Big|_0\,f(\mathrm{e}^{\epsilon}W)\,.\tag{3.5}$$

Problem III.2. Show that if

$$f(W)\ =\ f(O'WO)\tag{3.6}$$

for all $O,O'\in O(d)$ defining a function $\widetilde{f}(\boldsymbol{\rho})$ of the exponents $\rho_1,\dots\rho_d$ such that $W=O'\,\mathrm{diag}[\mathrm{e}^{\rho_1},\dots,\mathrm{e}^{\rho_d}]\,O$ then

$$\begin{aligned}(\mathcal{L}f)(W)\ =\ &\Big[\frac{\beta+\gamma}{2}\Big(\sum_{i=1}^d\frac{\partial^2}{\partial\rho_i^2}\ +\ \sum_{i\neq j}\coth(\rho_i-\rho_j)\frac{\partial}{\partial\rho_i}\Big)\\ &+\frac{\beta}{2}\Big(\sum_i\frac{\partial}{\partial\rho_i}\Big)^2\ -\ \frac{(d+1)\beta+\gamma}{2}\sum_i\frac{\partial}{\partial\rho_i}\Big]\,\widetilde{f}(\boldsymbol{\rho})\,.\end{aligned}\tag{3.7}$$

Problem III.3 (for volunteers). Show that for

$$\mathcal{F}(\boldsymbol{\rho})\ =\ \mathrm{e}^{-\sum\limits_i\rho_i/2}\Big(\prod_{i<j}\sinh(\rho_i-\rho_j)\Big)^{1/2},\tag{3.8}$$

in the action on functions $\widetilde{f}(\boldsymbol{\rho})$,

$$-\mathcal{H}_{CM} \equiv \mathcal{F}\,\mathcal{L}\,\mathcal{F}^{-1} \;=\; = \frac{\beta+\gamma}{2}\Big(\sum_{i=1}^{d}\frac{\partial^2}{\partial\rho_i^2} + \frac{1}{2}\sum_{i<j}\sinh^{-2}(\rho_i-\rho_j)\Big) + \frac{\beta}{2}\Big(\sum_i \frac{\partial}{\partial\rho_i}\Big)^2 - \text{const.} \tag{3.9}$$

Calculate the constant. What is the bottom of the spectrum of the resulting Calogero-Moser Hamiltonian $\mathcal{H}_{CM}$?