MA124: Maths by Computer - Week 3

Also available for download at http://www.warwick.ac.uk/~masbu

Regulations

- This assignment gives 15% of your final mark.
- The assignment is due on Monday 7 of February 2011, to be handed over to the drop-off box by the door of the Mathematics General Office by 3pm the latest.
- Even if you cannot come yourself, you have to arrange for your work to be handed in timely (e.g. ask a friend to bring it). No late work will be accepted.
- Your work should be on paper and stapled (no electronic form, unless you are disabled in a way making this necessary).
- You should put only your student identity number on your work, but not your name.
- Solutions can be by a single author, or by a pair in which case both authors will receive the same mark. In the latter case, do not forget to include both ID numbers.
- If you think you've found a typo, please consult my web page http://www.warwick.ac.uk/~masbu to see if has already been corrected before e-mailing me at S.V.Nazarenko@warwick.ac.uk.

1. **Random Fibonacci series**

   (a) Write an m-file to find $f(1), f(2), \ldots, f(1000)$ the first 1000 Fibonacci numbers defined by $f(1) = f(2) = 1$ and $f(k) = f(k-1) + f(k-2)$. Make your m-file plot the sequence. Enclose both the m-file and the plot with your work.

   (b) Now make it plot the sequence $\log(f(1)), \log(f(2)), \ldots, \log(f(1000))$, which should be more interesting. Solve the recurrence relation, and find the exact formula for Fibonacci numbers. Use this to check your output value of $log(f(1000))$. Enclose the plot.

   (c) Modify your program so that $f(1) = 1, f(2) = 2$ and $f(k) = f(k-1) \pm f(k-2)$, where at each $k \geq 3$ you choose $+$ or $-$ with equal probability. Run several times and find that the sequence often oscillates between positive and negative numbers. Include a representative plot to demonstrate this (and enclose the m-file).

(d) What is the exponential growth rate of the sequence, that is the growth of $\log(|f(n)|)$ as $n \to \infty$? There are theorems that say, roughly, that as n gets large then with high probability the terms should grow like a constant times $\pm c^n$. Take logs of your data and get an approximate value of this constant c. On repeated runs, how large is the range of values you get for the constant c? Investigate also taking longer runs. A research paper (Embree and Trefethen, Proc. Roy. Soc. London, 1999) has calculated similar growth constants c to high precision.

2. **Occupation time for a random walk**

A symmetric random walk will oscillate between positive values and negative values forever. However, this exercise shows that the proportion of time it spends taking non-negative values is usually near 1 or near 0.

(a) Create a vector of 10 (pseudo) random numbers, each either $+1$ or -1 with equal probability.

(b) A random walk takes independent steps up or down with equal probability. The position after n steps will be the nth partial sum of the random vector you created in the previous step. Plot a graph of the successive positions of the walk over the first 10 steps. (Hint: you might use the command cumsum). Enclose both the m-file and the plot with your work.

(c) Modify the m-file so that it counts the number of steps that the walk is in the non-negative integers, called the occupation time in $\{0, 1, 2, \ldots\}$. So if the successive positions are $-1, -2, -1, 0, 1, 0, -1, 0, 1, 2$ then it takes values in $\{0, 1, 2, \ldots\}$ in 6 of the 10 steps. (Hint: you might use the function sign(x), which returns the sign of the components of a vector x). Enclose the m-file with your work.

(d) Modify your m-file so that it takes walks with a large number N of steps, and make the program repeat the walk a large number M times, each time recording the occupation time in $\{0, 1, 2, \ldots\}$. Plot a histogram of the occupation times. Vary the values of M until you feel your histograms are not varying much, and give an accurate description of the occupation time. Enclose both the m-file and a representative plot with your work.

(e) *Lemma.* Let $X_0, X_1, \ldots, X_{2N}$ be the position of the walk after $0, 1, \ldots, 2N$ steps. Then

$$P[X_n \geq 0, \text{ for } n = 0, 1, \ldots, 2N] = P[X_{2N} = 0] = \binom{2N}{N} \frac{1}{2^{2N}}.$$

This lemma will be proved in ST111. Does your data support this lemma?