MA124: Maths by Computer - Week 10

Regulations

- This assignment gives 10% of your final mark.
- The assignment is due on Monday 21 March 2011, to be handed over to the special box in the lobby next to the Mathematics General Office by 3pm the latest. Obviously, if you away for holidays earlier, you can drop your assignment earlier (this assignment is lighter than usual, and you will have more than enough time to finish it by the end of week 10).
- Even if you cannot come yourself, you have to arrange for your work to be handed in timely (e.g. ask a friend to bring it). No late work will be accepted.
- Your work should be on paper and stapled (no electronic form, unless you are disabled in a way making this necessary).
- You should put only your student identity number on your work, but not your name.
- Solutions can be by a single author, or by a pair in which case both authors will receive the same mark. In the latter case, do not forget to include both ID numbers.
- If you think you've found a typo, please consult mathstuff to see if has already been corrected before e-mailing me at S.V.Nazarenko@warwick.ac.uk.

A bit of history (from Wiki). Carl Friedrich Gauss is credited with developing the fundamentals of the basis for least-squares analysis in 1795 at the age of eighteen. Legendre was the first to publish the method, however.

An early demonstration of the strength of Gauss's method came when it was used to predict the future location of the newly discovered asteroid Ceres. On January 1, 1801, the Italian astronomer Giuseppe Piazzi discovered Ceres and was able to track its path for 40 days before it was lost in the glare of the sun. Based on this data, it was desired to determine the location of Ceres after it emerged from behind the sun without solving the complicated Kepler's nonlinear equations of planetary motion. The only predictions that successfully allowed Hungarian astronomer Franz Xaver von Zach to relocate Ceres were those performed by the 24-year-old Gauss using least-squares analysis.

Gauss did not publish the method until 1809, when it appeared in volume two of his work on celestial mechanics, Theoria Motus Corporum Coelestium in sectionibus conicis solem ambientium. In 1829, Gauss was able to state that the least-squares approach to regression analysis is optimal in the sense that in a linear model where the errors have a mean of zero, are uncorrelated, and have equal variances, the best linear unbiased estimators of the coefficients is the least-squares estimators. This result is known as the GaussMarkov theorem.

The idea of least-squares analysis was also independently formulated by the Frenchman Adrien-Marie Legendre in 1805 and the American Robert Adrain in 1808.

10A **Least squares method for approximating a planetoid orbit.**

(a) The level set $\{(x, y) : f(x, y) = 0\}$ for the function

$$f(x, y) = ax^2 + bxy + cy^2 + dx + ey + 1$$

is a conic section. Look up the help for the command contour and use this to write an m-file that plots the conic section for some given values of a, b, c, d, e. Include a plot of a hyperbola, and the m-file that created it with the chosen values of a, b, c, d, e.

(b) A planetoid follows an elliptical orbit and ten observations of the position are given by the x and y co-ordinate vectors (in the plane of the orbit)

$$x = (1.02, 0.95, 0.87, 0.77, 0.67, 0.56, 0.44, 0.30, 0.16, 0.01),$$
$$y = (0.39, 0.32, 0.27, 0.22, 0.18, 0.15, 0.13, 0.12, 0.13, 0.15).$$

The aim is to find values of a, b, c, d, e so that all these observations lie on, or as close as possible to, the ellipse given by the zero level set in part (a). Solving for the values a, b, c, d, e reduces to a set of 10 equations in 5 unknowns. Writing this in the form $Av = b$, where v is the column vector $(a, b, c, d, e)^T$, input the matrix A into Matlab.

(c) Since solving the equations $Av = b$ exactly is impossible, one alternative is to try to find v that minimizes $|Av - b|^2 = \sum_i |(Av)_i - b_i|^2$, called the least squares minimizer. The following lemma is useful:

Lemma. Suppose $b \in \mathbf{R}^n$ and A is a n by m matrix, where $m < n$. Recall that A^T, the transpose of A, is the m by n matrix with entries given by $A^T_{ij} = A_{ji}$. Suppose also that $A^T A$ is an invertible matrix. Then the unique minimizer of the function $x \to |Ax - b|^2$ is given by $x_0 = (A^T A)^{-1} A^T b$.

Use this lemma to find the least squares minimizer for the values of a, b, c, d, e. Use them to plot the approximate orbit of the planetoid. Superimpose on the plot the 10 observed positions. Include the values of a, b, c, d, e and the plot in your write-up.