MA124: Maths by Computer - Week 7

Regulations

- This assignment gives 15% of your final mark.
- The assignment is due on Monday 28 February 2011, to be handed over to the drop-off box by the Mathematics General Office by 3pm the latest.
- Even if you cannot come yourself, you have to arrange for your work to be handed in timely (e.g. ask a friend to bring it). No late work will be accepted.
- Your work should be on paper and stapled (no electronic form, unless you are disabled in a way making this necessary).
- You should put only your student identity number on your work, but not your name.
- Solutions can be by a single author, or by a pair in which case both authors will receive the same mark. In the latter case, do not forget to include both ID numbers.
- If you think you've found a typo, please consult mathstuff to see if has already been corrected before e-mailing me at S.V.Nazarenko@warwick.ac.uk.

Question: Kapitsa pendulum. [15points.]

Kapitsa pendulum is a rigid-rod pendulum the suspension point of which is vibrated (Kapitsa, P.: Dynamic stability of the pendulum when the point of suspension oscillating. Journ. of Exper. and Theor. Physics, 21, (1951), 588 598). It turns out that vibration can stabilize the upper equilibrium point (which is unstable in absence of vibration). You can see a video of the Kapitsa pendulum experiment at http://video.sibnet.ru/alb6481/video32489/.

Here, we will study this effect using a simplified linear non-autonomous equation

$$\ddot{x} = x(1 + A\, sin(wt)), \tag{1}$$

where constants $A \gg 1$ and $w \gg 1$ model the amplitude and the cyclic frequency of the suspension point vibrations. Point $x_0 = 0$ represents the upper equilibrium of the pendulum.

Theory for Kapitsa pendulum can be found, e.g., in book *L.D. Landau, E.M. Lifshitz. Vol. 1. Mechanics (3ed., Pergamon, 1976).* Briefly, it predicts that the motion consists of two parts,

$$x(t) = X(t) + \tilde{x}(t), \tag{2}$$

with fast oscillations

$$\tilde{x}(t) = -\frac{A\, X(t)}{w^2} sin(wt), \tag{3}$$

and a slowly varying motion $X(t)$ which satisfies the following equation,

$$\ddot{X} = (1 - \frac{A^2}{2w^2})X. \tag{4}$$

1. Recall the last week's results about the analytical solutions of the linearized equation describing small deviations about the unstable and the stable equilibrium points. These solutions will come handy for understanding the behavior of the equation (4) in the unstable and the stable cases respectively.

2. Find the analytical solution of the equation (4) and derive the criterion of stability of the point $X = 0$ in terms of the parameters A and w. For the stable case, find the period of oscillations of $X(t)$ in terms of A and w, and the shape of the trajectory in the phase plane. For the unstable case, find the growth rate of deviation from the equilibrium point and find the trajectory in the $(X, \dot{X})$ phase plane.

3. Rewrite equation (1) as a system of two first order ODE's so that the Matlab programme ode45 could be used. Create a function file for the right-hand-side of this system to be used by ode45 to solve it.

4. Fix the values $A = 100$, $x(0) = 0.1$ and $\dot{x}(0) = 0$, and try several different values for the vibration frequency w (e.g. 20, 40, 60, 80 and 100) and use ode45 to to find numerical solutions for these cases. Plot the respective trajectories in the plane $(x, \dot{x})$. Present 2 or 3 typical plots (not too many please!) and analyze them:

   - Do these trajectories exhibit fast and slow components as implied by equation (2)?
   - Does the fast component oscillate and the amplitude of its oscillations depend on X as predicted by (3)?
   - For what value of w the trajectories indicate stability and for which ones they show instability, and does this agree with the stability criterion you obtained above (in part 2)? Do extra runs with more values of w between the stable and the unstable values, and thereby find the critical value w_{crit} separating the stable and the unstable behavior. Does this value agree with your stability criterion? (Note that it does not have to agree *exactly* because the Kapitsa theory is approximate).
   - Take one of your printouts of the $(x(t), \dot{x}(t))$ trajectory and use a pencil to draw an approximate curve corresponding to the oscillation-averaged slow component of motion (roughly, aim for the middle points of the oscillating curve). Does this curve agree with the analytical solutions of equation (4) that you obtained above (in part 2) for the unstable and the stable cases?

5. Consider the stable trajectories, with emphasis on the values of w close to the critical value w_{crit}. Find approximate periods for $X(t)$. For this, you will need to run solutions for different time intervals and find for the interval which leads to the first approximate return of the trajectory to the initial phase-space point (not necessarily an exact return, because recurrence in X does not imply exact recurrence in x). Do these periods agree with the analytical formula you obtained above (in part 2)? What happens to the period when $w \to w_{crit}$?