Absorption of Sound by Vortex Filaments

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The effect of an acoustic wave packet falling onto a thin 3D filament of vorticity is identified and analyzed. The wavelength of sound decreases to zero in a finite time in such a process. Therefore, even if viscosity is small the wave packets will reach the scales of strong viscous dissipation and get absorbed, transferring their energy to the thermal energy of the compressible vortex flow. The cross section of the sound absorption by multiple vortex filaments having an arbitrary 3D shape is derived. Applications to the theory of the second sound attenuation in He II are discussed.

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The vortex-acoustic interaction is one of the basic dynamical processes in compressible turbulent fluids. Acoustic waves can modify or even destabilize vortices [1–4]. In turn, the vortices can generate sound [5] and scatter it via the refractive and diffractive effects [6–10]. It will be shown in this Letter that thin vortex filaments can also absorb sound wave packets. Briefly, this effect can be described as follows. The convective distortion of acoustic wave fronts by the sheared velocity field produced by a vortex acts to turn wave packets more toward the vortex and increase their wave numbers. If the vortex is intense enough, then certain wave packets will approach the vortex core along a spiral trajectory (see Figs. 1 and 2) with their wave numbers rapidly increasing. When the wave numbers of such wave packets reach the viscous scale they will dissipate, transferring their energy to the internal energy of the vortex flow. Such a mechanism of the sound absorption is important for understanding the interaction of vortices and the high-frequency acoustic component in turbulence. It may also appear useful for explaining the second sound absorption by quantized vortices in He II, a problem which needs a better theoretical treatment. We will exploit the ray acoustics approach [9–11], thereby focusing on the refractive effects and neglecting the vortex-induced diffraction of sound. Equations for acoustic rays in a moving weakly inhomogeneous medium are:

$$\dot{\mathbf{r}} = \nabla_{\mathbf{k}} H, \tag{1}$$

$$\dot{\mathbf{k}} = -\nabla_{\mathbf{r}} H, \tag{2}$$

where

$$H = \omega_{\mathbf{k}} \equiv c_s|\mathbf{k}| + \mathbf{v} \cdot \mathbf{k} \tag{3}$$

is the Hamiltonian function (frequency), $\mathbf{r}$ and $\mathbf{k}$ are the coordinate and wave vector of the sound wave packet, and $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ and $c_s = (\mathbf{r}, t)$ are the vortex produced the velocity field and the local speed of sound, respectively. Let us consider a flow produced by an infinitely thin straight filament of vorticity,

$$v_\phi = \frac{\Gamma}{2\pi r}, \qquad v_r, v_z = 0, \tag{4}$$

where Γ is the total circulation of velocity; ϕ, r, and z are the polar coordinates. Assume also that the speed of sound c_s is constant everywhere except for the region of the vortex core which we will not consider. In this case, the Hamiltonian (3) is independent of z and ϕ, which results in conservation of the z component of the momentum, P_z, and the z component of the angular momentum, M_z,

$$P_z \equiv k_z = \text{const}, \tag{5}$$

$$M_z \equiv (\mathbf{k} \times \mathbf{r})_z = k_\phi r = \text{const}. \tag{6}$$

Together with the energy (3) this makes three motion integrals of the sixth order Hamiltonian system (1) and (2). Thus, taking into account that all the integrals (3), (5), and (6) are in involution, we arrive at the conclusion that the ray equations (1) and (2) are integrable. In fact, this result is valid for any axially symmetric vortex profiles $\mathbf{v}(r)$, $c_s(r)$, and, in particular, those having nonzero helicity. We will not consider this more general case in this Letter, leaving it for further consideration [4]. Substituting (4), (5), and (6) into (3), we express k_r^2 as a function of r:

$$k_r^2 = \frac{\Gamma^2 M_z^2}{4\pi^2 c_s^2}\frac{1}{r^4} - \left(\frac{H\Gamma M_z}{\pi c_s^2} + M_z^2\right)\frac{1}{r^2} + \frac{H^2}{c_s^2} - P_z^2. \tag{7}$$

Expression (7) for the projection of the acoustic rays on the plane (k_r, r) contains very important information about the dynamics of the sound wave packets. An example of the phase portrait of the system is shown in Fig. 3. When k_r turns into zero the wave packet experiences a reflection. Approaching $r = 0$ means falling of the wave packet onto the vortex filament. When r approaches zero, $|k_r|$ tends to infinity. Therefore, during the falling onto the vortex the wave packet will inevitably reach the small scales of viscous dissipation and get absorbed transferring its energy to the thermal energy of the mean flow. As seen in Fig. 3, an acoustic ray approaching the vortex

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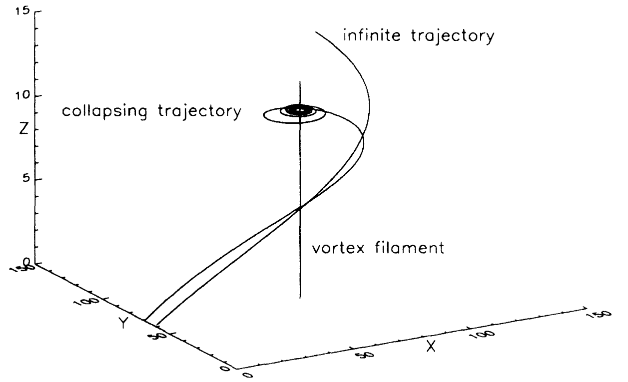

FIG. 1. Two three dimensional trajectories of an acoustic wave packet in the velocity field of a straight vortex filament corresponding to the wave packet's falling onto the vortex and to an infinite motion of the wave packet. The vorticity of the filament is concentrated on the line $\mathbf{r} = (64, 64, z)$. The values of the speed of sound and the circulation are $c_s = 1$ and $\Gamma/2\pi = 100$. Initial value of the wave number is $\mathbf{k}_0 = (1, 0, 0.3)$; the initial position is $\mathbf{r}_0 = (0, 70, 0)$ [criterion (8) is satisfied] for the collapsing trajectory and $\mathbf{r}_0 = (0, 60, 0)$ [criterion (8) is not satisfied] for the infinite trajectory.

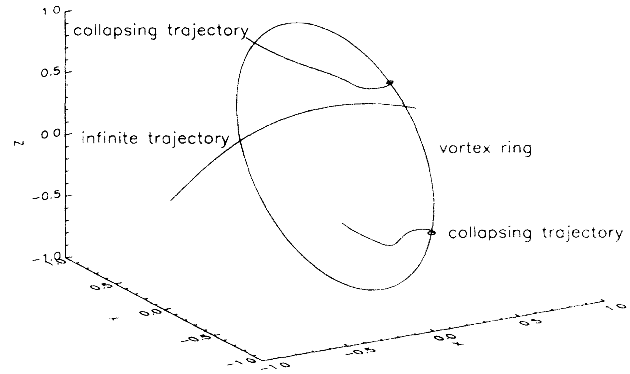

FIG. 2. The trajectories of the sound wave packet in the field produced by a filamentary ring for the initial values of the wave numbers for $k_{z0} = 1$, $k_{r0} = 0$, the angular momentum $M_z = 0.3$, and several different initial positions. Parameters of the ring are the following: The center position is at $\mathbf{r} = (0, 0, 0)$, the radius $R = 1$, the core radius $a = 0.001$, the circulation $\Gamma = 2\pi$, and the speed of sound $c_s = 6$.

from infinity will collapse onto it if it can pass from $r = \infty$ to $r = 0$ without reflection. This takes place when the minimal k_r^2 in (7) is greater than zero. Such a condition results in the following criterion for the collapse:

$$2\frac{k^\infty - k_\perp^\infty}{k_\perp^\infty} < \frac{2\pi r_\perp c_s}{\Gamma} < 2\frac{k^\infty + k_\perp^\infty}{k_\perp^\infty}, \tag{8}$$

where k^∞ is the absolute value of the wave vector at infinity, $k_\perp^\infty$ is its projection orthogonal to the z axis, and $r_\perp$ is the impact parameter, $r_\perp = -(\mathbf{k}^\infty \times \mathbf{r}^\infty)_z/k^\infty$. Note that $\Gamma/2\pi r_\perp c_s$ is the Mach number at $r = r_\perp$. According to the criterion (8) the collapse can occur even when $\mathrm{Ma}(r_\perp) \ll 1$ if $k^\infty \gg k_\perp^\infty$, i.e., if $k_z^2 \gg k_\perp^2$.

Let us show that the considered phenomenon is, in fact, of the collapse type, i.e., that the wave packet falls on the vortex in a finite time. Substituting (7), (5), and (6) into Eqs. (1) and (2) we arrive at

$$\dot r = c_s\frac{k_r}{|k|} = c_s\frac{[\alpha^2 - \beta r^2 + (\gamma^2 - P_z^2)r^4]^{1/2}}{\gamma r^2 - \alpha}, \tag{9}$$

where $\alpha = \Gamma M_z/2\pi c_s$, $\beta = M_z^2 + H\Gamma M_z/\pi c_s^2$, and $\gamma = H/c_s$. The solution of Eq. (9) can be written in an implicit form as

$$c_s t = \int \frac{\gamma r^2 - \alpha}{[\alpha^2 - \beta r^2 + (\gamma^2 - P_z^2)r^4]^{1/2}}\,dr. \tag{10}$$

The expression on the right hand side (rhs) is an elliptic integral, which can be written as a linear combination of the standard elliptic integrals of the first and the second kind [12]. Near the collapse point one can retain only the leading order in the expansion of the rhs of (10) with respect to small r. This gives

$$r = c_s(t_0 - t), \qquad \phi = \frac{\Gamma}{2\pi c_s^2(t - t_0)}, \tag{11}$$

$$k_r = \frac{\alpha}{c_s^2(t - t_0)^2}, \qquad k_\phi = \frac{M_z}{c_s(t - t_0)}. \tag{12}$$

As seen from (11), the wave packet moves in the negative radial direction with the speed of sound and makes an infinite number of rotations around the vortex filaments in a finite time. Further, from (12) we see that both the

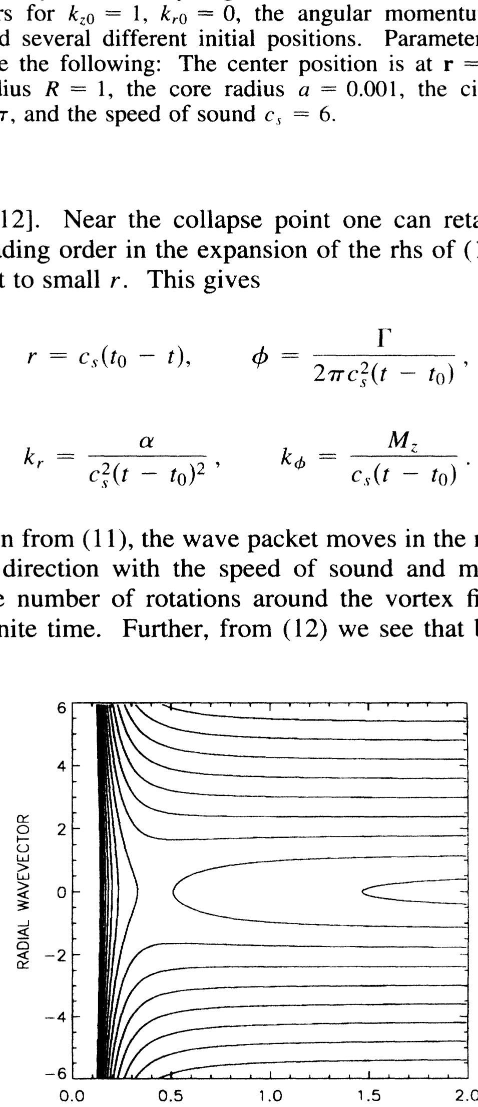

FIG. 3. Projection of the acoustic rays on the plane (k_r, r) calculated according to (7) for $k_z = 0.1$, $M_z = -1$, $c_s = 1$, $\Gamma/2\pi = 0.2$, and several different values of H, equally spaced in the interval from -2.985 to 8.359.

azimuthal and the radial wave numbers turn into infinity in a finite time, and that k_r grows faster than k_ϕ. An example of a collapsing acoustic ray is shown in Fig. 1. One can see that quite a small difference in the initial position of the wave packet can result in qualitatively different types of behavior. The physics behind the collapse phenomenon is that the convective distortion turns the acoustic wave front more toward the vortex. If the size of the vortex core is finite then the ray equations do not possess collapsing solutions. Nevertheless, in the case of thin vortices the wave packet can make many loops around and reach the scales of viscous dissipation before entering the vortex core.

What is remarkable is that the approximation of ray acoustics can remain valid up to the collapse time in spite of approaching the steep gradients of the velocity field. For this, the sound wavelength must remain smaller than the current radius of the spiral, $(k^2r^2)_{\min} \gg 1$, or, taking into account (3), (4), and (6),

$$\left(k_\infty r - \frac{\Gamma M_z}{2\pi c_s r}\right)^2_{\min} = \frac{2\Gamma k_\infty^2 r_\perp}{\pi c_s} \gg 1 . \qquad (13)$$

Thus, the ray acoustics is valid at the solutions corresponding to the collapse if the vortex circulation is large enough, $\Gamma \gg \pi c_s/2k_\infty^2 r_\perp$.

A prototype of the collapse in the 2D case has been found by Salant [10], who has shown that certain acoustic rays approach the point vortex along spirals. We have found and analyzed above three dimensional collapsing solutions. Now we suggest a hypothesis that the collapsing rays exist for any 3D shape of the vortex filament. Obviously, we cannot integrate the ray equations to prove this statement, because generally we lose all three integrals of motion (3), (5) and (6). Nevertheless, if the speed of sound in the medium is greater than the speed of the parts of the nonstationary filament then the wave packet can approach the filament quite closely without any significant deflection in its trajectory. But close to the filament the velocity field is such that the effects of the filament's curvature and nonstationarity are unimportant. Therefore, after approaching the filament the wave packet enters into the "self-similar" stage of its evolution and falls onto the filament provided that the criterion (8) is satisfied.

To illustrate this point we performed a numerical computation of the motion equations (1) and (2) for the wave packets moving in the velocity field of a filamentary vortex ring having the following structure [13]:

$$v_z = -\frac{1}{r}\frac{\partial \Psi}{\partial r} - v_0, \qquad v_r = \frac{1}{r}\frac{\partial \Psi}{\partial z}, \qquad v_\psi = 0, \qquad (14)$$

where

$$\Psi = -\frac{\Gamma}{2\pi}(r_1 + r_2)[K(\lambda) - E(\lambda)], \quad \lambda = (r_2 - r_1)/(r_2 + r_1), \quad r_{1,2}^2 = z^2 + (r \mp R)^2, \qquad (15)$$

K and E are the complete elliptic integrals of the first and the second kind [12], v_0 is the velocity of translation of the ring, $v_0 = (\Gamma/4\pi R)[\ln(8R/a) - 1/4]$, R is the radius of the ring, and a is the core radius.

The result of numerical integration of the ray equations for different initial positions of the wave packet is shown in Fig. 2. Here we have chosen $v_0 = 4.368\,59$ ($\Gamma/2\pi = 1$, $R/a = 1000$) and $c_s = 6$ to consider a case which is formally beyond the limits of applicability of the proof given above (i.e., $c_s \gg v_0$). We see that certain trajectories indeed collapse onto the vortex ring. Zooming on the collapse regions gives exactly the same spiral structure as in the case of the straight filament shown in Fig. 1. We have considered also the case $c_s \gg v_0$ and found that the acoustic rays have simpler structure in this case: The wave packets move along nearly straight lines until they approach quite close to the filament.

In Fig. 4 we show a typical behavior of the absolute value of the wave vector of the collapsing wave packet. We see that the most dramatic increase of the wave number takes place during a very short period $3.41 < t < 3.455$ which follows after a long period of gradual evolution. Obviously, after approaching quite close to the collapsing point the numerical method breaks down, which explains the irregular oscillations developing after $t = 3.455$ and a subsequent drop in $|k|$.

Criterion of the collapse (8) allows us to find the cross section for sound absorption by vortex filaments. Indeed, every wave packet having the impact parameter $r_\perp$ which satisfies condition (8) will fall onto the vortex and get absorbed due to the great increase in $|k|$. Therefore, each

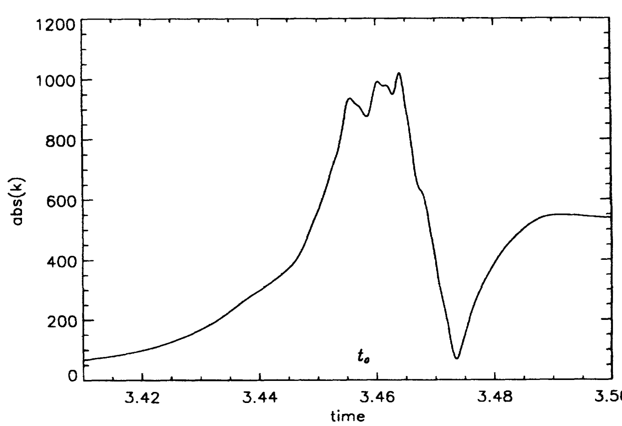

FIG. 4. The typical evolution of the absolute value of the wave vector during the collapse of the sound ray onto the ring. The evolution for $t < 3.41$ is very gradual and does not possess any peculiarities.

element of the vortex filament produces a nontransparent window, which width $\Delta r_\perp$, according to (8), is equal to $2\Gamma/\pi c_s$. Integrating over all the length elements of a filament, and summing over all the vortex filaments in the volume with 1 cm^2 base, we obtain the following expression for the cross section of the sound absorption:

$$\sigma = \frac{2}{\pi c_s} \sum_i \int_{\mathbf{l}_i} \Gamma_i \sin\alpha_i(\mathbf{l})\, dl\,, \tag{16}$$

where $\alpha_i(\mathbf{l})$ is the angle between the direction of incidence of the sound wave and the direction of the filament number i at the point $\mathbf{l}$ on this filament.

Finally, the above discussed mechanism of acoustic wave absorption by vortices may be very important in the hydrodynamics of liquid He II where vortex filaments are believed to be elementary structures of vortex excitations. In fact, second sound absorption by filaments is used in experiments on liquid helium as a diagnostics of intensity of vortex excitations in the system [14]. To date, no satisfactory theory has been proposed for the sound attenuation, and the cross section of the absorption is found empirically [15]. To apply the results of this paper to the second sound absorption by vortices we need to reformulate the problem in terms of two component hydrodynamics of liquid helium and take into account the dispersion which is proportional to $\hbar$. The latter is important because it determines the actual size of the vortex core and affects the acoustic component via a change of its group velocity.

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