The Implicit Function Theorem for Lipschitz Maps

A map \( f : X \to Y \) is Lipschitz if there is a constant \( C \) such that for all \( x_1, x_2 \in X \),
\[ d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2). \]
Every differentiable map from an open set in \( \mathbb{R}^n \) to \( \mathbb{R}^p \) is locally Lipschitz, but the converse is not true. For example, the function \( f(x) = |x| \) is Lipschitz but not differentiable at 0. Nevertheless, it does exhibit a remarkable feature of Lipschitz maps on \( \mathbb{R}^n \): they are almost everywhere differentiable. That is, the set of points where they are not differentiable is of measure 0. Despite their not being differentiable, there is an inverse function theorem for Lipschitz maps, due to F.H.Clarke. It makes use of the generalised derivative of a Lipschitz map, which is defined as follows: since the set of points where the map \( f \) is differentiable has full measure, even if \( f \) is not differentiable at \( x_0 \) there are many sequences of points \( (x_n) \) tending to \( x_0 \), at each of which \( f \) is differentiable, and for which the sequence \( (d_{x_n} f) \) of linear maps does tend to a limit. The set of all limits obtained in this way forms a compact subset of matrix space. The generalised derivative of \( f \) at \( x_0 \) is the convex hull of this set. It is denoted \( \delta_{x_0} f \).

**Theorem 0.1.** Let \( f : U \subset \mathbb{R}^n \to \mathbb{R}^n \) be Lipschitz, and suppose that every matrix \( A \in \delta_{x_0} f \) is invertible. Then there are neighbourhoods \( U \) of \( x_0 \) and \( V \) of \( f(x_0) \) in \( \mathbb{R}^n \) and a Lipschitz map \( g : V \to U \) such that \( f \circ g = 1_V \) and \( g \circ f = 1_U \).

In the theory of \( C^1 \) maps, the Implicit Function Theorem can easily be derived from the Inverse Function Theorem, and it is easy to imagine that an implicit function theorem for Lipschitz functions might follow from the Inverse Function Theorem in the same way. However, there turns out to be a difficulty. The most natural hypothesis for a Lipschitz implicit function theorem would seem to be that every matrix \( A \in \delta_{x_0} f \) should be an epimorphism. In the \( C^1 \) case, given a map \( f : \mathbb{R}^n \to \mathbb{R}^p \) with epimorphic derivative at \( x_0 \), in order to invoke the inverse function theorem one looks for a linear map \( L : \mathbb{R}^n \to \mathbb{R}^{n-p} \) such that the map \( (d_{x_0} f, L) : \mathbb{R}^n \to \mathbb{R}^p \times \mathbb{R}^{n-p} \simeq \mathbb{R}^n \) is an isomorphism. Of course this is not hard: as \( d_{x_0} f \) has rank \( p \), we can select \( p \) linearly independent columns of the matrix \( d_{x_0} f \). These columns correspond to \( p \) basis vectors. We can take \( L \) to be any linear map mapping the subspace of \( \mathbb{R}^n \) spanned by the remaining \( n-p \) basis vectors isomorphically onto \( \mathbb{R}^{n-p} \). For example, \( L \) could be projection onto this subspace. If we try to apply the same argument in the case of Lipschitz maps, we would have to find a linear map \( L \) with the property

\[ (*) \text{ for all } A \in \delta_{x_0} f, \text{ the map } (A, L) \text{ should be an isomorphism.} \]
This is equivalent to

(**) there are integers $1 \leq i_1 \leq \ldots \leq i_p \leq n$ such that for every $A \in \delta_{x_0} f$, the $i_1$'st, \ldots, $i_p$'th columns of $A$ are linearly independent.

It is not clear whether this is possible. In fact it is not hard how to see that it can be done if $p = 1$, but for $p > 1$ the answer seems not to be known. It seems that F.H. Clarke himself is aware of the difficulty: in his book on convexity, in which he proves an implicit function theorem for Lipschitz functions, he makes (**) part of the hypothesis.

The aim of the project is to answer this question. That is, if every matrix $A \in \delta_{x_0} f$ has rank $p$, does (**) hold? It is thus a project in convex and linear geometry. The project should of course begin with some background on Lipschitz functions. It could then focus on the simplest case not already dealt with, namely Lipschitz maps $f : U \to \mathbb{R}^2$, where $U$ is open in $\mathbb{R}^3$. It might consider examples of non-differentiable Lipschitz maps in these dimensions and compute their generalised derivative at some non-$C^1$ points. Or it might by-pass the origin of the problem in Lipschitz maps and consider directly examples of compact convex sets of $2 \times 3$ matrices of maximal rank, and try to determine whether they have property (**).