MA124: Maths by Computer - Week 6

Regulations

- This assignment gives 15% of your final mark.
- The assignment is due on Monday 21 February 2011 to be handed over to the drop-in box by the Mathematics General Office by 3pm the latest.
- Even if you cannot come yourself, you have to arrange for your work to be handed in timely (e.g. ask a friend to bring it). No late work will be accepted.
- Your work should be on paper and stapled (no electronic form, unless you are disabled in a way making this necessary).
- You should put only your student identity number on your work, but not your name.
- Solutions can be by a single author, or by a pair in which case both authors will receive the same mark. In the latter case, do not forget to include both ID numbers.
- If you think you've found a typo, please consult mathstuff to see if has already been corrected before e-mailing me at S.V.Nazarenko@warwick.ac.uk.

Question: Rigid-rod pendulum. [15points.]

Let us consider a pendulum suspended on a rigid rod. It has two equilibrium points: a stable one at the lowest position and an unstable one at the highest position. It is described by the following ODE,

$$\ddot{\theta} = -\sin\theta, \tag{1}$$

where θ is the angle between the rod and its stable equilibrium position.

1. Rewrite equation (1) as a system of two first order ODE's so that the Matlab programme ode45 could be used. Create a function file for the right-hand-side of this system to be used by ode45 to solve it.
2. Let us now find two solutions, both with zero initial velocity, $\dot{\theta}(0) = 0$, and with initial positions near the stable equilibrium in one case and the unstable equilibrium in the other case; e.g. $\theta(0) = 0.1$ and $\theta(0) = \pi - 0.1$ respectively. Plot the trajectories on the phase plane $(\theta, \dot{\theta})$ corresponding to these two solutions. Which of the two trajectories remain close to the initial position, and which one deviates during the initial stages of motion? Comment on this in terms of the stability and instability.
3. To study trajectories near equilibrium points, one can *linearize* the equation (1), i.e. Taylor expand its right-hand side in small $\tilde{\theta} = \theta - \theta_0$ (with $\theta_0 = 0$ or π) and leaving only the first non-vanishing term in this expansion.
   - Linearizing the equation (1) with respect to θ we have $\ddot{\theta} = -\theta$. Find analytical solutions corresponding to linear oscillations about the stable equilibrium point. What is the period of these oscillations? What is the shape of the phase-space trajectory $(\theta(t), \dot{\theta}(t))$?

- Linearizing the equation (1) with respect to $\tilde{\theta} = \pi - \theta$, we get $\ddot{\tilde{\theta}} = \tilde{\theta}$. Find analytical solutions corresponding to the exponential growth of $\tilde{\theta}$ (i.e. to the exponential deviation of the trajectory from the unstable equilibrium point $\theta = \pi$). What shape of the phase-space trajectory $(\theta(t), \dot{\theta}(t))$ do we have in this case?
- Now draw several trajectories for $\dot{\theta}(0) = 0$ and for several intermediate starting positions, $\theta(0) = 0.5; 1; 1.5; 2, 2.5$. Comment on the periodicity, and measure the period of the oscillations in each of these cases. [For this, do simulations with several different time intervals, and find the length of the time interval corresponding to exactly one round in the phase plane.] Do the periods for small values of $\theta(0)$ agree with your analytical solutions obtained via the linearization? How does the period change for oscillations of larger amplitudes? What happens to the period when $\theta(0)$ gets close to π?

4. Now start somewhere close to the upper (unstable) equlibrium, and give your pendulum a "push" (initial velocity), e.g. start with $\theta(0) = \pi - 0.1$ and $\dot{\theta}(0) = 0.5$. Plot the corresponding phase space trajectory and describe what you see. What kind of pendulum motion does this trajectory correspond to?

5. The most efficient way to understand the overall dynamics described by ODE's is to draw a phase portrait for this system, as we discussed in class. Modify the phase portrait script we used in class (given below), and plot the phase portrait for the pendulum system. Follow the vectorfield directions (with a pencil) and draw several trajectories of the types discussed above, i.e. small and large initial angles, closed loops and open trajectories. What types of the pendulum motion do they correspond to (particularly the open trajectories)?

   Script we used in class (type help for the meshgrid and the quiver commands to see how they work):

```
[y1,y2]=meshgrid(-2*pi:0.6:2*pi,-4:0.6:4);
dy1dt=y2;
dy2dt=-y1;
quiver(y1,y2,dy1dt,dy2dt)
```