Singularities of Mappings

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Chapter 1

Introduction

The crucial notion is of course the *derivative* of a smooth or analytic mapping: if $f : X \to Y$ is a map of manifolds and $x \in X$ then $d_x f : T_x X \to T_{f(x)} Y$ is the derivative, defined by

$$d_x f(\hat{x}) = \lim_{h \to 0} \frac{f(x + h\hat{x}) - f(x)}{h}$$

if X and Y are open sets in linear spaces. If X and Y are contained, but not open, in linear spaces, $d_x f$ can be defined by restricting to $T_x X$ the derivative of a suitable extension of f to an open set in the linear ambient space; otherwise one uses charts. It is also worth recalling that every tangent vector $\hat{x} \in T_x X$ is the tangent vector $\gamma'(0)$ to a parameterised curve $\gamma : (\mathbb{R}, 0) \to (X, x)$ (or $\gamma : (\mathbb{C}, 0) \to (X, x)$ in the complex analytic category), and that $d_x f$ satisfies

$$d_x f(\gamma'(0)) = (f \circ \gamma)'(0). \tag{1.0.1}$$

This may be taken as the definition. It is particularly useful in infinite dimensional cases, such as where X is a group of diffeomorphisms.

A point $x \in X$ is a *regular point* of f if $d_x f$ is surjective, and a *critical point* if it is not. The image of a critical point is a *critical value* of f; any point in Y which is not a critical value is a *regular value* (even if it has no preimages). The set of all critical values is often called the *discriminant* of the map f. If x_0 is a regular point then f is said to be a *submersion* at x_0. If x_0 is a regular point, then a simple argument based on the inverse function theorem establishes

Theorem 1.0.1. *(Normal form for submersions) Suppose that* $\dim X = n \geq k = \dim Y$ *and* x_0 *is a regular point of* $f : X \to Y$*. Then one can choose coordinates* $x_1, \ldots, x_n$ *on* X *around* x_0*, and* $y_1, \ldots, y_k$ *on* Y *around* $f(x_0)$*, such that* f *takes the form* $f(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$*.*

These notions are only of interest when $\dim X \geq \dim Y$; when $\dim X < \dim Y$, all points of X are critical points, and the set of critical values of f is the whole image of f. In this case one is interested in whether or not $d_x f$ is *injective.* If it is, f is an *immersion* at x_0, and one has

Theorem 1.0.2. *(Normal form for immersions) Suppose that* $\dim X = n \leq k = \dim Y$ *and that* $f : X \to Y$ *is an immersion at* x_0*. Then one can choose coordinates around* x_0 *and* $f(x_0)$ *such that* f *takes the form* $f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0)$*.*

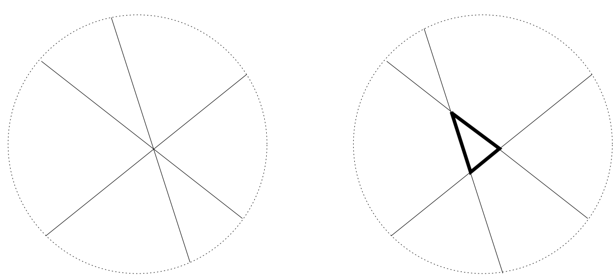
Exercise 1.0.3. *Find proofs of 1.0.1 and 1.0.2. Both follow from the inverse function theorem, by incorporating* f *into a suitable auxiliary mapping whose derivative is invertible. Clear proofs scan be found in the first chapter of [GG73].*

Singularity theory begins where these two theorems end: it is concerned with what happens at points where f is neither a submersions nor an immersion. It concentrates on the local behaviour of mappings, and for this reason uses the notion of *germ of mapping*, which we study briefly in Subsection 1.1. Geometrical singularity theory for the two cases $\dim X \geq \dim Y$ and $\dim X < \dim Y$ is rather different. In the first case, classical singularity theory is interested in *preimages* $f^{-1}(y_0)$, and there is also a theory of the discriminant, initiated by Teissier in [Tei76]. In the second case, to which much less attention has been devoted, one studies the *images* of maps. In fact very little is known about the geometry of maps in case $\dim X < \dim Y - 1$, and the theory for the case $\dim X = \dim Y - 1$ has an embarassing gap, in the form of an unproved (and unrefuted) conjecture which I made twenty five years ago.

This book concentrates on two key invariants for singularities of mappings, and the relation between them. The first comes from deformation theory: it is the *deformation-theoretic codimension*, and is the subject of Section 3. Until then, one can use the following relatively non-technical working definition: it is the minimal number of parameters for a family of mappings in which a singularity equivalent to the one in question occurs 'stably' or 'irremovably'. The second, studied in Section 3, comes from topology: it is the "rank of the vanishing homology (of a nearby stable object)". This vague phrase will be made more precise; for now, we make do with two examples. The first is the non-degenerate critical point of a polynomial or analytic function, equivalent, by the Morse Lemma, to the germ defined by

$$f(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2.$$

Here $f^{-1}(0)$ is contractible, but for $t \neq 0$, $f^{-1}(t)$ has the homotopy-type of an n-sphere [1]. When t returns to 0, the rank of the homology of $f^{-1}(t)$ diminishes by 1; this is the 'rank of the vanishing homology' for this example. The second is the three pieces of plane curve which meet at a point in the Reidemeister move of type III. This configuration is evidently unstable: one can move any one of the three to form a triangle. Since now all intersections are transverse, this configuration is stable. It is the 'nearby stable object' for this example, and its vanishing homology, generated by the 1-cycle highlighted in the drawing on the right, once again has rank 1.



Unstable *Stable*

The deformation-theoretic codimension in the second example is also equal to 1; therein lies its importance in knot theory. Given two plane projections of the same knot, one can be deformed to the other in such a way that during the deformation, only three types of qualitative change occur. These are the three 'Reidemeister moves', and our example shows the third of these. They cannot be avoided in a 1-parameter family of projections; other more complicated singularities can be.

[1] This is true for any $t \neq 0$ when $k = \mathbb{C}$; when $k = \mathbb{R}$ it holds for $t > 0$. Indeed in this case the inclusion of real in complex is a homotopy equivalence. It is an example of a "good real picture".

Notation and Terminology 1.0.4. *Let X and Y be manifolds, and $f : X \to Y$ a differentiable map.*

1. *A singular point, or singularity of f is a point where f is not a submersion, in case* $\dim X \geq \dim Y$*, and not an immersion, in case* $\dim X \leq \dim Y$*.*

2. *A map $X \to Y$ has corank r at x_0 if the rank of $d_{x_0}f$ is r less than the maximum possible,* $\min\{\dim X, \dim Y\}$*. Thus if* $\dim X \leq \dim Y$ *then f has corank r at x_0 if r is the dimension of the kernel of $d_{x_0}f$, and if* $\dim X \geq \dim Y$ *then the corank is the dimension of the cokernel of $d_{x_0}f$.*

3. *If $Z \subset X$ then a singular point of Z is a point at which Z is not a submanifold of X.*

The following proposition plays an important technical role.

Proposition 1.0.5. *Suppose that $f : X^n \to Y^p$ is smooth, and that at the point $x_0 \in X$, the derivative $d_{x_0}f$ has rank r. Then there are coordinates around x_0 in X and $f(x_0)$ in Y with respect to which f takes the form*

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_r, f_{r+1}(x), \ldots, f_p(x)), \tag{1.0.2}$$

where all first order partials of $f_{r+1}, \ldots, f_p$ vanish at x_0.

Proof. First suppose that X and Y are open sets in $\mathbb{K}^n$ and $\mathbb{K}^p$ respectively. Let L be the (r-dimensional) image of $d_{x_0}f$ in $\mathbb{K}^p$, let $M \subset \mathbb{K}^p$ be a complement to L, and let $\pi_L : \mathbb{K}^p = L \oplus M \to L$ and $\pi_M : \mathbb{K}^p \to M$ be the projections. Then $\pi_L \circ f$ is a submersion, and so by 1.0.2 there are coordinates on X and L around x_0 and $\pi(f(x_0))$ with respect to which $\pi \circ f$ is the standard submersion. Pick any coordinate system on M; then the coordinates on L and M together give a coordinate system on $\mathbb{K}^p$, and f takes the desired form with respect to this coordinate system and the chosen coordinates on X around x_0.

The general case follows by taking charts on X and Y around x_0 and $f(x_0)$. □

Coordinates with respect to which f takes the form 1.0.2 are known as *linearly adapted coordinates.*

1.1 Germs, cones and local rings

Definition 1.1.1. Let $f, g : X \to Y$ be maps of topological spaces, and let $S \subset X$.

1. We say that f and g *have the same germ at* S (or *along* S if S is not a finite point set), if there is a neighbourhood U of S in X such that f and g coincide on U. This is evidently an equivalence relation, and a *germ of mapping at* S is an equivalence class under this relation.

2. Two subsets X_1 and X_2 of X *have the same germ at* (or *along*) S if there is a neighbourhood U of S in X such that $X_1 \cap U = X_2 \cap U$. A *germ at S of subset of* X is an equivalence class of subset under this relation.

We denote a germ at S of mapping $X \to Y$ by $f : (X,S) \to Y$, or $f : (X,S) \to (Y,T)$ if $f(S) \subset T \subset Y$. To determine a germ of mapping at S, it is enough to specify the behaviour of f on some neighbourhood of S in X. Usually X is $\mathbb{C}^n$ or an analytic variety embedded in $\mathbb{C}^n$, S is a single point or a finite set, and we specify f by means of power series which converge in some neighbouhood of the points of S. Not every power series can be extended to a globally defined map $X \to Y$, so really our subject is not 'germs at S of maps $X \to Y$', but 'germs at S of maps to Y from some neighbourhood of S'. In practice this will not cause any difficulty.

Germs of maps to $\mathbb{C}$ can be added and multiplied, and the set of germs at x_0 of analytic functions on X is a $\mathbb{C}$-algebra. It is denoted $\mathcal{O}_{X,x_0}$.

The notion of germ is particularly natural in the complex-analytic category, because of uniqueness of analytic continuation: if U_1 and U_2 are connected open sets in $\mathbb{C}^n$ and $f_i : U_i \to \mathbb{C}^p$ are complex analytic maps, then if f_1 and f_2 coincide on some open $V \subset U_1 \cap U_2$, they coincide on all of $U_1 \cap U_2$.

Exercise 1.1.2. *Show that the same is not true of real C^∞ maps.*

If X and Y are spaces, and we select some class of germs of maps $X \to Y$ – e.g. germs of continuous maps, or germs of complex analytic maps in case X and Y are complex analytic varieties – then we can put together all of the germs into a global object, a *sheaf*. This notion is crucial in algebraic and analytic geometry, but I do not want to make it a prerequisite for this course. Instead, we will develop the notion as it is needed. We begin with a working definition sufficient to make some of the necessary theorems at least vaguely comprehensible.

The definition of sheaf requires an algebraic structure, so we take, as our target space Y, the field $\mathbb{C}$. It is natural to associate to each open $U \subset X$ the set

$$\mathcal{O}_X(U) := \{f : U \to \mathbb{C} : f \text{ is complex analytic}\}$$

and make it into a $\mathbb{C}$-algebra by defining the operations pointwise

$$(f+g)(x) = f(x) + g(x), \quad (fg)(x) = f(x)g(x), \quad (\lambda f)(x) = \lambda f(x) \text{ for } \lambda \in \mathbb{C}.$$

If $U \subset V$ there is a restriction map $\rho_{U,V} : \mathcal{O}(V) \to \mathcal{O}(U)$ which is a homomorphism of $\mathbb{C}$-algebras, and if $U \subset V \subset W$ then evidently

$$\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}. \tag{1.1.1}$$

Let $\mathcal{U}_x$ be the collection of all neighbourhoods of a point x. The equivalence relation by which we arrived at the notion of germ of function or mapping becomes a relation on the disjoint union $\coprod_{U \in \mathcal{U}_x} \mathcal{O}(U)$:

$$f \in \mathcal{O}(U) \text{ and } g \in \mathcal{O}(V) \text{ are equivalent if there exists } W \in \mathcal{U}_x \text{ such that } \rho_{W,U}(f) = \rho_{W,V}(g). \tag{1.1.2}$$

The set of equivalence classes, $\mathcal{O}_{X,x_0}$, is in a natural way a $\mathbb{C}$-algebra: if $f, g \in \mathcal{O}_{X,x_0}$ then they can be represented by some $f_1 \in \mathcal{O}(U)$ and $g_1 \in \mathcal{O}(V)$, for some open neighbourhoods U, V of x_0, and then the restrictions $\rho_{U\cap V,U}(f)$ and $\rho_{U\cap V,V}(g)$ in $U \cap V$ can be added or multiplied in the usual way. The sum and product of these restrictions then determine germs at x_0, which, as one can easily check, are independent of the choices of representative f_1, g_1.

Exercise 1.1.3. *Show this.*

The map $\rho_{x_0,U} : \mathcal{O}(U) \to \mathcal{O}_{X,x_0}$ defined by sending $f \in \mathcal{O}(U)$ to its germ at x_0 is a $\mathbb{C}$-algebra homomorphism. Evidently

$$\rho_{x_0,V} = \rho_{x_0,U} \circ \rho_{U,V}.$$

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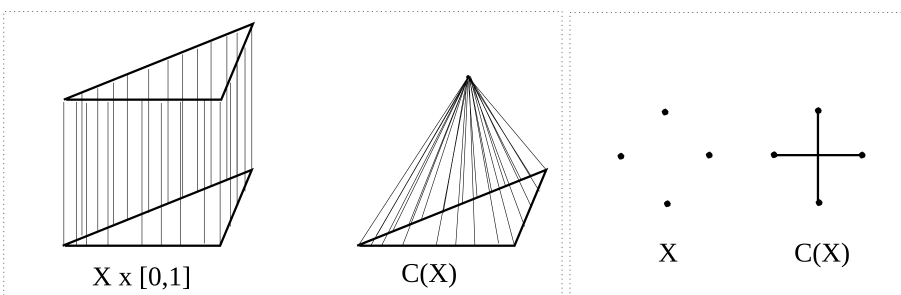
Exercise 1.1.4. *Is $\rho_{x_0,U}$ surjective? Injective?*

The procedure we have outlined can be applied equally well to functions of other types: continuous, or C^∞, or real analytic, etc. It also makes sense in a wider context:

Exercise 1.1.5. Let $f : X \to Y$ be a map of topological spaces. For $U \subset Y$ define $\mathcal{H}^p(U) := H^p(f^{-1}(U))$ (the p-th topological cohomology of $f^{-1}(U)$).

1. Given $U \subset V \subset X$, show how to define $\rho_{U,V} : \mathcal{H}^p(V) \to \mathcal{H}^p(U)$ so that (1.1.1) holds.
2. Show that if f is a locally trivial fibre bundle then for $U \in \mathcal{U}_x$ sufficiently small and contractible, $\mathcal{H}^p(U) \simeq \mathcal{H}^p(\{x\})$.

A further justification for the use of the notion of germ in singularity theory comes from the fact that closed analytic spaces are 'locally conical'. This is particularly important in the definition of the vanishing homology, so we go into some detail here. If X is any topological space, the *cone on* X, which we denote by $C(X)$, is obtained by forming the Cartesian product $X \times [0,1]$ and then identifying all of the points of $X \times \{1\}$ with one another. One writes $C(X) = (X \times [0,1])/(X \times \{1\})$, where the notation B/A, for A a subset of B, means the quotient of B by the equivalence relation which identifies all the points of A to one another. If X is embedded in some $\mathbb{R}^n$ then the cone $C(X)$ can be described more concretely as follows: if v is an (arbitrary) point in $\mathbb{R}^n \times \{1\}$ then $C(X)$ is homeomorphic to the union of all of the line segments in $\mathbb{R}^n \times [0,1]$ joining v to a point $(x,0)$, for $x \in X$.



Exercise 1.1.6. For any space X, $C(X)$ can be contracted to its vertex.

Because cones are contractible, their homology is equal to that of a point.

For $x_0 \in \mathbb{C}^n$, let $S_\varepsilon(x_0)$ be the sphere of radius ε centred at x_0, and let $B_\varepsilon(x_0)$ be the ball of radius ε centred at x_0.

Theorem 1.1.7. *Let $U \subset \mathbb{C}^n$ be open and let $X \subset U$ be the set of common zeros of k analytic functions $f_1, \ldots, f_k \in \mathcal{O}(U)$. If $x_0 \in X$, there exists $\varepsilon > 0$ such that $X \cap B_\varepsilon(x_0)$ is homeomorphic to the cone on its boundary $X \cap S_\varepsilon(x_0)$.*

Exercise 1.1.8. *Show that this is true in the trivial case that $X = \mathbb{C}^n$., and therefore if X is a smooth manifold at x_0.*

For $x_0 \in \mathbb{C}^n$, let $S_\varepsilon(x_0)$ be the sphere of radius ε centred at x_0, and let $B_\varepsilon(x_0)$ be the ball of radius ε centred at x_0. Write $X_\varepsilon := S_\varepsilon(x_0) \cap X$ and $X_{\leq\varepsilon} := X \cap B_\varepsilon(x_0)$. If X is a k-dimensional manifold except at x_0 (i.e. X has isolated singularity at x_0) then the theorem can be proved by

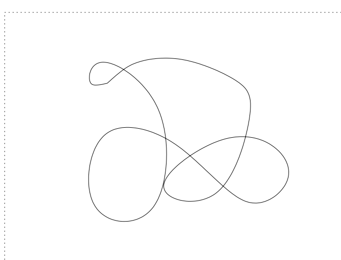
1. constructing a 'radial' vector field v, pointing in towards x_0, on a neighbourhood of x_0 in X, and adjusting the length of the vectors so that for each point $x \in X_\varepsilon$, the trajectory $\varphi_t(x)$ starting at x arrives at x_0 at time $t = 1$, and
2. defining a homeomorphism $H : X_\varepsilon \times [0, 1) \to X_{\leq\varepsilon} \smallsetminus \{x_0\}$ by
$$H(x, t) = \varphi_t(x),$$
3. which (automatically) extends to a homeomorphism $(X_\varepsilon \times [0, 1])/(X_\varepsilon \times \{1\}) \to X_{\leq\varepsilon}$.

The theorem holds also for locally closed real analytic subsets of $\mathbb{R}^n$ with isolated singularities, but not in general for the zero loci of C^∞ functions. A more involved argument, using Whitney regular stratifications, proves the theorem for the case where X is a (real or complex) analytic set with arbitrary singularity at x_0 – see [BV72].

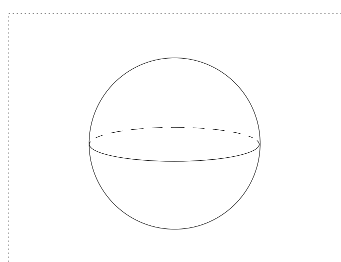
Exercises 1.1.9.

1. Give an example to show that the zero-loci of C^∞ functions need not be locally conical.
2. Suppose that X has isolated singularity at 0, and that there is a function $\rho : X \to \mathbb{R}_{\geq 0}$ such that
   (a) ρ has no critical point in $X_{\leq\varepsilon} \smallsetminus \{x_0\}$, and
   (b) $\rho^{-1}(0) = \{x_0\}$.

   Use the gradient vector of ρ to construct the vector field of the sketched proof of 1.1.7. [2]
3. Show that ρ_E satisfies condition 1. of the previous exercise iff for all ε' with $0 < \varepsilon' \leq \varepsilon$, $X \pitchfork S_{\varepsilon'}(x_0)$.
4. Divide up the objects pictured below into subsets which are homeomorphic to cones on their boundary.



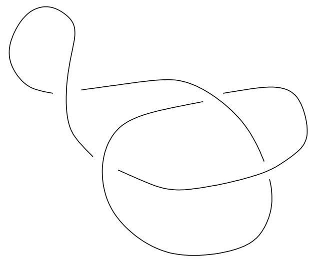
Planar projection of a knot



Sphere

[2] The hardest part of the proof of 1.1.7 comes in showing that such a function exists. In fact *any* real analytic function $\rho : X \to \mathbb{R}_{\geq 0}$ satisfying 1.1.92(2) will do; one uses the curve selection lemma (*cf* [Mil68]) to show that it also satisfies 1.1.92(1) for some $\varepsilon > 0$. In particular, one can use the Euclidean distance-squared function $\rho_E(x) := \|x - x_0\|^2$.

5. Take a thin copper wire, easy to bend but thick enough to form a self-supporting structure, and join the two ends after bending it to form a knot – which (making allowances for the fact that the wire is not infinitely thin) should be a C^∞ embedding of the circle in $\mathbb{R}^3$. You should obtain something looking like



The view shown here is "a generic projection' – the only singular points on the image are transverse crossings of two branches. If you look at the knot from different points of view, you see different projections

6. What is the appropriate version of locally conical structure for a *mapping*? It's worth trying to make up your own definition. For a useful discussion, see [Fuk82].

The local conical structure is crucially important in singularity theory. It gives a clear meaning to the term "local", and it makes possible the idea of local changes in a deformation. The simplest example along these lines is the Milnor fibre of an isolated hypersurface singularity. We have already seen that if f is an analytic function on some open set in $\mathbb{C}^n$ and has isolated singularity at x_0, then there exists $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subset U$ and $f^{-1}(y_0) \cap B_\varepsilon(x_0)$ is homeomorphic to the cone on $f^{-1}(y_0) \cap S_\varepsilon(x_0)$ – indeed, that $f^{-1}(y_0) \pitchfork S_{\varepsilon'}(x_0)$ for all ε' with $0 < \varepsilon' \leq \varepsilon$. An argument involving properness shows also that

Proposition 1.1.10. *In this case, there exists $\eta > 0$ (depending on the choice of ε) such that provided $|y_0 - y| \leq \eta$ then $f^{-1}(y) \pitchfork S_\varepsilon(x_0)$. For such ε and η, the map*

$$f| : B_\varepsilon(x_0) \cap f^{-1}(B^*_\eta(y_0)) \to B^*_\eta(y_0)$$

is a locally trivial fibre bundle.

The same principle gives us the notion of the "nearby stable object" (near to a singularity with isolated instability) in other situations. The details may be more complicated but the basic idea is the same.

1.2 Background in commutative algebra

If X is any analytic space and $p \in X$, then the evaluation map

$$\mathcal{O}_{X,p} \to \mathbb{C}, \quad f \mapsto f(p)$$

is surjective, so that its image is the field $\mathbb{C}$. Its kernel is therefore a maximal ideal in $\mathcal{O}_{X,p}$,which is denoted by $\mathfrak{m}_{X,p}$. Indeed it is the only maximal ideal, since if $f \in \mathcal{O}_{X,p}$ is not in $\mathfrak{m}_{X,p}$ then $1/f \in \mathcal{O}_{X,p}$, so that any ideal containing f also contains 1 and therefore all of $\mathcal{O}_{X,p}$. This shows that every proper ideal of $\mathcal{O}_{X,p}$ is contained in $\mathfrak{m}_{X,p}$. Rings with a single maximal ideal are called *local rings*. Their properties play a very large rôle in singularity theory.

We will frequently abbreviate $\mathfrak{m}_{X,p}$ simply to $\mathfrak{m}$. If $x_1, \ldots x_n$ are coordinates on X around p, and $p = (p_1, \ldots, p_n)$ in these coordinates, then every germ $f \in \mathcal{O}_{X,p}$ can be written as a convergent power series in $x_1 - p_1, \ldots, x_n - p_n$. It follows that

$$\mathfrak{m}_{X,p} = (x_1 - p_1, \ldots, x_n - p_n) \tag{1.2.1}$$

(the ideal generated by $x_1 - p_1, \ldots, x_n - p_n$).

In any ring R, the sum and product of ideals I and J are defined simply by

$$I + J = \{r + s : r \in I, s \in J\}, \qquad IJ = \{\sum_{i=0}^{m} r_i s_i : m \in \mathbb{N}, r_i \in I, s_i \in J \text{ for all } i\}.$$

Exercise 1.2.1. *1. Show that in any ring R, if I and J are ideals then so are $I + J$ and IJ.*

2. Let $X = \mathbb{C}^n$ and $p = 0$.

(a) Show that $\mathfrak{m}^2 = \{f \in \mathcal{O}_{\mathbb{C}^n,0} : f(0) = \partial f/\partial x_i(0) = 0 \ \text{ for } i = 1, \ldots, n\}$.

(b) Show more generally that

$$\mathfrak{m}^k = \{f \in \mathcal{O}_{\mathbb{C}^n,0} : \partial^\alpha f/\partial x^\alpha(0) = 0 \ \text{ for } 0 \leq |\alpha| \leq k - 1\}$$

where α is a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and by $\partial^0 f/\partial x^0$ we mean simply f.

In the C^∞ category, (1.2.1) and 6.2.41(a) and (b) also hold. However (1.2.1) is no longer completely obvious, and is known as *Hadamard's Lemma* – see Martinet's book [Mar82], Chapter 1.

We will make much use of the following statement.

Lemma 1.2.2. (Nakayama's Lemma) *Let M be a finitely generated module over a Noetherian local ring R with maximal ideal $\mathfrak{m}$. If $\mathfrak{m} M = M$ then $M = 0$.*

Corollary 1.2.3. *Let M and N be submodules of an R-module P, with M finitely generated, and suppose that*

$$M \subset N + \mathfrak{m} M. \tag{1.2.2}$$

Then $M \subset N$.

Proof Let $m_1, \ldots, m_r$ generate M over R. Since $M = \mathfrak{m} M$, for each i there exist $\alpha_{ij} \in \mathfrak{m}$ such that for $i = 1, \ldots, r$,

$$m_i = \alpha_{11} m_1 + \cdots + \alpha_{1r} m_r.$$

Rewriting these r equations as a single matrix equation we get

$$\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{n1} \\ \vdots & \cdots & \vdots \\ \alpha_{1n} & \cdots & \alpha_{nn} \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}$$

and therefore

$$(I_n - A)\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0,$$

where I_n is the $n \times n$ identity matrix and A is the matrix $[\alpha_{ij}]$. Multiplying both sides by the matrix of cofactors of $I_n - A$ we deduce that

$$\det[I_n - A] m_i = 0$$

for all i. But $\det[I_n - A]$ is a unit in the ring R, since it is equal to $1 + \alpha$ for some $\alpha \in \mathfrak{m}$. Hence $m_i = 0$ for $i = 1, \ldots, r$, and so $M = 0$. $\square$

Proof of Corollary Let $M_0 = (M + N)/N$. The hypothesis $M \subset N + \mathfrak{m}\, M$ implies that $M_0 = \mathfrak{m}\, M_0$. It follows by the Lemma that $M_0 = 0$, so that $M \subset N$. $\square$

1.3 Conservation of multiplicity

Suppose that U is open in $\mathbb{C}^n$, that$f : U \to \mathbb{C}^n$ is analytic, that $f(a) = b$, and that a is isolated in $f^{-1}(b)$ – that is, there exists $\varepsilon > 0$ such that $f^{-1}(b) \cap B_\varepsilon(a) = \{a\}$. Then the ideal $f^*\, \mathfrak{m}_{\mathbb{C}^n,b} := (f_1 - b_1, \ldots, f_n - b_n)$ must contain a power of the maximal ideal $\mathfrak{m}_{\mathbb{C}^n,a}$, since $\sqrt{f^*\, \mathfrak{m}_{\mathbb{C}^n,b}} = \mathfrak{m}_{\mathbb{C}^n,a}$. In fact

Proposition 1.3.1. *The following three statements are equivalent:*

1. *a is isolated in $f^{-1}(b)$;*
2. *$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,a} / f^*\, \mathfrak{m}_{\mathbb{C}^n,b} < \infty$;*
3. *$f^*\, \mathfrak{m}_{\mathbb{C}^n,b} \supset \mathfrak{m}^k$ for some $k < \infty$.*

Proof. That 3 implies 2 implies 1 is obvious. The converse follows from Ruckert's *Nullstellensatz*: that for any ideal $I \subset \mathcal{O}_{\mathbb{C}^n,a}$, the ideal of all functions vanishing on $V(I)$ is the radical $\sqrt{I} := \{f \in \mathcal{O}_{\mathbb{C}^n,a} : f^k \in I \text{ for some } k\}$. Since each coordinate function $x_i - a_i$ vanishes on $V(f^*\, \mathfrak{m}_{\mathbb{C}^n,b})$ it follows that $(x_i - a_i)^{k_i} \in f^*\, \mathfrak{m}_{\mathbb{C}^n,b}$ for some k_i. Then 3 holds with $k = n \max_i\{k_i\} - 1$. $\square$

Exercise 1.3.2. Show that if I is any ideal in $\mathcal{O}_{\mathbb{C}^n,x_0}$ such that $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,x_0} / I = k < \infty$ then $I \supset \mathfrak{m}^k$.

The dimension of $\mathcal{O}_{\mathbb{C}^n,a} / f^*\, \mathfrak{m}_{\mathbb{C}^n,b}$ is the *multiplicity* of f at a; we will denote it by $\mathrm{mult}_a(f)$.

Theorem 1.3.3. *Let U be open in $\mathbb{C}^n$, let $f : U \to \mathbb{C}^n$ be analytic, and let x_0 be isolated in $f^{-1}(y_0)$. Then there exists $\varepsilon > 0$ and $\eta > 0$ such that for all $y \in B_\eta(y_0)$,*

$$\sum_{x \in f^{-1}(y) \cap B_\varepsilon(x_0)} \mathit{mult}_x(f) = \mathit{mult}_{x_0} f. \tag{1.3.1}$$

The equality (1.3.1) is the basis for a number of statements about conservation of multiplicity. Here are some examples.

Conservation of Milnor number: If U is open in $\mathbb{C}^n$ and $f : U \to \mathbb{C}$ has isolated singularity at x_0 then the *Milnor number* of f at x_0 is defined to be $\text{mult}_{x_0}(j^1 f)$ where $j^1 f : (\mathbb{C}^n, x_0) \to (\mathbb{C}^n, 0)$ is the map with component functions $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$. That is,

$$\mu_{x_0}(f) = \dim \mathcal{O}_{\mathbb{C}^n, x_0} / J_f,$$

where J_f is the jacobian ideal $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$.

Corollary 1.3.4. *Let U be open in $\mathbb{C}^n$ and let $f : U \to \mathbb{C}$ have isolated singularity at x_0 with Milnor number $\mu < \infty$. Then in any deformation $F : U \times \mathbb{C}^d \to \mathbb{C}$ of f, there exists $\varepsilon > 0$ and $\eta > 0$ such that for $|u| < \eta$,*

$$\sum_{x \in B_\varepsilon(x_0)} \mu_x(f_u) = \mu_{x_0}(f).$$

Proof. Suppose first that the set

$$S_F^{\text{rel}} := \{(x, u) : \partial F/\partial x_1 = \cdots = \partial F/\partial x_n = 0 \text{ at } (x, u)\}$$

is smooth. Its dimension is necessarily equal to d, since $j^1 f$ must be a submersion outside x_0.

Let $\pi : S_F^{\text{rel}} \to U$ be projection. Since S_F^{rel} is locally isomorphic to $\mathbb{C}^{\dim U}$, we can apply 1.3.3 to the map π. If $(u, x) \in S_F^{\text{rel}}$ then

$$\mathcal{O}_{S_F^{\text{rel}}, (u,x)} / \pi^* \mathfrak{m}_{U,(v,u)} \simeq \mathcal{O}_{\mathbb{C}^n, x} / J_{f_u} \tag{1.3.2}$$

and thus

$$\text{mult}_{(u,x)}(\pi) = \mu_x f_u.$$

It follows from 1.3.3 that there exists $\varepsilon > 0$ and $\eta > 0$ such that for $|u| < \eta$,

$$\sum_{x \in B_\varepsilon(x_0)} \mu_x(f_u) = \mu_{x_0}(f).$$

If S_F^{rel} is not smooth, one can further deform F by a deformation $G : U \times \mathbb{C}^d \times \mathbb{C}^e$ such that S_G^{rel} is smooth of the requisite dimension – for example $G(x, u, v) = F(u, x) + \sum_i v_i x_i$. The first part of the argument applies to G, and the conclusion is obtained by restricting to $\{v = 0\}$. $\square$

Exercise 1.3.5.

1. Prove the equality (1.3.2).
2. Show that if S_F^{rel} is smooth then u is a regular value of π if and only if f_u has only non-degenerate critical points.

Conservation of intersection number of plane curves: If $C = \{f = 0\}$ and $D = \{g = 0\}$ are plane analytic curves meeting at x_0, their *intersection number* at x_0, $I_{x_0}(C, D)$, is defined to be the multiplicity at x_0 of the map (f, g).

Corollary 1.3.6. *Suppose the two curves C and D meet at x_0 with $I_{x_0}(C,D) < \infty$, and let C_t and D_t be parameterised families of plane curves with $C_0 = C, D_0 = D$. Then there exist $\varepsilon > 0$ and $\eta > 0$ such that for $|t| < \eta$,*

$$\sum_{x \in C_t \cap D_t \cap B_\varepsilon(x_0)} I_x(C_t, D_t) = I_{x_0}(C,D).$$

Proof. **Exercise** □

Conservation of cross-cap number: Suppose $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is given by $f(x,y) = (x, f_2(x,y), f_3(x,y))$. Its non-immersve locus S_f is determined by the equations $\partial f_2/\partial y = \partial f_3/\partial y = 0$. Suppose this set consists just of 0. We define the *cross-cap number of f, $C_0(f)$*, as $\mathrm{mult}_0(\partial f_2/\partial y, \partial f_3/\partial y)$.

Exercise 1.3.7. 1. Find $C_0(f)$ in each of the following cases:

(a) $f(x,y) = (x, y^2, xy)$ (this is the parameterisation of the Whitney umbrella, and is known as the *cross-cap*);

(b) $f(x,y) = (x, y^2, y^3 + x^{k+1}y)$

(c) $f(x,y) = (x, y^3, xy + y^{3k-1})$.

2. Suppose that $F(x,y,u) = (x, F_2(x,y,u), F_3(x,y,u), u)$ is an unfolding of f with $u \in \mathbb{C}^d$, and for fixed u let $f_u(x,y) = (x, F_2(x,y,u), F_3(x,y,u))$. Let S_F be the non-immersive locus of F, and consider the projection $\pi : S_F \to \mathbb{C}^d$. Show that

(a) It is possible to choose F so that S_F is smooth of codimension 2 in $\mathbb{C}^2 \times \mathbb{C}^d$.

(b) In this case $\mathrm{mult}_{(x,y,u)}(\pi) = C_{(x,y)}(f_u)$.

(c) There exist $\varepsilon > 0$ and $\eta > 0$ such that for $|u| < \eta$,

$$\sum_{(x,y) \in S_{f_u} \cap B_\varepsilon(0)} C_{(x,y)}(f_u) = C_0(f).$$

(d) One can show that if $C_0(f) = 1$ then f is $\mathcal{A}$-equivalent to the cross-cap, the germ of 1(a). Conclude that there exist deformations f_u of f with $C_0(f)$ cross-caps.

3. Suppose that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ has corank 1. Show that the ideal of $(n-1) \times (n-1)$ minor determinants of the matrix of df (the *ramification ideal* of f, $\mathcal{R}_f$) is generated by some two of these minors. Hint: do this first when $n = 2$, where it's easier to see what is going on. How many generators does $\mathcal{R}_f$ need when f has corank 2? corank 3?

We will see other applications of 1.3.3 to prove conservation of multiplicity of one kind or another. However 1.3.3 is not sufficient in all cases. In the examples we have just seen, we applied 1.3.3 to the projection π from the singular or relative critical space S_F of a deformation F, to the parameter space $\mathbb{C}^d$. This relied upon being able to choose F such that S_F is smooth. However there are situations where this is not possible. For example, the non-immersive locus of an unfolding $F(x,y,u) = (F_1(x,y,u), F_2(x,y,u), F_3(x,y,u), u)$ has equations

$$\det \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix} = \det \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} \end{vmatrix} = \det \begin{vmatrix} \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} \end{vmatrix} = 0 \qquad (1.3.3)$$

and if F is an unfolding of a map-germ of corank 2, then all three determinants lie in the square of the maximal ideal, so that their locus of common zeroes is unavoidably singular.

Nevertheless, it is still true that, just as shown in Exercise 1.3.72(d) above, for a finitely determined map-germ $(\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$, the number of cross-caps appearing in a stable perturbation is equal to

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0}/\mathcal{R}_f,$$

where $\mathcal{R}_f$ is the *ramification ideal* of f, generated by the three 2×2 minors of the matrix of df (as for F in (1.3.3) above). The proof of this makes use of the notion of *Cohen-Macaulay* rings and spaces, and involves some quite serious, though by now rather standardised, commutative algebra arguments. Instead of 1.3.3 we use

Theorem 1.3.8. *Let U be open in an n-dimensional Cohen Macaulay variety $X \subset \mathbb{C}^N$, let $f : U \to \mathbb{C}^n$ be analytic, and let x_0 be isolated in $f^{-1}(y_0)$. Then there exists $\varepsilon > 0$ and $\eta > 0$ such that for all $y \in B_\eta(y_0)$,*

$$\sum_{x\in f^{-1}(y)\cap B_\varepsilon(x_0)\cap X} mult_x(f) = mult_{x_0} f. \tag{1.3.4}$$

In the example described above, $V(\mathcal{R}_f)$ is Cohen Macaulay provided its codimension in the domain of the unfolding F is equal to 2. This is a consequence of Theorem 1.6.2 below.

The proofs of Theorems 1.3.3 and 1.3.8 run along the same lines. The first step is to show that $\mathcal{O}_{X,x_0}$ is a finitely generated module over $\mathcal{O}_{\mathbb{C}^n,0}$. For this one uses the Preparation Theorem, 1.4.1 below. The second step is to use the Cohen-Macaulayness of $\mathcal{O}_{X,x_0}$ to show that it is not only finitely generated but free over $\mathcal{O}_{\mathbb{C}^n,0}$.

Proof that $\mathcal{O}_{XC,x_0}$ is Cohen Macaulay generally uses the technique of "pulling back algebraic structures" discussed in Subsection 1.6 below.

1.4 The preparation theorem

The following theorem has rather an algebraic appearance, but is in fact a theorem of analysis. The classical Weierstrass Preparation Theorem on which it is based concerns division of analytic functions, and is more evidently "analytic".

Theorem 1.4.1. *Let X and Y be complex manifolds (or, more generally, analytic spaces) and let $f : (X,x_0) \to (Y,y_0)$ be an analytic map germ. Let M be a finitely generated module over $\mathcal{O}_{X,0}$. The following statements are equivalent.*

1. *M is also finitely generated over $\mathcal{O}_{Y,y_0}$ via f.*

2. *$\dim_{\mathbb{C}} M/f^*\mathfrak{m}_{Y,y_0} M < \infty$.*

It is extensively used in analytic geometry and singularity theory. The statement also holds, verbatim, for C^∞ mappings and modules over the ring $\mathscr{E}_n$ of C^∞ germs. This much harder theorem was proved by Bernard Malgrange, at the urging of René Thom, in the 1960's, and made possible Thom's Catastrophe Theory, and Mather's celebrated series of papers on the stability of C^∞ mappings, [Mat68a], [Mat69a], [Mat68b], [Mat69b], [Mat70], [Mat71]. Lojasiewicz and Mather himself published alternative proofs.

We also include here a more general version of the Preparation Theorem which can be also useful when we consider multi-germs of analytic mappings instead of just mono-germs. We recall that a multi-germ of analytic spaces is a finite disjoint union $(X,S)=(X_1,x_1)\cup\cdots\cup(X_r,x_r)$ of germs of analytic spaces. By definition, the ring $\mathcal{O}_{X,S}$ is equal to $\oplus_{i=1}^r\mathcal{O}_{X_i,x_i}$.

A multi-germ of analytic mapping $f:(X,S)\to(Y,y_0)$ is given by a system of analytic map germs $f_i:(X_i,x_i)\to(Y,y_0)$, $i=1,\dots,r$. Such a map induces a morphism of $\mathbb{C}$-algebras $f^*:\mathcal{O}_{Y,y_0}\to\mathcal{O}_{X,S}$ in the obvious way.

Corollary 1.4.2. *Let $f:(X,S)\to(Y,y_0)$ be a multi-germ of analytic mapping. Let M be a finitely generated module over $\mathcal{O}_{X,S}$. The following statements are equivalent:*

1. *M is also finitely generated over $\mathcal{O}_{Y,y_0}$ via f.*

2. *$\dim_{\mathbb{C}}M/f^*\mathfrak{m}_{Y,y_0}M<\infty$.*

Proof. For each $i=1,\dots,r$ we denote by $u_i\in\mathcal{O}_{X,x}$ the germ which is the constant function 1 in $\mathcal{O}_{X_j,x_j}$ and 0 in $\mathcal{O}_{X_i,x_i}$ for $i\neq j$. Then $M_i:=\{u_i\}\cdot M$ is an $\mathcal{O}_{X_i,x_i}$-module as well as an $\mathcal{O}_{X,S}$-module, and $M=M_1\oplus\cdots\oplus M_r$. Moreover, we have the following obvious equivalences:

1. M is generated over $\mathcal{O}_{X,S}$ by $\{m_1,\dots,m_k\}$ if and only if for each $i=1,\dots,r$, M_i is generated over $\mathcal{O}_{X_i,x_i}$ by $\{u_im_1,\dots,u_im_k\}$.

2. M is generated over $\mathcal{O}_{Y,y_0}$ via f by $\{m_1,\dots,m_k\}$ if and only if for each $i=1,\dots,r$, M_i is generated over $\mathcal{O}_{Y,y_0}$ via f_i by $\{u_im_1,\dots,u_im_k\}$.

3. $M/f^*\mathfrak{m}_{Y,y_0}M$ is generated over $\mathbb{C}$ by $\{\overline{m}_1,\dots,\overline{m}_k\}$ if and only if for each $i=1,\dots,r$, $M_i/f_i^*\mathfrak{m}_{Y,y_0}M_i$ is generated over $\mathbb{C}$ by $\{\overline{u_im_1},\dots,\overline{u_im_k}\}$.

Now the corollary is an easy consequence of Theorem 1.4.1. □

1.5 Jet spaces and jet bundles

We denote by $J^k(n,p)$ the space of p-tuples of polynomials of degree $\le k$ in n variables with no constant term. A map-germ : $(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ determines a germ of map $j^kf:(\mathbb{C}^n,0)\to J^k(n,p)$, the k-jet extension of f, defined by

$$j^kf(x)=\text{degree k Taylor polynomial of } f \text{ at } x\text{, without its constant term.}$$

The Taylor polynomial of f is determined by partial derivatives of order $\le k$ of the component functions of f at x, so the k-jet can be thought of as simply recording these partial derivatives. There is a also a *jet bundle* $J^k(X,Y)$ over any pair of manifolds X and Y, whose fibre over $(x_0,y_0)\in X\times Y$, which we denote by $J^k(X,Y)_{(x,y)}$, is the set of k-jets of germs of maps $(X,x_0)\to(Y,y_0)$. Two such map-germs determine the same k-jet at x if they have the same partials of order $\le k$ at x, with respect to some, and therefore to any, local coordinate systems on X and Y. So in coordinate free terms, a k-jet is an equivalence class of map-germs $(X,x)\to(Y,y)$.

Although $J^k(n,p)$ is a vector space, the fibre $J^k(X,Y)_{(x_0,y_0)}$ is not; for the identifications between the two spaces depends on a choice of coordinate system, and when we change coordinates the higher derivatives of f change in a non-linear way. Thus there is no natural way of providing

$J^k(X,Y)_{(x_0,y_0)}$ with the operations of a vector space, and $J^k(X,Y)$ is not a vector bundle over $X \times Y$.

Nevertheless, $J^k(X,Y)$ is a locally trivial fibre bundle over $X \times Y$.

Its importance for us is because of its role as a kind of Platonic Heaven which houses ideal versions of all of the singularities which appear in mappings. I will spend the rest of this section justifying this metaphysical remark.

Consider first the 1-jet-bundle $J^1(X,Y)$. By a choice of local coordinates on $U_X \subset X$ and $U_Y \subset Y$ we can identify $\pi^{-1}(U_X \times U_Y)$ with a product $V_X \times V_Y \times J^1(n,p)$ where $V_X \subset \mathbb{C}^n, V_Y \subset \mathbb{C}^p$ are open sets. The information contained in the 1-jet $j^1f(x)$ is just the values of the first order partials of f, so we can think of j^1f as the map

$$x \mapsto (x, f(x), [d_xf]) \in \mathbb{C}^n \times \mathbb{C}^p \times \mathrm{Mat}_{p\times n}(\mathbb{C})$$

where $[d_xf]$ is the $n \times p$ jacobian matrix of f at x. Let us suppose, to fix ideas, that $n \leq p$, and define $\Sigma^k(n,p)$ (or Σ^k when the dimensions are clear from the context) to be the set of $p \times n$ complex matrices of kernel rank k.

Exercise 1.5.1. *$\Sigma^k(n,p)$ is a submanifold of $Mat_{p\times n}(\mathbb{C})$ of codimension $k(p-n+k)$. The formula for the codimension can be recalled as follows: a $p \times n$ matrix of the form*

$$\begin{pmatrix} I_{n-k} & B \\ 0 & D \end{pmatrix}$$

has kernel rank k if and only if $D = 0$. The same is true if we have an invertible $(n-k)\times(n-k)$ matrix A in place of I_{n-k}. A more general matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

in which A is of size $(n-k)\times(n-k)$ and invertible can be brought to this form by left-multiplying by

$$\begin{pmatrix} I_{n-k} & 0 \\ -CA^{-1} & I_{p-n+k} \end{pmatrix}$$

The matrix is in Σ^k if all entries in the transformed D are equal to zero. This gives $(p-n+k)k$ equations.

Let $f : X \to Y$ be a mapping, and denote now by $\Sigma^k(f)$ the set of points in X where d_xf has kernel rank k. Then $\Sigma^k(f) = (j^1f)^{-1}(\Sigma^k)$. Note, incidentally, that if we change coordinates on X then of course j^1f also changes, but $(j^1f)^{-1}(\Sigma^k)$ is, evidently, unchanged. This is because Σ^k has the important property that it is preserved by the action of coordinate changes on X (or on Y).

Observation: suppose $x_0 \in \Sigma^k(f)$ and $j^1f \pitchfork \Sigma^k$ at x_0. Then

- $\Sigma^k(f)$ is a smooth submanifold of X of codimension $k(p-n+k)$.
- Slightly less obvious: for $\ell < k$, $j^1f \pitchfork \Sigma^\ell$ also.
- Indeed, writing $m_0 := j^1f(x_0)$, there is a local diffeomorphism of germs of filtered spaces

$$(\mathrm{Mat}_{p\times n}, m_0) \supset (\overline{\Sigma^1}, m_0) \supset \cdots \supset (\overline{\Sigma^{k-1}}, m_0) \supset (\Sigma^k, m_0)$$

and

$$\Big((X,x_0) \supset (\overline{\Sigma^1(f)},x_0) \supset \cdots \supset (\overline{\Sigma^{k-1}(f)},x_0) \supset (\Sigma^k(f),x_0)\Big) \times \text{ smooth factor}$$

The second statement is a consequence of the fact that the corresponding stratification

$$\mathrm{Mat}_{p\times n}(\mathbb{C}) \supset (\Sigma^1 \smallsetminus \overline{\Sigma^2}) \supset \cdots \supset (\Sigma^\ell \smallsetminus \overline{\Sigma^{\ell+1}})\cdots$$

is Whitney regular. We do not dwell on this now. The aim is simply to make clear that the transversality of j^1f to certain submanifolds of the jet bundle $J^k(X,Y)$ gives us a lot of information about submanifolds (subsets) of X determined by the geometry of f. The subsets that we are interested in are those which are preserved by the action of the group of diffeomorphisms of X and Y – the so-called left-right invariant subsets of $J^k(X,Y)$. The hypothesis on the transversality of j^1f to Σ^k that we invoked in our observation is motivated by the following statement.

Proposition 1.5.2. *Let $W \subset J^k(X,Y)$ be a left-right invariant submanifold. Then*

1. *If $f : X \to Y$ is a stable[3] map, then $j^kf \pitchfork W$.*
2. *If $f : (X,x_0) \to (Y,y_0)$ is a map-germ of finite $\mathcal{A}_e$-codimension, then $j^kf \pitchfork W$ on $X \smallsetminus \{x_0\}$.*

Proof. Suppose f is stable.

Step 1: Suppose that $j^kf(x_0) \in W$. There exists a germ of unfolding $F : (X \times S,(x_0,0)) \to (Y \times S,(f(x_0),0))$ of f such that the "relative" jet extension map $j^k_xF : X \times S \to J^k(X,Y)$ is transverse to W at $(x_0,0)$. This can be arranged by choosing coordinates on X and Y around x_0 and y_0, and then taking as parameter space $S = J^k(n,p)$, and regarding its members as polynomial maps, which can be added to f. The resulting family is defined by $F(x,u) = f(x) + u(x)$, and $j^k_xF|_{\{x_0\}\times S} \to J^k(X,Y)_{(x_0,y_0)}$ is the identity map. It is thus transverse to W.

Step 2: f is stable, so F is a trivial unfolding. Thus, there exist germs of diffeomorphisms Φ of $(X \times S,(x_0,0))$ with $\Phi(x,u) = (\varphi_u(x),u)$ and Ψ of $(Y \times S,(y_0,0))$ with $\Psi(y,u) = \psi_u(y),u)$ such that $\Psi \circ (f \times \ \mathrm{id}_S) \circ \Phi = F$. As $j^k_xF \pitchfork W$, we have $j^k_x\Psi \circ F \circ \Phi \pitchfork W$. As W is left-right invariant, it follows that $j^kf \pitchfork W$ (Exercise).

The second statement follows by the geometric criterion for finite codimension, Theorem 3.7.3. Since f is stable outside x_0, j^kf is transverse to W outside x_0. □

Using an auxiliary map such as j^kf to pull back a universal object from jet space can give useful information. Provided the codimension of the pulled back object is the same as the codimension of the universal object, much of the associated algebraic structure pulls back also. We will see this in Subsection 1.6.

A second important application of jet-space is through the Thom Transversality Theorem, which concerns the behaviour of smooth maps between smooth manifolds. A *residual* subset of a topological space is the intersection of a countable number of dense open sets, and a property is *generic* if it is held by all members of a residual subset. If M and N are smooth manifolds,

[3]The notions of stability and $\mathcal{A}_\rceil$-codimension are discussed in Chapter 3 below. See also Theorem 3.7.3 for the geometrical import of finite codimension - essentially it means "isolated instabiity".

the *Whitney C^k Topology* on the space $C^\infty(M,N)$ of smooth maps from M to N has as base the collection of subsets modelled on open sets $U \subset J^k(M,N)$:

$$C_U = \{f \in C^\infty(M,N) : j^k f(M) \subset U\}$$

and the Whitney C^∞ topology allows such sets for all values of k. We will always consider $C^\infty(M,N)$ with this topology. It is a Baire Space – residual sets are dense. A property of mappings $M \to N$ is said to be *generic* if it is held by the members of a residual subset of $C^\infty(M,N)$.

Exercise 1.5.3. If A is a residual subset of a Baire space S, can $S \setminus A$ contain a residual subset of S?

Theorem 1.5.4. (Thom Transversality Theorem) *Let M and N be C^∞ manifolds, let $W \subset J^k(M,N)$ be a smooth submanifold, and let $T(W)$ be the set of smooth maps $f : M \to N$ such that $j^k f \pitchfork W$. Then*

1. *$T(W)$ is residual in $C^\infty(M,N)$.*
2. *If W is closed in $J^k(M,N)$ then $T(W)$ is open in $C^\infty(M,N)$.*

Note that if $\operatorname{codim} W > \dim M$, the only way that $j^k f : M \to J^k(M,N)$ can be transverse to W is if $(j^k f)^{-1}(W) = \emptyset$. This is often the way that one proves that sets of mappings with certain properties are residual.

An immersion is an *embedding* if it is a diffeomorphism onto its image.

It is just a short step to prove Whitney's 'easy" embedding theorem from 1.5.4:

Theorem 1.5.5. *Let M be an n-dimensional smooth manifold. If $p \geq 2n+1$ then the set of embeddings $M \to \mathbb{R}^p$ is residual in $C^\infty(M, \mathbb{R}^p)$.*

If the domain M is compact, one has only to prove that immersions are residual, and that injective maps are residual. Properness (that the preimage of every compact set is compact) is a global property with some subtlety, and we will not discuss it except to say that it is automatic if the domain is compact. Injectivity, on the other hand, is a property of jets, and can be arranged, if the dimensions are right, by requiring transversality to a suitable submanifold of the *multi-jet space* ${}_rJ^k(M,N)$, which is defined as follows: there is a natural map $p : J^k(M,N) \to M$ giving the source of each jet; ${}_rJ^k(M,N)$ is the preimage in $\left(J^k(M,N)\right)^r$ of the set

$$M^{(r)} = \{(x_1, \ldots, x_r) \in M^r : x_i \neq x_j \text{ if } i \neq j\},$$

under the r-fold Cartesian product map $p^r : \left(J^k(M,N)\right)^r \to M^r$. Each map $f : M \to N$ gives rise to a natural map ${}_rj^k f : M^{(r)} \to {}_rJ^k(M,N)$.

Theorem 1.5.6. *Let M and N be C^∞ manifolds, and let $W \subset {}_rJ^k(M,N)$ be a smooth submanifold. Then the set of smooth maps $f : M \to N$ such that ${}_rj^k f \pitchfork W$ is residual in $C^\infty(M,N)$ with the Whitney topology.*

Exercises 1.5.7. 1. The "Elementary Transversality Theorem" says that if W is a smooth submanifold of N then the set $\{f \in C^\infty(M,N) : f \pitchfork W\}$ is residual. Show how to deduce this from the Thom Transversality Theorem 1.5.4.

2. Show that an immersion which is a homeomorphism onto its image is a diffeomorphism.

3. Show that if M is compact then an injective immersion is an embedding.

4. Give an example of an injective immersion of $\mathbb{R}$ in $\mathbb{R}^2$ which is not an embedding.

5. Prove Whitney's easy embedding theorem 1.5.5 for compact manifolds M. The theorem does not *require* the hypothesis of compactness, but explaining this would lead us too far away from the main thrust of the lectures.

6. (a) Let $W = \{(x, 0, 0) \in \mathbb{R}^3 : -1 < x < 1\}$. Show that the set $\{f \in C^\infty(\mathbb{R}, \mathbb{R}^3) : f \pitchfork W\}$ is *not* open. Hint: consider $f(t) = (-1, t, 0)$.

   (b) Let $W = \{(x, 0) \in \mathbb{R}^2 : -1 < x < 1\}$. Show that the set $\{f \in C^\infty(S^1, \mathbb{R}^3) : f \pitchfork W\}$ is not open.

7. If $n < 6$, the set of mappings $M^n \to N^{n+1}$ for which all singularities have corank 1 is residual (see 1.0.4 for the definition of corank). Is it open?

8. What is the smallest value of n for which a stable map $M^n \to N^{n+1}$ can have a corank 2 singularity? A corank 3 singularity?

9. A critical point x_0 of a smooth function $f : M^m \to \mathbb{R}$ is *non-degenerate* if the Hessian matrix $\det([\frac{\partial^2 f}{\partial x_i x_j}(x_0)]_{1 \le i,j \le m}])$ (with respect to some, and hence any, set of local coordinate) is invertible. A function $M \to \mathbb{R}$ is a *Morse function* if all of its critical points are non-degenerate and no two critical points share the same critical value. Show that for any smooth manifold M, Morse functions form a residual set in $C^\infty(M, \mathbb{R})$.

10. A fixed point x_0 of a smooth map $f : M \to M$ is *non-degenerate* if $d_{x_0} f$ does not have 1 as an eigenvalue. Show that this condition can be expressed in terms of the transversality of some jet extension map to a suitable submanifold of jet space, and deduce that the set of maps $f : M \to M$ with only non-degenerate fixed points is residual in $C^\infty(M, M)$.

Further reading: Chapter II of the textbook [GG73] of Guillemin and Golubitsky.

1.6 Pulling back algebraic structures

The following result fits well with the idea that in singularity theory we study ideal objects, in the sense of Plato, and then attempt to wrestle their properties back to the reality of our concrete examples by some kind of pull-back procedure. The ideal objects are usually contained in spaces of $p \times q$ matrices, or jet spaces $J^k(N, P)$. The condition for the success of this strategy is usually that the codimension of the concrete object in its ambient space is the same as the codimension of the ideal object in its ambient space.

Theorem 1.6.1. *Let $f : X \to Y$ be a map of complex manifolds and let $W \subset Y$ be an analytic subspace.*

1. *If $f^{-1}(W) \neq \emptyset$ then*

$$\mathit{codim}_X f^{-1}(W) \le \mathit{codim}_Y(W). \tag{1.6.1}$$

2. *If W is Cohen-Macaulay, and the inequality in* (1.6.1) *is an equality, then*
   (a) *$f^{-1}(W)$ is Cohen-Macaulay, and*
   (b) *If $\mathbf{L}_\bullet$ is a free resolution of the germ of $\mathcal{O}_{W,w_0}$ as $\mathcal{O}_{Y,w_0}$-module, then for each $x \in f^{-1}(W)$ with $f(x) = w_0$, $\mathbf{L}_\bullet \otimes_{\mathcal{O}_{Y,w_0}} \mathcal{O}_{X,x}$ is a free resolution of $\mathcal{O}_{f^{-1}(W),x}$ as $\mathcal{O}_{X,x}$ module.*

Later we will need a version of 2(b) of Theorem 1.6.1 with $M \otimes_{\mathcal{O}_{Y,y_0}} \mathcal{O}_{X,x_0}$, where M is an $\mathcal{O}_{Y,y}$-module, in place of $\mathcal{O}_{f^{-1}(W),x}$. Its proof is very similar to the proof of 1.6.1, and is left to the reader.

Before proving 1.6.1, let us look at an example of its application.

Corollary 1.6.2. *Let M be a $p \times n$ matrix of functions in $\mathcal{O}_{\mathbb{C}^n,x_0}$, with $p \geq n$. If the codimension in $\mathbb{C}^n$ of $V(\min_k(M))$ is equal to $(p-k+1)(n-k+1)$ then $V(\min_k(M))$ is Cohen-Macaulay.*

Proof. Denote the entries of M by m_{ij}. Let ψ_M denote the map

$$\mathbb{C}^n \to \mathrm{Mat}_{p\times q}(\mathbb{C}), \quad x \mapsto (m_{ij}(x) : 1 \leq i \leq p,\, 1 \leq j \leq q).$$

Then $V(\min_k(M)) = \psi_M^{-1}(W_k)$. A well known theorem of Eagon and Hochster in [EH71] tells us that the space W defined by the $k \times k$ minors of the generic matrix in matrix space $\mathrm{Mat}_{p\times n}(\mathbb{C})$ is Cohen-Macaulay of codimension $(p-k+1)(n-k+1)$. Now apply Theorem 1.6.1. □

Corollary 1.6.3. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic germ, with $n < p$, and denote by $\sum_f$ the non-immersive locus of f. Then*

$$\mathit{codim}\,(\sum_f) \leq p - n + 1,$$

and in case of equality, $\sum_f$ is Cohen-Macaulay.

Proof. $\sum_f$ is defined by the maximal ($= n \times n$) minors of the Jacobian matrix

$$\left(\frac{\partial f_i}{\partial x_j}\; 1 \leq i \leq p,\; 1 \leq j \leq n\right)$$

of f. So the corollary is just an application of 1.6.2. □

To prove 1.6.1 we need some preparatory lemmas.

Lemma 1.6.4. *Let M be a Cohen-Macaulay module over the ring R and let $a_1, \ldots, a_m \in R$. If* $\dim\, M/(a_1, \ldots, a_m)M = \dim\, M - m$ *then $a_1, \ldots, a_m$ is an M-sequence.*

Proof. We prove this by induction on m. Let $M_j = M/(a_1, \ldots, a_j)M = M_{j-1}/a_jM_{j-1}$. The hypothesis implies that $\dim\, M_j/a_{j+1}M_j = \dim\, M_j - 1$. We claim that a_{j+1} cannot be a member of any associated prime of M_j. For $\mathrm{Ass}(M_j)$ is the set of minimal members (with respect to inclusion) of $\mathrm{Supp}(M_j)$. The fact that M_j is Cohen-Macaulay means in particular that all of these have the same height, equal to $\dim R - \dim M_j$. Because $\dim\, M_j/a_{j+1}M_j < \dim\, M_j$, the minimal members of $\mathrm{Ass}(M_j/a_{j+1}M_j) = \mathrm{Supp}(M_j) \cap V(a_{j+1})$ are all of greater height than the minimal members of $\mathrm{supp}(M_j)$. Thus

$$\text{miminal members of } \mathrm{Supp}(M_j) \bigcap V(a_{j+1})$$

contains none of the minimal members of $\mathrm{Supp}(M_j)$. In other words, a_{j+1} lies in none of the minimal members of $\mathrm{Supp}(M_j)$, i.e. in none of the associated primes of M_j. This means that a_{j+1} is regular on M_j. $\square$

Lemma 1.6.5. *Suppose that M is a Cohen-Macaulay module over R and that the elements $a_1, \ldots, a_m$ in R form an M-sequence. Let I be the ideal in R generated by $a_1, \ldots, a_m$. If $\mathbf{L}_\bullet$ is a free resolution of M over R, then $\mathbf{L}_\bullet \otimes R/I$ is a free resolution of M/IM.*

Proof. Again we use induction on m, and the sequence M_j, $j = 0, \ldots, m$ of modules defined in the previous proof. Let $R_0 = R$ and $R_j = R/(a_1, \ldots, a_j)$ for $j = 1, \ldots, m$. Suppose that $\mathbf{L}_\bullet \otimes_R R_j$ is exact. Then it is a resolution of M_j. We have

$$H_i(\mathbf{L}_\bullet \otimes R_{j+1}) = \mathrm{Tor}_i^{R_j}\big(M_j, R_j/(a_{j+1})\big)$$

so to prove exactness we have to show that these Tor modules vanish. We calculate $\mathrm{Tor}^{R_j}\big(M_j, R_j/(a_{j+1})\big)$ by tensoring the short exact sequence

$$0 \longrightarrow R_j \xrightarrow{a_{j+1}} R_j \longrightarrow R_{j+1} \longrightarrow 0$$

with M_j. This gives the long exact sequence

$$\cdots \to \mathrm{Tor}_{i+1}(M_j, R_j) \to \mathrm{Tor}_i(M_j, R_{j+1}) \to \mathrm{Tor}_i(M_j, R_j) \to \cdots$$

$$\cdots \to \mathrm{Tor}_1(M_j, R_{j+1}) \longrightarrow M_j \xrightarrow{a_{j+1}} M_j \longrightarrow M_{j+1} \longrightarrow 0\,.$$

From this it we immediately obtain the vanishing of $\mathrm{Tor}_i^{R_j}(M_j, R_{j+1}) = 0$ for $i > 1$, since this module appears in the sequence flanked by Tor modules which are trivially zero. Vanishing of $\mathrm{Tor}_1^{R_j}(M_j, R_{j+1})$ follows from the vanishing of $\mathrm{Tor}_1^{R_j}(M_j, R_j)$ and the injectivity of $M_j \xrightarrow{a_{j+1}} M_j$. $\square$

Proof. of Theorem 1.6.1 The map

$$f^{-1}(W) \xrightarrow{\mathrm{graph} f} \widetilde{f^{-1}(W)} := \{(x, w) \in X \times W : w = f(x)\}$$

defined by $\mathrm{graph}(f)(x) = (x, f(x))$ has inverse given by the restriction to $\widetilde{f^{-1}(W)}$ of the usual projection $X \times W \to X$. Thus $f^{-1}(W)$ and $\widetilde{f^{-1}(W)}$ are isomorphic, and it is enough to prove that $\widetilde{f^{-1}(W)}$ is Cohen Macaulay. As the product of a smooth space with a Cohen Macaulay space, $X \times W$ is Cohen Macaulay of dimension $\dim W + \dim X$. Taking local coordinates $y_1, \ldots, y_p$ on Y around w_0, we can then view $\widetilde{f^{-1}(W)}$ as the fibre over $0 \in \mathbb{C}^p$ of the map $\pi : X \times W \to Y$ given by $\pi(x) = (y_1 - f_1(x), \ldots, y_p - f_p(x))$. By the *hauptidealsatz*, $\dim X \times W - \dim \widetilde{f^{-1}(W)} \le p = \dim Y$, from which (1.6.1) follows.

Now suppose that (1.6.1) is an equality. Then by Lemma 1.6.4 the components of π form a regular sequence in $\mathcal{O}_{X\times W}$. Since $\mathcal{O}_{X\times W}$ is Cohen-Macaulay, so is $\mathcal{O}_{X\times W}/(y_1 - f_1(x), \ldots, y_p - f_p(x)) = \mathcal{O}_{\widetilde{f^{-1}(W)}} \simeq \mathcal{O}_{f^{-1}(W)}$. This proves that $f^{-1}(W)$ is Cohen-Macaulay. The remaining statement is just Lemma 1.6.5 applied to the $\mathcal{O}_{X\times W}$-module $\mathcal{O}_{\widetilde{f^{-1}(W)}}$. $\square$

Chapter 2

Equivalence of germs of mappings

Let $f, g : (X, S) \to (Y, y_0)$ be germs of analytic maps. They are

1. *right-equivalent* if there exists a germ of bianalytic map $\varphi : (X, S) \to (X, S)$ such that $f_2 = f_1 \circ \varphi$;

2. *left-equivalent*, if there exists a germ of bianalytic map $\psi : (Y, y_0) \to (Y, y_0)$ such that $f_2 = \psi \circ f_1$;

3. *left-right-equivalent*, if there exist germs of bianalytic maps $\varphi : (X, S) \to (X, S)$ and $\psi : (Y, y_0) \to (Y, y_0)$ such that $\psi \circ f \circ \varphi^{-1} = g$. This is the most natural equivalence relation if one is interested in the maps themselves.

4. *contact equivalent*, if there exists a germ of diffeomeorphism $\Phi : (X \times Y, S \times \{y_0\}) \to (X \times Y, S \times \{y_0\})$, of the form $\Phi(x, y) = (\varphi_1(x), \varphi_2(x, y))$, such that $\Phi(\text{graph }(f_1)) = \text{graph }(f_2)$.

The term "bianalytic map" is usually replaced by "diffeomorphism", because of the fact that a great deal of the theory works unchanged for C^∞ maps. In each case there is a group of germs of diffeomorphisms acting on the set of mappings. The groups (or, more precisely, their actions) are denoted by $\mathcal{R}, \mathcal{L}, \mathcal{A}$ and $\mathcal{K}$ respectively. We will be most interested in $\mathcal{A}$: it is the most natural if one is interested in the geometry of maps between complex spaces.

Exercise 2.0.6. *1. Show that $\mathcal{R} \subset \mathcal{K}$, in the sense that $\mathcal{R}$-equivalence implies $\mathcal{K}$-equivalence.*

2. Show that if $f \sim_{\mathcal{K}} g$ then $f^{-1}(y_0)$ and $g^{-1}(y_0)$ are diffeomorphic.

For a very good survey of these groups and their actions, see [Wal81].

A big part of singularity theory has always been concerned with the problem of classification. Generally one classifies germs of analytic maps $(\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ up to $\mathcal{A}$-equivalence, and up to $\mathcal{R}$-equivalence if $p = 1$. Contact equivalence is a technical device which is of interest primarily if one is concerned with preimages of y_0, but also plays an important role in the theory of left-right equivalence, as we will see.

A key ingredient in classification is the notion of *finite determinacy*. Let us assume that $X = \mathbb{C}^n$, $Y = \mathbb{C}^p$ and $S = \{x_0\}$.

Definition 2.0.7. Let $f : (X, x_0) \to (Y, y_0)$ be a complex analytic or $\mathcal{C}^\infty$ map, and let $\mathcal{G}$ be one of the groups listed above. We say f is *k-determined for $\mathcal{G}$-equivalence* if whenever the Taylor series of another germ g coincides with that of f up to degree k, then $f \sim_{\mathcal{G}} g$, and *finitely determined* if it is k-determined for some $k \in \mathbb{N}$.

The notion has an obvious generalisation to the case where S consists of more than a single point, but has only been used in practice in case S is a finite point set. Here we will look only at the case where S is a single point.

In [Mat68b], John Mather showed that for all of the groups listed above, finite determinacy is equivalent to *isolated instability.* We will not prove this, but will explain the main ideas of the proof. The key is to understand how to construct diffeomorphisms. In all of singularity theory this is done by integrating vector fields. With very few exceptions, there is no other method!

2.1 Integration of vector fields

Proposition 2.1.1. *Let χ be an analytic vector field on the open set $U \subset \mathbb{C}^n$. Then for each $x_0 \in U$ there is an open neighbourhood $U(x_0) \subset U$, a disc $B_\eta(0)$ of radius $\eta > 0$ centred at $0 \in \mathbb{C}$, and an analytic map $\Phi : U(x_0) \times B_\eta(0) \to U$ such that for all (x,t)*

1. $\Phi(x, 0) = x$;
2. $\frac{d}{dt}\Phi(x,t) = \chi(\Phi(x,t))$.

The curve described by $\Phi(x,t)$, for fixed x, as t varies, is called a *trajectory* of the vector field χ, and (2) above says that the tangent vector to this trajectory at the point $\Phi(x,t)$ is the vector $\chi(\Phi(x,t))$. Writing $\gamma_x(t)$ in place of $\Phi(x,t)$, and keeping x fixed, this becomes

$$\gamma_x'(t) = \chi(\gamma_x(t)).$$

If instead we fix t, we get a map $\varphi_t : U(x_0) \to U$. Notice that (1) above says that φ_0 is the identity map. From the theorem of existence and uniqueness of solutions of ordinary differential equations, one easily deduces

Corollary 2.1.2. *1. Wherever the composite is defined, one has*

$$\varphi_s \circ \varphi_t = \varphi_{s+t}.$$

2. For each $x_0 \in U$ and each fixed value of $t \in B_\eta(0)$, the map $\varphi_t : U(x_0) \to \varphi_t(U(x_0))$ is a diffeomorphism (bianalytic isomorphism), with inverse φ_{-t}.

The family of diffeomorphisms φ_t is called the *integral flow* of the vector field χ. All arguments involving the integration of vector fields to construct diffeomorphisms go via the following *Thom-Levine* theorem:

Corollary 2.1.3. *Suppose that $F : X \to Y$ is an analytic map of complex manifolds, and that χ and $\tilde{\chi}$ are vector fields on Y and X such that for each $x \in X$ one has*

$$d_x F(\tilde{\chi}(x)) = \chi(F(x)). \tag{2.1.1}$$

Then the integral flows Φ and $\tilde{\Phi}$ of χ and $\tilde{\chi}$ satisfy

$$F \circ \tilde{\varphi}_t = \varphi_t \circ F \tag{2.1.2}$$

wherever the composites are defined.

The two equations (2.1.1) and (2.1.2) can be expressed in terms of commutative diagrams. The vector fields χ and $\tilde{\chi}$ are sections of the tangent bundles TY and TX respectively, and (2.1.1) and (2.1.2) say that the diagrams

$$\begin{tikzcd} TX \arrow[r, "dF"] \arrow[d, dashed] & TY \arrow[d, dashed] \\ X \arrow[r, "F"] \arrow[u, bend left, "\tilde{\chi}"] & Y \arrow[u, bend right, "\chi"'] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} X \arrow[r, "F"] \arrow[d, "\tilde{\varphi}_t"'] & Y \arrow[d, "\varphi_t"] \\ X \arrow[r, "F"'] & Y \end{tikzcd} \tag{2.1.3}$$

commute.

The Thom-Levine theorem shows how an "infinitesimal condition" gives rise to a family of diffeomorphisms. Equalities like (2.1.1) are linear in χ and $\tilde{\chi}$, and these vector fields can often be constructed by the methods of commutative algebra. This is the entry-point of commutative algebra, which, through it, has a huge input into Singularity Theory.

As an example of what is involved, let us prove the simplest of the determinacy theorems of John Mather. If $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ is an analytic germ of function, then the first order partial derivatives $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$ generate the *jacobian ideal* J_f in the ring $\mathcal{O}_{\mathbb{C}^n,0}$.

Example 2.1.4. *1. If $f(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$ then J_f is the maximal ideal $\mathfrak{m} := \mathfrak{m}_{\mathbb{C}^n,0}$.*

2. If $f(x_1, x_2) = x_1^2 + x_2^{k+1}$ then $J_f = (x_1, x_2^k)$.

3. If $f(x_1, x_2) = x_1^2 x_2 + x_2^{k-1}$ then $J_f = (x_1 x_2, x_1^2 + (k-1)x_2^{k-2})$.

4. If $f(x_1, x_2) = x_1^2 x_2$ then $J_f = (x_1 x_2, x_1^2)$.

Theorem 2.1.5. *(i) Suppose that $f \in \mathcal{O}_{\mathbb{C}^n,0}$ is k-determined for right equivalence. Then $\mathfrak{m}\, J_f \supset \mathfrak{m}^{k+1}$.*
(ii) Conversely, suppose that $f \in \mathcal{O}_{\mathbb{C}^n,0}$ and

$$\mathfrak{m}\, J_f \supset \mathfrak{m}^k\,. \tag{2.1.4}$$

Then f is k-determined for $\mathcal{R}$-equivalence.

Exercise 2.1.6. *Find the lowest value of k for which* (2.1.4) *holds for each of the functions in Example 2.1.4.*

Proof of 2.1.5.(i) Let $h \in \mathfrak{m}^{k+1}$. Then for all t there exists $\varphi_t \in \mathrm{Diff}(\mathbb{C}^n, 0)$ such that $f + th = f \circ \varphi_t$. If we could assume the existence of a smoothly parametrised family of diffeomorphisms φ_t with $\varphi_0 = \mathrm{id}$ such that $f \circ \varphi_t = f + th$ then we could reason as follows:

$$h = \frac{\partial(f + th)}{dt} = \frac{\partial(f \circ \varphi_t)}{dt} = \sum_i \left(\frac{\partial f}{\partial x_i} \circ \varphi_t\right) \frac{\partial \varphi_{t,i}}{\partial t}. \tag{2.1.5}$$

Note that since $\varphi_t(0) = 0$ for all t it follows that $\partial\varphi_{t,i}/\partial t \in \mathfrak{m}$. When $t = 0$, since $\varphi_0 = \mathrm{id}$, this gives

$$h = \sum_i \frac{\partial f}{\partial x_i} \frac{\partial \varphi_{t,i}}{\partial t} \in \mathfrak{m}\, J_f \tag{2.1.6}$$

so that $\mathfrak{m}^{k+1} \subset \mathfrak{m}\, J_f$ as required.

However, our hypothesis does not allow us immediately to assert that the diffeomorphisms φ_t fit together to give a smooth family. So instead we look in jet space $J^{k+1}(n,1) = \mathfrak{m}_n \,/\, \mathfrak{m}_n^{k+2}$. As f is k-determined, the set

$$L := \{j^{k+1}(f+h) : h \in \mathfrak{m}^{k+1}\} \subset J^{k+1}(n,1)$$

lies entirely in the $\mathcal{R}^{(k+1)}$-orbit of f, where $\mathcal{R}^{(k+1)}$ is the finite dimensional quotient of $\mathrm{Diff}(\mathbb{C}^n,0)$ acting on jet space. Now $\mathcal{R}^{(k+1)}$ can be identified with the set

$$\{j^{k+1}\varphi(0) : \varphi \in \mathrm{Diff}(\mathbb{C}^n,0)\}$$

and has a natural structure of algebraic group: the composite of two polynomial mappings depends polynomially on their coefficients, and in $\mathcal{R}^{(k+1)}$ one composes and then truncates at degree $k+1$. This group acts algebraically on $J^{k+1}(n,1)$. Thus, as the set L lies in the orbit of $j^{k+1}f(0)$, writing $z = j^{k+1}f(0)$, and $\mathcal{R}^{(k+1)}z$ for the $\mathcal{R}^{(k+1)}$-orbit of z, one has

$$\frac{\mathfrak{m}^{k+1}}{\mathfrak{m}^{k+2}} = T_z L \subset T_z(\mathcal{R}^{(k+1)}z) = \frac{\mathfrak{m}\, J_f + \mathfrak{m}^{k+2}}{\mathfrak{m}^{k+2}}, \tag{2.1.7}$$

and thus

$$\mathfrak{m}^{k+1} \subset \mathfrak{m}\, J_f + \mathfrak{m}^{k+2}\,.$$

The conclusion we want follows by Nakayama's Lemma, 1.2.2.

The second equality in (2.1.7) is important and not completely obvious. It can be obtained along the lines of the argument leading up to (2.1.6), but using the crucial fact that if the Lie group G acts on the manifold M and for $x \in M$ we denote by α_x the orbit map $g \in G \mapsto gx$, then for each $x \in M$ with smooth orbit Gx,

$$T_x Gx = d_e\alpha_x(T_eG).$$

Now

$$d_e\alpha_x(T_eG) = \{\frac{d}{dt}(\gamma(t)\cdot x)|_{t=0} : \ \gamma : (\mathbb{C},0) \to (G,e) \text{ is a curve germ}\};$$

every curve in $(\mathcal{R}^{(k+1)}, \mathrm{id})$ is of the form $j^{k+1}\varphi_t$ for a 1-parameter family of diffeomorphisms φ_t, so now it really is true that

$$\begin{aligned} T_z\mathcal{R}^{(k+1)}z &= \{\frac{d}{dt}j^{k+1}(f\circ\varphi_t)_{t=0} : \varphi_t \text{ is a 1-parameter family in } \mathrm{Diff}(\mathbb{C}^n,0) \text{ with } \varphi_0 = \mathrm{id}\} \\ &= \{j^{k+1}\left(\frac{d}{dt}(f\circ\varphi_t)|_{t=0}\right) : \varphi_t \text{ is a 1-parameter family in } \mathrm{Diff}(\mathbb{C}^n,0) \text{ with } \varphi_0 = \mathrm{id}\}. \end{aligned}$$

(ii) Suppose that g has the same degree k Taylor polynomial as f. Then $g - f \in \mathfrak{m}^{k+1}$. Let $F(x,t) = f(x) + t(g(x) - f(x))$, and write $f_t(x) = F(t,x)$. The idea of the proof is to show that for each value t_0 of t, there is a neighbourhood $U(t_0)$ of t_0 in $\mathbb{C}$ such that f_t and f_{t_0} are $\mathcal{R}$-equivalent for all $t \in U(t_0)$. A finite number of these neighbourhoods cover the compact interval $[0,1] \subset \mathbb{C}$, so by transitivity $f = f_0 \sim_{\mathcal{R}} f_1 = g$.

We do this first for $t_0 = 0$. As F is a function of the $n+1$ variables $x_1, \ldots, x_n, t$, we consider the germ $F \in \mathcal{O}_{\mathbb{C}^{n+1},0}$. Notice that $\partial F/\partial t = g - f \in \mathfrak{m}_n^{k+1}$, where by $\mathfrak{m}_n$ we mean the ideal in

$\mathcal{O}_{\mathbb{C}^{n+1},0}$ generated by $(x_1, \ldots, x_n)$. This is of course *not* the maximal ideal of $\mathcal{O}_{\mathbb{C}^{n+1},0}$. In any case, it follows from our hypothesis on f that

$$\frac{\partial F}{\partial t} \in \mathfrak{m}_n \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right). \tag{2.1.8}$$

We would like to show

$$\frac{\partial F}{\partial t} \in \mathfrak{m}_n \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right). \tag{2.1.9}$$

For if we have

$$\frac{\partial F}{\partial t} = \tilde{\chi}_1 \frac{\partial F}{\partial x_1} + \cdots + \tilde{\chi}_n \frac{\partial F}{\partial x_n} \tag{2.1.10}$$

for some functions $\tilde{\chi}_i \in \mathfrak{m}\,\mathcal{O}_{\mathbb{C}^{n+1},0}$, then defining a vector field $\tilde{\chi}$ on $\mathbb{C}^{n+1}$ by

$$\tilde{\chi} = \frac{\partial}{\partial t} - \sum_i \chi_i \frac{\partial}{\partial x_i},$$

(2.1.10) becomes

$$dF(\tilde{\chi}) = 0.$$

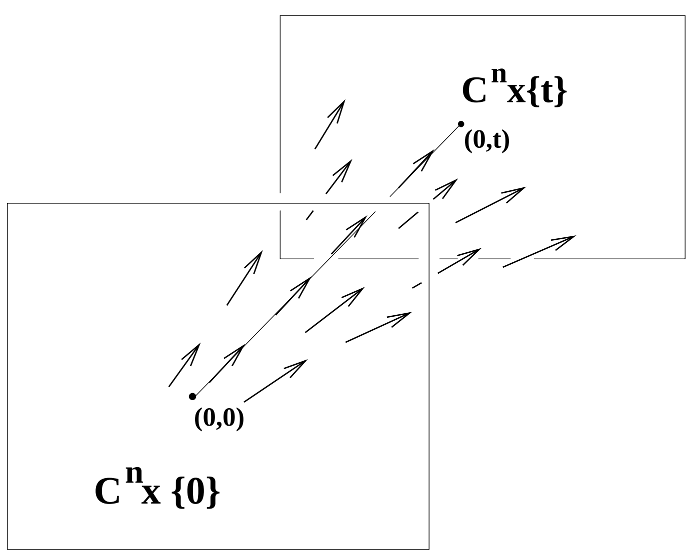
This is exactly (2.1.1) with $\chi = 0$. Let $\tilde{\Phi}(x,t) = (\tilde{\phi}_t(x), t)$ be the integral flow of the vector field $\tilde{\chi}$. The integral flow of the zero vector field is the identity map, and therefore by the Thom-Levine lemma we have

$$F \circ \tilde{\Phi} = F. \tag{2.1.11}$$

Since the component of $\tilde{\chi}$ in the t-direction has constant length 1, it follows that $\tilde{\varphi}_t$ maps $\mathbb{C}^n \times \{0\}$ to $\mathbb{C}^n \times \{t\}$. Restricting both sides of (2.1.11) to $\mathbb{C}^n \times \{0\}$ we therefore get

$$f_t \circ \tilde{\varphi}_t = f.$$

This is not quite enough to show that the germs at 0 of f and of f_t are right-equivalent: we need to show also that $\varphi_t(0) = 0$. But this holds, because $\tilde{\chi}_i \in \mathfrak{m}_n$ for all i. Thus $\tilde{\varphi}_t \in \mathcal{R}$ and $f_t \sim_{\mathcal{R}} f$ as required.



The arrows show a real version of the vector field $\tilde{\chi}$ of the proof. At all points of the t-axis, the vector field is tangent to the axis, so any trajectory beginning at a point on the axis remains on the axis. Thus $\varphi_t(0) = 0$.

Now we set about deducing (2.1.9) from (2.1.8). Since $\partial F/\partial t = g - f \in \mathfrak{m}_n^k$, to show (2.1.9), it will be enough to show

$$\mathfrak{m}_n^k \subset \mathfrak{m}_n \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right). \tag{2.1.12}$$

We know that

$$\mathfrak{m}_n^k \subset \mathfrak{m}_n \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) \tag{2.1.13}$$

(as ideals in $\mathcal{O}_{\mathbb{C}^{n+1},0}$, as in $\mathcal{O}_{\mathbb{C}^n,0}$). Because

$$\frac{\partial f}{\partial x_i} = \frac{\partial F}{\partial x_i} - t\frac{\partial (g-f)}{\partial x_i} \tag{2.1.14}$$

it follows that

$$\frac{\partial f}{\partial x_i} \in \mathfrak{m}_n \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right) + \mathfrak{m}_{n+1}\, \mathfrak{m}^k$$

and therefore

$$\mathfrak{m}_n^k \subset \mathfrak{m}_n \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) \subset \mathfrak{m}_n \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right) + \mathfrak{m}_{n+1}\, \mathfrak{m}^k\,. \tag{2.1.15}$$

Now some commutative algebra comes to our aid. By *Nakayama's Lemma*, 1.2.2, proved in Subsection 1.2, (2.1.15) implies at once that (2.1.12) holds: we apply it taking as R the local ring $\mathcal{O}_{\mathbb{C}^{n+1}}$ with maximal ideal $\mathfrak{m}_{n+1}$, and taking $M = \mathfrak{m}_n^k$ and $N = \mathfrak{m}_n\, J_f$ (where, as before $\mathfrak{m}_n$ means the ideal in $\mathcal{O}_{\mathbb{C}^{n+1}}$ generated by $x_1, \ldots, x_n$).

This completes the proof that the deformation $f + t(g-f)$ is trivial for t in some neighbourhood of 0. The remainder of the proof involves showing that the same procedure can be employed for every value of t: we want to show that for any t_0 the deformation $f + t(g-f)$ is trivial in a neighbourhood of t_0. This deformation can be written in the form $(f + t_0(g-f)) + (t-t_0)(g-f)$, and taking as new parameter $s = t - t_0$, the problem reduces to what we have already discussed, except that instead of our original f we now have a new function, $f_{t_0} := f + t_0(g-f)$. In order that our earlier argument should apply, we have to show that f_{t_0} also satisfies the hypothesis of this argument: that

$$\mathfrak{m}\, J_{f_{t_0}} \supset \mathfrak{m}^k \tag{2.1.16}$$

Once again this is done by a simple argument involving Nakayama's Lemma, which I leave as an exercise.

Exercise 2.1.7. *Show that if $\mathfrak{m}\, J_f \supset \mathfrak{m}^k$ and $g - f \in \mathfrak{m}^{k+1}$ then $\mathfrak{m}\, J_{f_{t_0}} \supset \mathfrak{m}^k$.*

□

The first part of the proof of Theorem 2.1.5 justifies part (i) of the following definition.

Definition 2.1.8.

$$\textit{(i)} \quad T\mathcal{R}f = \mathfrak{m}_n J_f$$

$$\textit{(ii)} \quad T\mathcal{R}_e f = J_f$$

The second tangent space is the *extended* right tangent space. Its heuristic justification is less clear than that of $T\mathcal{R}f$; it can be obtained by the argument of the proof of Theorem 2.1.5(i) *if we remove the requirement that* $\varphi_t(0) = 0$ *for all* t.

Chapter 3

Left-right equivalence and stability

In these lectures we are interested in left-right equivalence more than right equivalence. But Theorem 2.1.5 is a good indication of what is true and how, in principle, one goes about proving it. For left-right equivalence, the proof is necessarily more complicated, since one has simultaneously to produce families of diffeomorphisms of source and target. However the overall strategy is the same. First we need to define a suitable tangent space for $\mathcal{A}$-equivalence.

Mather and Thom, in their work in the 60's on smooth maps, thought in global terms: a C^∞ map $f : N \to P$ is *stable* if its orbit under the natural action of $\mathrm{Diff}(N) \times \mathrm{Diff}(P)$ is open in $C^\infty(N, P)$, with respect to a suitable topology. Here we are interested in local geometry, and so we give a local version of this definition: a map-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is stable if every deformation is trivial: roughly speaking, if f_t is a deformation of f then there should exist deformations of the identity maps of $(\mathbb{C}^n, S)$ and $(\mathbb{C}^p, 0)$, φ_t and ψ_t, such that

$$f_t = \psi_t \circ f \circ \varphi_t^{-1}. \tag{3.0.1}$$

A substantial part of Mather's six papers on the stability of C^∞ mappings [Mat68a]-[Mat71] is devoted to showing that if all the germs of a proper mapping f are stable in this local sense then f is stable in the global sense. We will not discuss global stability any further.

Definition 3.0.9. (1) A *d-parameter unfolding* of f is a map-germ

$$F : (\mathbb{C}^n \times \mathbb{C}^d, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$$

of the form

$$F(x, u) = (\tilde{f}(x, u), u)$$

such that $\tilde{f}(x, 0) = f(x)$. If we denote the map $x \mapsto \tilde{f}(x, u)$ by f_u, then the above condition becomes simply $f_0 = f$.

Retaining the parameters u in the second component of the map makes the following definition easier to write down:
(2) Two unfoldings F, G of f are *equivalent* if there exist germs of diffeomorphisms

$$\Phi : (\mathbb{C}^n \times \mathbb{C}^d, S \times \{0\}) \to (\mathbb{C}^n \times \mathbb{C}^d, S \times \{0\})$$

and

$$\Psi : (\mathbb{C}^p \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$$

which are unfoldings of the identity in $\mathbb{C}^n$ and $\mathbb{C}^p$ respectively, such that

$$G = \Psi \circ F \circ \Phi^{-1}. \tag{3.0.2}$$

(3) The unfolding F is *trivial* if it is equivalent to $f \times \mathrm{id}$ (the 'constant' unfolding $(x, u) \mapsto (f(x), u)$).
(4) The map-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is *stable* if every unfolding of f is trivial.

By writing $\Phi(x, u) = (\varphi_u(x), x)$ and $\Psi(y, u) = (\psi_u(y), u)$, from (3.0.2) we recover the heuristic definition (3.0.1) in the particular case $d = 1$. We do not insist that the mappings φ_u and ψ_u preserve the origin of $\mathbb{C}^n$ and $\mathbb{C}^p$ respectively. After all, if the interesting behaviour merely changes its location, we should not regard the unfolding as non-trivial.

Example 3.0.10. Consider the map-germ $f(x) = x^2$, and its unfolding $F(x, u) = (x^2 + ux, u)$. This is trivialised by the families of diffeomorphisms $\Phi(x, u) = (x + u/2, u)$, $\Psi(y, u) = (y - u^2/4, u)$. Both Φ and Ψ are just families of translations.

Exercise 3.0.11. Check that in the previous example $F = \Psi \circ (f \times \mathrm{id}) \circ \Phi^{-1}$.

Fortunately, there exists a simple and computable criterion for stability. If f is stable, then the quotient

$$T^1(f) := \frac{\{\frac{d}{dt} f_t|_{t=0} : f_0 = f\}}{\{\frac{d}{dt}(\psi_t \circ f \circ \varphi_t^{-1})|_{t=0} : \varphi_0 = \mathrm{id},\ \psi_0 = \mathrm{id}\}}, \tag{3.0.3}$$

is equal to 0: very deformation is trivial, so the f_t in the numerator is equal to $\psi_t \circ f \circ \varphi_t^{-1}$ for suitable families of diffeomorphisms ψ_t and φ_t. In general $T^1(f)$ is a vector space whose dimension, the $\mathcal{A}_e$-*codimension of* f, measures the failure of stability. Mather ([Mat69a]) proved

Theorem 3.0.12. Infinitesimal stability is equivalent to stability: f *is stable if and only if* $T^1(f) = 0$.

One of the main aims of this chapter is to develop techniques for calculating $T^1(f)$, and apply them in some examples.

Exercise 3.0.13. Germs of submersions and immersions are infinitesimally stable and therefore stable. This is an easy calculation using the normal forms of Theorems 1.0.1 and 1.0.2

Before continuing, we note that the denominator in (3.0.3) is very close to being the tangent space to the orbit of f under the group $\mathscr{A} = \mathrm{Diff}(\mathbb{C}^n, S \times \mathrm{Diff}(\mathbb{C}^p, 0)$. It is not quite equal to it, because we are allowing ϕ_t and ψ_t to move the base points (so they are not "paths in $\mathrm{Diff}(\mathbb{C}^n, S)$ and $\mathrm{Diff}(\mathbb{C}^p, 0)$"). For this reason we call the denominator in (3.0.3) the 'extended' tangent space and denote it by $T\mathscr{A}_e f$. The tangent space to the $\mathscr{A}$-orbit of f is denoted $T\mathscr{A} f$. It is the subspace of $T\mathscr{A}_e f$ corresponding to families φ_t and ψ_t for which $\varphi_t(x_i) = x_i$, $\forall x_i \in S$ and $\psi_t(0) = 0$ for all t.

The numerator of (3.0.3) is denoted by $\theta(f)$.

By the chain rule,

$$\frac{d}{dt}\Big(\psi_t \circ f \circ \varphi_t^{-1}\Big)|_{t=0} = df(\frac{d\varphi_t^{-1}}{dt}|_{t=0}) + (\frac{d\psi_t}{dt}|_{t=0}) \circ f.$$

Both $(d\varphi_t/dt)|_{t=0}$ and $(d\psi_t/dt)|_{t=0}$ are germs of vector fields, on $(\mathbb{C}^n, S)$ and $(\mathbb{C}^p, 0)$ respectively: quite simply, $(d\varphi_t(x)/dt)|_{t=0}$ is the tangent vector at x to the trajectory $\varphi_t(x)$. In the same way, the elements of the numerator $\theta(f)$ of (3.0.3) should be thought of as 'vector fields along f'; $(df_t/dt)|_{t=0}$ is the tangent vector at $f(x)$ to the trajectory $x \mapsto f_t(x)$. By associating to $(df_t/dt)|_{t=0}$ the map

$$\hat{f} : x \mapsto (x, (d/dt)f_t|_{t=0}) \in T\mathbb{C}^p,$$

we obtain a commutative diagram:

$$\begin{tikzcd} T\mathbb{C}^n \arrow[r, "df"] \arrow[d] & T\mathbb{C}^p \arrow[d] \\ \mathbb{C}^n \arrow[r, "f"] \arrow[ur, dashed, "\hat{f}"] & \mathbb{C}^p \end{tikzcd} \tag{3.0.4}$$

in which the vertical maps are the bundle projections.

The set of all germs at 0 of vector fields on $\mathbb{C}^n$ is denoted by $\theta_{\mathbb{C}^n,0}$. Elements of $\theta_{\mathbb{C}^n,0}$ can be written in various ways: as n-tuples,

$$\xi(x) = (\xi_1(x), \ldots, \xi_n(x))$$

(sometimes as columns rather than rows), or as sums:

$$\xi(x) = \sum_{j=1}^{n} \xi_j(x)\partial/\partial x_j.$$

The second notation emphasizes the role of the coordinate system on $\mathbb{C}^n, S$. Similarly, elements of $\theta(f)$ can be written as row vectors or column vectors, or as sums:

$$\hat{f}(x) = \sum_{j=1}^{p} \hat{f}_j(x)\partial/\partial y_j.$$

In calculations below we will usually write the elements of $\theta(f)$ and of $\theta_{\mathbb{C}^n,0}$ as columns.

We denote by

$\theta(f)$	the numerator of (3.0.3)
$\theta_{\mathbb{C}^n,S}$	the space of germs at S of vector fields on $\mathbb{C}^n$
$\theta_{\mathbb{C}^p,0}$	the space of germs at 0 of vector fields on $\mathbb{C}^p$
$tf : \theta_{\mathbb{C}^n,S} \to \theta(f)$	the map $\xi \mapsto df \circ \xi$
$\omega f : \theta_{\mathbb{C}^p,0} \to \theta(f)$	the map $\eta \mapsto \eta \circ f$

The notation "tf" is slightly fussy. We use it instead of df here because we think of df as the bundle map between tangent bundles induced by f, as in the diagram (3.0.4), whereas tf is the map "left composition with df" from $\theta_{\mathbb{C}^n,S}$ to $\theta(f)$. Some authors use "df" for both. In any case,

$$T\mathscr{A}_e f = tf(\theta_{\mathbb{C}^n,S}) + \omega f(\theta_{\mathbb{C}^p,0}), \quad T\mathscr{A}_f = tf(\mathfrak{m}_n\, \theta_{\mathbb{C}^n,S}) + \omega f(\mathfrak{m}_p\, \theta_{\mathbb{C}^p,0}). \tag{3.0.5}$$

These spaces are not just vector spaces:

$\theta_{\mathbb{C}^n,S}$	is an $\mathcal{O}_{\mathbb{C}^n,S}$-module
$\theta(f)$	is an $\mathcal{O}_{\mathbb{C}^n,S}$-module
$tf : \theta_{\mathbb{C}^n,S} \to \theta(f)$	is $\mathcal{O}_{\mathbb{C}^n,S}$-linear, so
$\theta(f)/tf(\theta_{\mathbb{C}^n,S})$	is an $\mathcal{O}_{\mathbb{C}^n,S}$-module

But $T^1(f)$ is not an $\mathcal{O}_{\mathbb{C}^n,S}$-module, because $\mathcal{O}_{\mathbb{C}^p,0}$ is not. It is, however, an $\mathcal{O}_{\mathbb{C}^p,0}$-module; for via composition with f, $\mathcal{O}_{\mathbb{C}^n,S}$ becomes an $\mathcal{O}_{\mathbb{C}^p,0}$- module: we can 'multiply' $g \in \mathcal{O}_{\mathbb{C}^n,S}$ by $h \in \mathcal{O}_{\mathbb{C}^p,0}$ using composition with f to transport $h \in \mathcal{O}_{\mathbb{C}^p,0}$ to $h \circ f \in \mathcal{O}_{\mathbb{C}^n,S}$:

$$h \cdot g := (h \circ f)g.$$

By this 'extension of scalars', every $\mathcal{O}_{\mathbb{C}^n,S}$-module becomes an $\mathcal{O}_{\mathbb{C}^p,0}$-module. This is where commutative algebra enters the picture. But we will not open the door to it in any serious way just yet. We simply note that

$\theta_{\mathbb{C}^p,0}$	is an $\mathcal{O}_{\mathbb{C}^p,0}$-module
$\omega f : \theta_{\mathbb{C}^p,0} \to \theta(f)$	is $\mathcal{O}_{\mathbb{C}^p,0}$-linear, so
$T^1(f)$	is an $\mathcal{O}_{\mathbb{C}^p,0}$-module

Exercise 3.0.14. *Given a germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ and diffeomorphisms $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and $\psi : (\mathbb{C}^p, 0) \to (\mathbb{C}^p, 0)$, find*

1. *a natural isomorphism $L : \theta(f) \to \theta(f \circ \varphi)$, for which $L(T\mathcal{A}_e f) = T\mathcal{A}_e(f \circ \varphi)$, and*

2. *a natural isomorphism $K : \theta(f) \to \theta(f \circ \varphi)$, for which $K(T\mathcal{A}_e f) = T\mathcal{A}_e(\psi \circ f)$.*

For (a), the diagram

$$\begin{tikzcd}
T\mathbb{C}^n \arrow[r, "T\varphi"] \arrow[d] & T\mathbb{C}^n \arrow[r, "Tf"] \arrow[d] & T\mathbb{C}^p \arrow[d] \\
\mathbb{C}^n \arrow[r, "\varphi"] \arrow[urr, dashed] & \mathbb{C}^n \arrow[r, "f"] \arrow[ur, dashed] & \mathbb{C}^p
\end{tikzcd}$$

in which elements of $\theta(f)$ and $\theta(f \circ \varphi)$ are shown as dashed arrows, can help to guide the definition of L. A similar diagram will help with (b).

Conclude that if f and g are $\mathcal{G}$-equivalent, for $\mathcal{G} = \mathcal{R}, \mathcal{L}$ or $\mathcal{A}$, then $\theta(f)/T\mathcal{G}_e f$ and $\theta(g)/T\mathcal{G}_e g$ are isomorphic.

3.1 First calculations

Example 3.1.1. (1) Consider $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ defined by $f(x) = (x^2, x^3)$. We write elements of $\theta_{\mathbb{R}^2,0}$ and $\theta(f)$ as column vectors. We work first at the level of formal power series. This will be justified by a theorem at the end of the chapter – see Remark 3.6.3. Every monomial x^k except x itself can be written as a product of powers of x^2 and x^3 and hence as a composite $a \circ f$. It follows that

$$\omega f(\theta_{\mathbb{R}^2,0}) + \mathrm{Sp}_{\mathbb{R}}\{\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}\} = \theta(f).$$

Since

$$tf\left(\frac{\partial}{\partial x}\right) = \begin{pmatrix} 2x \\ 3x^2 \end{pmatrix},$$

it follows that

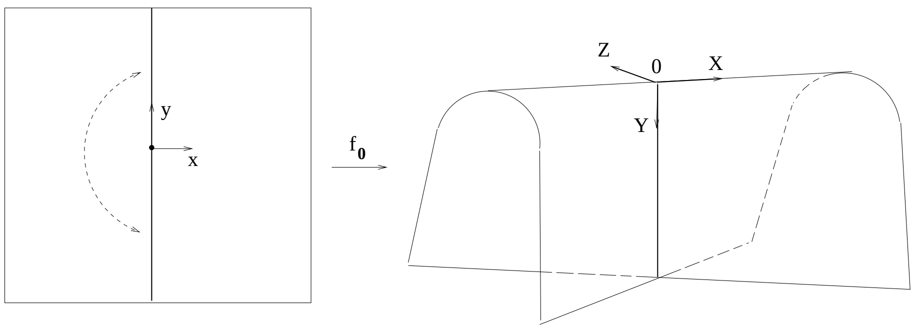
$$tf(\theta_{\mathbb{R},0}) + \omega f(\theta_{\mathbb{R}^2,0}) + \mathrm{Sp}_{\mathbb{R}}\{\begin{pmatrix} 0 \\ x \end{pmatrix}\} = \theta(f).$$

Finally, it is easy to see that the missing term $\begin{pmatrix} 0 \\ x \end{pmatrix}$ does not lie in $T\mathscr{A}_e f$, since the order of the second component of any element in $T\mathscr{A}_e f$ is at least 2. Hence $T^1(f)$ has as basis the class of $\begin{pmatrix} 0 \\ x \end{pmatrix}$.

(2) The map-germ

$$f(x,y) = (x, y^2, xy)$$

parametrises the cross-cap (also known as pinch point and Whitney umbrella). We will show that it is stable by showing that $T^1(f) = 0$. We use coordinates (x,y) on the source and (X,Y,Z) on the target.



Elements of $\theta_{\mathbb{R}^2,0}, \theta_{\mathbb{R}^3,0}$ and $\theta(f)$ will be written as column vectors. We divide $\mathcal{O}_{\mathbb{C}^2,0}$ into *even* and *odd* parts with respect to the y variable, and denote them by $\mathcal{O}^e$ and $\mathcal{O}^o$. Every element of $\mathcal{O}^e$ can be written in the form $a(x,y^2)$, and every element of $\mathcal{O}^o$ in the form $ya(x,y^2)$. This is obvious in the analytic case. It is also true for C^∞ germs, and will be justified by a theorem at the end of the chapter.

Then (we hope the notation is self-explanatory)

$$\theta(f) = \begin{pmatrix} \mathcal{O}^e \oplus \mathcal{O}^o \\ \mathcal{O}^e \oplus \mathcal{O}^o \\ \mathcal{O}^e \oplus \mathcal{O}^o \end{pmatrix}$$

and since

$$\omega f \begin{pmatrix} a(X,Y) \\ b(X,Y) \\ c(X,Y) \end{pmatrix} = \begin{pmatrix} a(x,y^2) \\ b(x,y^2) \\ c(x,y^2) \end{pmatrix} \tag{3.1.1}$$

we see that the even part of $\theta(f)$ is indeed contained in $T\mathscr{A}_e f$, and we need worry only about the odd part. Since

$$tf\left(a(x,y^2)\frac{\partial}{\partial x}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 2y \\ y & x \end{pmatrix} \begin{pmatrix} a(x,y^2) \\ 0 \end{pmatrix} = \begin{pmatrix} a(x,y^2) \\ 0 \\ ya(x,y^2) \end{pmatrix} \tag{3.1.2}$$

we get all of the odd part of the third row. Since

$$tf\left(a(x,y^2)\frac{\partial}{\partial y}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 2y \\ y & x \end{pmatrix} \begin{pmatrix} 0 \\ a(x,y^2) \end{pmatrix} = \begin{pmatrix} 0 \\ 2ya(x,y^2) \\ xa(x,y^2) \end{pmatrix} \tag{3.1.3}$$

we get all of the odd part of the second row. Since

$$tf\left(ya(x,y^2)\frac{\partial}{\partial x}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 2y \\ y & x \end{pmatrix}\begin{pmatrix} ya(x,y^2) \\ 0 \end{pmatrix} = \begin{pmatrix} ya(x,y^2) \\ 0 \\ y^2a(x,y^2) \end{pmatrix} \tag{3.1.4}$$

we get all of the odd part of the first row. So $T\mathscr{A}_e f = \theta(f)$, $T^1(f) = 0$ and f is stable.

(3) The map-germ $f(x,y) = (x, y^2, y^3 + x^2y)$ is not stable. The calculation of (3.1.1), (3.1.3) and (3.1.4) still apply, with insignificant modifications. The only change from (2) is that (3.1.2) now shows that

$$T\mathscr{A}_e f \supset \big(x\,\mathcal{O}^o\big)\partial/\partial Z \tag{3.1.5}$$

and we need an extra calculation

$$tf\left(ya(x,y^2)\frac{\partial}{\partial y}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 2y \\ 2xy & x^2+3y^2 \end{pmatrix}\begin{pmatrix} 0 \\ ya(x,y^2) \end{pmatrix} = \begin{pmatrix} 0 \\ 2y^2a(x,y^2) \\ x^2ya(x,y^2) + 3y^3a(x,y^2) \end{pmatrix} \tag{3.1.6}$$

In view of (3.1.5) and what we know about the even terms, this completes the proof that

$$T^1(f) = \begin{pmatrix} \mathcal{O}^e + \mathcal{O}^o \\ \mathcal{O}^e + \mathcal{O}^o \\ \mathcal{O}^e + x\,\mathcal{O}^o + y^2\,\mathcal{O}^o \end{pmatrix} \tag{3.1.7}$$

It follows that $T^1(f)$ is generated, as a vector space over $\mathbb{C}$, by $y\partial/\partial Z$.

Definition 3.1.2. The $\mathscr{A}_e$-codimension of $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is the dimension, as a $\mathbb{C}$-vector space, of $T^1(f)$.

Exercise 3.1.3.

1. Calculate the $\mathscr{A}_e$-codimension, and a $\mathbb{C}$-basis for $T^1(f)$, when

   (a) $f(x) = (x^2, x^5)$ (b) $f(x) = (x^2, x^{2k+1})$ (c) $f(x) = (x^3, x^4)$

   (d) $f(x,y) = (x, y^2, y^3 + x^{k+1}y)$ (e) $f(x,y) = (x, y^2, x^2y + y^5)$

   (f) $f(x,y) = (x, y^2, x^2y + y^{2k+1})$.

2. Show that the Whitney cusp $f(x,y) = (x, y^3 + xy)$ and the fold map $f(x_1, \ldots, x_n) = (x_1, \ldots, x_k, x_{k+1}^2 + \cdots + x_n^2)$ are stable.

3. Show that the $\mathscr{A}_e$-codimension of f is 1, and find a $\mathbb{C}$-basis for $T^1(f)$, when

   $f(x,y) = (x, y^3 \pm x^2y)$.

4. The calculation of a basis for $T^1(f)$ in Example 3.1.1(1) above suggests that the unfolding $F(x,u) = (x^2, x^3 + ux, u)$ should be interesting. Make drawings of the images of f_u for $u < 0$, $u = 0$ and $u > 0$, and show that as u passes through 0 the family f_u describes the first Reidemeister move of knot theory.

5. Make an analogous sequence of drawings of the images of the maps in the family

$$f_u(x, y) = (x, y^2, y^3 + x^2 y + uy)$$

suggested by Example 3.1.1(3). We will return to this example in Chapter.

6. Show that if $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ is a non-immersive germ such that the two vectors $f''(0)$ and $f'''(0)$ are linearly independent then f is $\mathscr{A}$-equivalent to the cusp $t \mapsto (t^2, t^3)$.

7. Let $f(x, y) = (x, y^2, xy)$.

   (a) Check that $j^1 f : (\mathbb{R}^2, 0) \to L(\mathbb{R}^2, \mathbb{R}^3)$ meets, and is transverse to, the submanifold Σ^1 of linear maps of corank 1.

   (b) In fact this property *characterises* the cross-cap: any map-germ $g : (\mathbb{R}^2, a) \to (\mathbb{R}^3, b)$ with this property is $\mathscr{A}$-equivalent to f. To prove this, begin by choosing coordinates in which g takes the form $g(u, v) = (u, g_2(u, v), g_3(u, v))$ (*cf* Exercise 1.0.3). It is then not hard to find explicit coordinate changes which reduce g to the form $g(u, v) = f(u, v) +$ higher order terms.

8. If $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is not an immersion then the ideal $f^* m_{\mathbb{C}^3,0}$ generated in $\mathcal{O}_{\mathbb{C}^2,0}$ by the three component functions of f is strictly contained in $m_{\mathbb{C}^2,0} = (x, y)$. It follows that $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0} / f^* m_{\mathbb{C}^3,0} \geq 2$. Show that every germ for which this dimension is exactly 2 (as in all the examples above) is $\mathscr{A}$-equivalent to one of the form $f(x, y) = (x, y^2, yp(x, y^2))$. Details can be found in [Mon85]. What is the significance here of the involution $(x, y) \mapsto (x, -y)$?

3.2 Multi-germs

We have spoken only of 'mono'-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. But many of the interesting phenomena associated with deformations of mono-germs require description in terms of multi-germs, so they cannot sensibly be avoided. For example, a parametrised plane curve singularity splits into a certain number of nodes on deformation; each of these is stable; their number is an important invariant of the singularity.

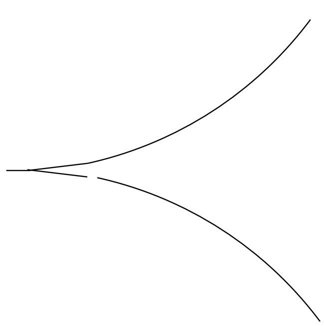
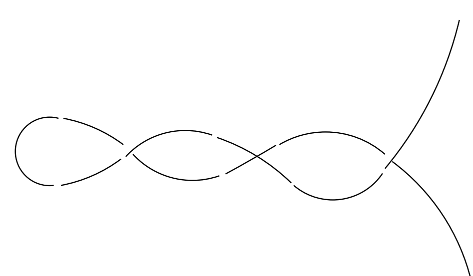


Figure 3.1: $t \mapsto (t^2, t^7)$



$t \mapsto (t^2, t(t^2 - 4u)(t^2 - 9u)(t^2 - 16u))$

Notation

Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a bi-germ. In general we will choose independent coordinate systems $x_1^{(1)}, \dots, x_n^{(1)}$ and $x_1^{(2)}, \dots, x_n^{(2)}$ centred on the two base points (points of S) which we denote by 0 and $0'$. Then $\theta_{\mathbb{C}^n,S} = \theta_{\mathbb{C}^n,0} \oplus \theta_{\mathbb{C}^n,0'}$, and we write its elements as pairs $(\xi^{(1)}, \xi^{(2)})$. Similarly $\theta(f) = \theta(f_1) \oplus \theta(f_2)$. In calculations we represent its elements as $p \times 2$-matrices, in which the first column is in $\theta(f_1)$ and the second in $\theta(f_2)$. Then $tf : \theta_{\mathbb{C}^n,\{0,0'\}} \to \theta(f)$ is equal to $tf^{(1)} \oplus tf^{(2)}$, and $\omega f : \theta_{\mathbb{C}^p,0} \to \theta(f)$ is given by $\eta \mapsto (\eta \circ f_1, \eta \circ f_2)$.

For multi-germs $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ with $|S| > 2$ we extend the same idea. Thus, elements of $\theta(f)$ are represented by $p \times |S|$ matrices.

Example 3.2.1. Where $n = 1$ we usually replace $x^{(1)}$ and $x^{(2)}$ simply by s and t. Consider the bi-germ $f : (\mathbb{C}, S) \to (\mathbb{C}^2, 0)$ consisting of two germs of immersion from $\mathbb{C}$ to $\mathbb{C}^2$ which meet tangentially. In suitable coordinates such a germ can be written

$$\begin{cases} f_1 : s \mapsto (s, 0) \\ f_2 : t \mapsto (t, h(t)) \end{cases} \tag{3.2.1}$$

The two branches meet tangentially if $h \in (t^2)$. Let us calculate $T\mathscr{A}_e f$. We have

$$tf\left(a(s)\frac{\partial}{\partial s}, b(t)\frac{\partial}{\partial t}\right) = \left(a(s)\frac{\partial f_1}{\partial s}, b(t)\frac{\partial f_2}{\partial t}\right).$$

Thus

$$tf\left(a(s)\frac{\partial}{\partial s}, 0\right) = \begin{pmatrix} a(s) & 0 \\ 0 & 0 \end{pmatrix}$$

so

$$T\mathscr{A}_e f \supset \begin{pmatrix} \mathscr{O}_{\mathbb{C},0} & 0 \\ 0 & 0 \end{pmatrix}; \tag{3.2.2}$$

now since

$$\omega f\left(a(x)\frac{\partial}{\partial x}\right) = \begin{pmatrix} a(s) & a(t) \\ 0 & 0 \end{pmatrix}$$

it follows from (3.2.2) that

$$T\mathscr{A}_e f \supseteq \begin{pmatrix} 0 & \mathscr{O}_{\mathbb{C},0'} \\ 0 & 0 \end{pmatrix} \tag{3.2.3}$$

also.

Since

$$tf\left(0, b(t)\frac{\partial}{\partial t}\right) = \begin{pmatrix} 0 & b(t) \\ 0 & h'(t)b(t) \end{pmatrix}, \tag{3.2.4}$$

it follows from (3.2.3) that

$$T\mathscr{A}_e f \supset \begin{pmatrix} 0 & 0 \\ 0 & J_h \end{pmatrix} \tag{3.2.5}$$

where J_h is the Jacobian ideal of h. Contributions to the bottom left hand entry in $T\mathscr{A}_e f$ come only from ωf:

$$\omega f\left(\eta_2\frac{\partial}{\partial y}\right) = \begin{pmatrix} 0 & 0 \\ \eta_2(s, 0) & \eta_2(t, h(t)) \end{pmatrix}. \tag{3.2.6}$$

We have $\eta_2(s,0)=0$ if and only if η_2 is divisible by y, in which case $\eta_2(t,h(t))\in(h)\subset J_h$ (recall that we are working in 1 dimension here!). Thus

$$T\mathscr{A}_e f\bigcap\begin{pmatrix}0 & 0\\ 0 & \mathcal{O}_{\mathbb{C},0'}\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & J_h\end{pmatrix} \tag{3.2.7}$$

A map

$$\ell:\frac{\mathcal{O}_{\mathbb{C},0'}}{J_h}\to\frac{\theta(f)}{T\mathscr{A}_e f}$$

may now be defined as follows. For $a\in\mathcal{O}_{\mathbb{C},0'}$ denote the class of a modulo J_h by $\bar{a}$. Then

$$\ell(\bar{a})=\begin{pmatrix}0 & 0\\ 0 & a\end{pmatrix}+T\mathscr{A}_e f.$$

It is well defined and injective by (3.2.9). It is surjective, by (3.2.2), (3.2.3) and (3.2.6).

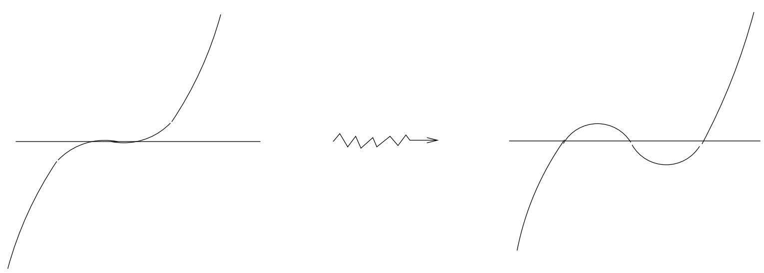
Note that ℓ is $\mathcal{O}_{\mathbb{C}^2,0}$-linear, where $\mathcal{O}_{\mathbb{C},0'}$ is an $\mathcal{O}_{\mathbb{C}^2,0}$-module via f_2. We have proved

Proposition 3.2.2.

$$\theta(f)/T\mathscr{A}_e f\simeq\mathcal{O}_{\mathbb{C},0'}/J_h$$

□

Since h can be perturbed to have $\nu=\operatorname{order}h$ non-degenerate zeros, the germ f of this example can be perturbed to a bi-germ with ν nodes.



So the number of nodes is one more than the codimension. In fact the image of a ν-nodal perturbation is homotopy-equivalent to a wedge of $\nu-1$ circles, and so we conclude

Corollary 3.2.3. *The bi-germ f of Example 3.2.1 can be perturbed to a germ whose image has the homotopy-type of a wedge of circles, with the number of circles in the image equal to the $\mathscr{A}_e$-codimension of f.*

We will return to this theme many times! The relation between the $\mathcal{A}_e$-codimension of a map-germ and the geometry and topology of a stable perturbation is one of the most interesting aspects of the subject, and will be explored further below.

Exercise 3.2.4. Consider the bi-germ of Example 3.2.1 for which $h(t)=t^2$. It is known as the *tacnode.*

1. Show that $T\mathscr{A}_e f+\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}=\theta(f)$.

2. Let f_u be the unfolding

$$\begin{cases} s \mapsto (s,0) \\ t \mapsto (t,t^2+ut) \end{cases}$$

(this is obtained by adding, to f, u times the basis element $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ for $T^1(f)$). Draw the images of the mappings f_u for $u<0, u=0$ and $u>0$, and thus show that as u passes through 0, this family describes the second Reidemeister move.

3. The third Reidemester move is obtained by deforming the tri-germ

$$f : \begin{cases} s \mapsto (s,0) \\ t \mapsto (0,t) \\ u \mapsto (u,u) \end{cases}$$

Find $g \in \theta(f)$ projecting to a basis for $T^1(f)$, and draw the sequence of images of the mappings f_v defined by adding vg to f, as v passes through 0. Check that this corresponds to the third Reidemeister move. The tri-germ f here is known as a *triple point*.

4. Are there any more map-germs $(\mathbb{C},S) \to (\mathbb{C}^2,0)$of $\mathscr{A}_e$-codimension 1, beyond the cusp (Example 3.1.1(1)), the tacnode and the triple point?

5. *Classifying bi-germs of immersions*: we measure the contact between the two branches of a bi-germ

$$g : \begin{cases} s \mapsto g_1(s) \\ t \mapsto g_2(t) \end{cases}$$

as follows: pick an equation h_1 for the image of g_1, and define the order of contact $\nu(g_1,g_2)$ to be the order (lowest non-zero derivative at 0) of $g_2^*(h_1)$.

(i) Find ν for the bi-germs

$$\text{(a)} \begin{cases} s \mapsto (s,0) \\ t \mapsto (t,r(t)) \end{cases} \quad \text{(b)} \begin{cases} s \mapsto (s^2,s^3) \\ t \mapsto (t,r(t)) \end{cases} \quad \text{(c)} \begin{cases} s \mapsto (s,r(s)) \\ t \mapsto (t^2,t^3) \end{cases}$$

(ii) Show

(a) The definition of $\nu(g_1,g_2)$ is independent of choice of h_1;

(b) $\nu(g_2,g_1) = \nu(g_1,g_2)$;

(c) If g_1 and g_2 are both immersions then g is $\mathscr{A}$-equivalent to the germ

$$g^{(\nu)} : \begin{cases} s \mapsto (s,0) \\ t \mapsto (t,t^\nu) \end{cases}$$

6. If $f : (\mathbb{C}^n,S) \to (\mathbb{C}^{n+1},0)$ is a bi-germ of immersions whose images meet tangentially at 0 then in suitable coordinate f can be written in the form

$$\begin{cases} x^{(1)} \mapsto (x^{(1)},0)) \\ x^{(2)} \mapsto (x^{(2)},h(x^{(2)})) \end{cases} \tag{3.2.8}$$

where $h \in \mathfrak{m}^2$. The calculations of Example 3.2.1 go through unchanged until the line corresponding to (3.2.9). Show that in fact

$$T\mathscr{A}_e f \bigcap \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{O}_{\mathbb{C}^n,0'} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & (h) + J_h \end{pmatrix} \tag{3.2.9}$$

and deduce that $T^1(f) \simeq \mathcal{O}_{\mathbb{C}^n,0'} / (h) + J_h$. We call the function h the *separation function* of the bi-germ f.

7. Show that bi-germs of the form (3.2.8) are $\mathscr{A}$-equivalent to one another if and only if their separation functions are $\mathscr{K}$-equivalent.

8. (Assumes familiarity with the Milnor fibre of an isolated hypersurface singularity)

   (a) Show that the image X_t of a stable perturbation of a bi-germ of the form (3.2.8) has reduced homology satisfying

   $$\widetilde{H}_q(X_t) = \begin{cases} \mathbb{Z}^{\mu(h)} & \text{if } q = n \\ 0 & \text{otherwise} \end{cases} \tag{3.2.10}$$

   (b) Show further that this image is homotopy-equivalent to a wedge of $\mu(h)$ n-spheres.

3.3 Infinitesimal stability implies stability

Next, we prove some lemmas which are needed in order to prove Theorem 3.0.12. The first is a Thom-Levine type result similar to Corollary 2.1.1, namely, an infinitesimal condition for the triviality of a 1-parameter unfolding. Given any k, we consider germs of vector fields Z on $\mathbb{C}^k \times \mathbb{C}$ such that $Z(t) = 1$, that is, of the form

$$Z = \sum_{i=1}^{k} Z_i(x,t)\frac{\partial}{\partial x_i} + \frac{\partial}{\partial t}. \tag{3.3.1}$$

Lemma 3.3.1. *A 1-parameter unfolding F is trivial if and only if there exist germs of vector fields X on $(\mathbb{C}^n \times \mathbb{C}, S \times \{0\})$ and Y on $(\mathbb{C}^p \times \mathbb{C}, 0)$ such that $X(t) = 1$, $Y(t) = 1$ and*

$$dF \circ X = Y \circ F.$$

Proof. Suppose first that F is trivial. Then there exist diffeomorphisms Φ, Ψ which are unfoldings of the identity in $\mathbb{C}^n, \mathbb{C}^p$ respectively, such that $F = \Psi \circ G \circ \Phi^{-1}$, where $G = f \times \mathrm{id}$. We define X, Y as the vector fields given by

$$X = d\Phi \circ \frac{\partial}{\partial t} \circ \Phi^{-1}, \quad Y = d\Psi \circ \frac{\partial}{\partial t} \circ \Psi^{-1}. \tag{3.3.2}$$

Since Φ, Ψ are unfoldings of the identity, we have $X(t) = 1$, $Y(t) = 1$. We use the fact that

$$dG \circ \frac{\partial}{\partial t} = \frac{\partial}{\partial t} \circ G,$$

and the chain rule:

$$\begin{aligned} dF \circ X &= dF \circ d\Phi \circ \frac{\partial}{\partial t} \circ \Phi^{-1} = d(F \circ \Phi) \circ \frac{\partial}{\partial t} \circ \Phi^{-1} = d(\Psi \circ G) \circ \frac{\partial}{\partial t} \circ \Phi^{-1} \\ &= d\Psi \circ dG \circ \frac{\partial}{\partial t} \circ \Phi^{-1} = d\Psi \circ \frac{\partial}{\partial t} \circ G \circ \Phi^{-1} = Y \circ \Psi \circ G \circ \Phi^{-1} = Y \circ F. \end{aligned}$$

Conversely, assume there exist vector fields X, Y such that $X(t) = 1$, $Y(t) = 1$ and $dF \circ X = Y \circ F$. By taking the integral flows of X, Y we define Φ, Ψ as the unique diffeomorphisms which satisfy equations (3.3.2). Since $X(t) = 1$ and $Y(t) = 1$, it follows that Φ, Ψ are unfoldings of the identity on $\mathbb{C}^n, \mathbb{C}^p$ respectively.

We define now $G = \Psi^{-1} \circ F \circ \Phi$. Again, by the chain rule we have:

$$\begin{aligned} dG \circ \frac{\partial}{\partial t} &= d(\Psi^{-1} \circ F \circ \Phi) \circ \frac{\partial}{\partial t} = d\Psi^{-1} \circ dF \circ d\Phi \circ \frac{\partial}{\partial t} = d\Psi^{-1} \circ dF \circ X \circ \Phi \\ &= d\Psi^{-1} \circ Y \circ F \circ \Phi = d\Psi^{-1} \circ Y \circ \Psi \circ G = \frac{\partial}{\partial t} \circ G. \end{aligned}$$

If the unfolding G is written as $G(x,t) = (\tilde{g}(x,t), t)$, then the above condition means that $\partial\tilde{g}/\partial t = 0$. Thus, G is the constant unfolding. □

If Z is a germ of vector field in $\mathbb{C}^k \times \mathbb{C}$ as in (3.3.1), then we write

$$\tilde{Z} = Z - \frac{\partial}{\partial t} = \sum_{i=1}^{k} Z_i(x,t) \frac{\partial}{\partial x_i}.$$

We can see $\tilde{Z}$ as a vector field along the projection $\pi_k : \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^k$ onto the first factor. We can also see Z as a time-dependent vector field on $\mathbb{C}^k$: given a representative of Z in an open neighbourhood $U \times D \subset \mathbb{C}^k \times \mathbb{C}$, for each $t \in D$, Z_t is the vector field on U given by $(Z_t)_x = \tilde{Z}_{(x,t)}$.

Let $F : (\mathbb{C}^n \times \mathbb{C}, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}, 0)$ be a 1-parameter unfolding given by $F(x,t) = (\tilde{f}(x,t), t)$. The infinitesimal condition $dF \circ X = Y \circ F$ can be written now in matrix notation as:

$$\begin{pmatrix} 1 & 0 \\ \frac{\partial \tilde{f}}{\partial t} & df_t \end{pmatrix} \begin{pmatrix} 1 \\ X_t \end{pmatrix} = \begin{pmatrix} 1 \\ Y_t \circ f_t \end{pmatrix}$$

which turns out to be equivalent to:

$$\frac{\partial \tilde{f}}{\partial t} + df_t \circ X_t = Y_t \circ f_t. \tag{3.3.3}$$

We have two new morphisms:

1. $\tilde{t}F : \theta(\pi_n) \to \theta(\tilde{f})$, defined by $\tilde{t}F(\tilde{X}) = df_t \circ X_t$, and
2. $\tilde{\omega}F : \theta(\pi_p) \to \theta(\tilde{f})$, defined by $\tilde{\omega}F(\tilde{Y}) = Y_t \circ f_t$.

Finally, we set:

$$\widetilde{T}^1(F) = \frac{\theta(\tilde{f})}{\tilde{t}F(\theta(\pi_n)) + \tilde{\omega}F(\theta(\pi_p))}.$$

This is an $\mathcal{O}_{\mathbb{C}^p \times \mathbb{C},0}$-module via F, in the same way that $T^1(f)$ is an $\mathcal{O}_{\mathbb{C}^p,0}$-module via f.

Observe that $\tilde{T}^1(\tilde{f})$ is just a version of $T^1(f)$ with the additional variable t. Indeed, $\theta(\pi_n)$, $\theta(\pi_p)$ and $\theta(\tilde{f})$ can be viewed as the sets of vector fields in $\theta_{\mathbb{C}^n\times\mathbb{C},S\times\{0\}}$, $\theta_{\mathbb{C}^p\times\mathbb{C},(0,0)}$ and $\theta(F)$ whose $\partial/\partial t$ component is equal to zero.

In particular,

$$\tilde{T}^1(F)/\{t\}\tilde{T}^1(F) = T^1(f). \tag{3.3.4}$$

Lemma 3.3.2. *For any* 1*-parameter unfolding* F *of* f*, we have:*

$$T^1(f) = 0 \Longleftrightarrow \widetilde{T}^1(F) = 0.$$

Proof. The implication from right to left is immediate from (3.3.4).

To see the converse, we define

$$M := \frac{\theta(\tilde{f})}{\tilde{t}F(\theta(\pi_n))}, \quad M_0 := \frac{M}{\{t\}\cdot M} \cong \frac{\theta(f)}{tf(\theta_{\mathbb{C}^n,S})}.$$

We apply the multi-germ version of the Preparation Theorem (Corollary 1.4.2) simultaneously to M, F and M_0, f. Note that both M and M_0 are finitely generated over $\mathcal{O}_{\mathbb{C}^n\times\mathbb{C},S\times\{0\}}$ and $\mathcal{O}_{\mathbb{C}^n,S}$ respectively (since in fact $\theta(\tilde{f}) \cong (\mathcal{O}_{\mathbb{C}^n\times\mathbb{C},S\times\{0\}})^p$ and $\theta(f) \cong (\mathcal{O}_{\mathbb{C}^n,S})^p$).

If $T^1(f) = 0$, then

$$\theta(f) = tf(\theta_{\mathbb{C}^n,S}) + \omega f(\theta_{\mathbb{C}^p,0}) = tf(\theta_{\mathbb{C}^n,S}) + \mathcal{O}_{\mathbb{C}^p,0}\cdot\{e_1,\dots,e_p\},$$

which is equivalent to the fact that M_0 is generated over $\mathcal{O}_{\mathbb{C}^p,0}$ by the classes $\{\overline{e}_1,\dots,\overline{e}_p\}$. It follows that $M_0/(f^*\mathfrak{m}_{\mathbb{C}^p,0}M_0)$ is generated over $\mathbb{R}$ by $\{\overline{e}_1,\dots,\overline{e}_p\}$. However,

$$\frac{M}{F^*\mathfrak{m}_{\mathbb{C}^p\times\mathbb{C},0}M} \cong \frac{M_0}{f^*\mathfrak{m}_{\mathbb{C}^p,0}M_0},$$

so by the Preparation Theorem M also is generated (over $\mathcal{O}_{\mathbb{C}^p\times\mathbb{C},0}$) by $\{\overline{e}_1,\dots,\overline{e}_p\}$. Again this is equivalent to

$$\tilde{\theta}(F) = \tilde{t}F(\theta(\pi_n)) + \mathcal{O}_{\mathbb{C}^p\times\mathbb{C},0}\cdot\{e_1,\dots,e_p\} = \tilde{t}F(\theta(\pi_n)) + \tilde{\omega}F(\theta(\pi_p)),$$

and hence, $\widetilde{T}^1(F) = 0$. □

Remark 3.3.3. By (3.3.4), if $T^1(f) = 0$ then $\tilde{T}^1(F) = \{t\}\tilde{T}^1(F)$, so if we knew that $\tilde{T}^1(F)$ was finitely generated over $\mathcal{O}_{\mathbb{C},0}$ we could conclude immediately from Nakayama's Lemma that $\tilde{T}^1(F) = 0$. In fact the last equality implies $\tilde{T}^1(F) = \mathfrak{m}_{\mathbb{C}^p\times\mathbb{C},(0,0)}\tilde{T}^1(F)$, and so finiteness of $\tilde{T}^1(F)$ over $\mathcal{O}_{\mathbb{C}^p\times\mathbb{C},(0,0)}$ would be enough to imply vanishing of $\tilde{T}^!(F)$. This is in effect what the proof of 3.3.2 shows.

Exercise 3.3.4. *Show that if* $F(x,u) = (\tilde{f}(x,u),u)$ *is a* d*-parameter unfolding of* f *and* $g_1,\dots,g_k \in \theta(\tilde{f})$ *then* $T^1(f) = \mathrm{Sp}_{\mathbb{R}}\{\bar{g}_1,\dots,\bar{g}_k\}$ *if and only if* $\tilde{T}^1(F) = \mathrm{Sp}_{\mathcal{O}_{\mathbb{C}^d,0}}\{g_1,\dots,g_k\}$.

Proof of Theorem 3.0.12. Assume f is stable. For each $Z \in \theta(f)$, we consider the 1-parameter unfolding $F(x,t) = (f_t(x),t)$ given by $f_t = f + tZ$. Because f is stable, F is trivial and by 3.3.1 there exist vector fields X, Y such that $X(t) = 1$, $Y(t) = 1$ and $dF\circ X = Y\circ F$. By (3.3.3) this means that

$$\frac{\partial f_t}{\partial t} + df_t\circ X_t = Y_t\circ f_t,$$

and evaluating at $t = 0$,

$$Z = \frac{\partial f_t}{\partial t}|_{t=0} = -df \circ X_0 + Y_0 \circ f \in T\mathscr{A}_e f.$$

Conversely, assume that $T^1(f) = 0$. We first prove that any 1-parameter unfolding $F(x,t) = (f_t(x), t)$ is trivial. In fact, we know by 3.3.3 that $\widetilde{T}^1(F) = 0$. Hence, there exist vector fields $\tilde{X}$ and $\tilde{Y}$ such that

$$\frac{\partial f_t}{\partial t} = df_t \circ X_t + Y_t \circ f_t.$$

Again by (3.3.3) we have $dF \circ X = Y \circ F$, where $X = -\tilde{X} + \partial/\partial t$ and $Y = \tilde{Y} + \partial/\partial t$, so F is trivial by (3.3.1).

We show now that any r-parameter unfolding $F(x,u) = (f_u(x), u)$ is trivial, by induction on r. We have already proved the case $r = 1$ and we assume the result is true for $r - 1$. Consider the $(r-1)$-parameter unfolding F_1 obtained from F by taking $u_r = 0$. By induction hypothesis, F_1 is trivial and hence, equivalent to $f \times \mathrm{id}$. But this implies that F_1 is also $\mathscr{A}$-equivalent to $f \times \mathrm{id}$ as a map germ and hence $T^1(F_1) = 0$. Since F is a 1-parameter unfolding of F_1 we deduce that F is a trivial unfolding of F_1 and hence, a trivial unfolding of f. □

Suppose $S = \{x_1, \dots, x_r\}$. A natural question is how the stability of $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is related to the stability of each branch $f_i := f|_{(\mathbb{C}^n, x_i)} : (\mathbb{C}^n, x_i) \to (\mathbb{C}^p, 0)$, $i = 1, \dots, r$. To answer this question we need to introduce a new concept.

Definition 3.3.5. For each $i = 1, \dots, r$ we define:

$$\tau(f_i) = \mathrm{ev}\left((\omega f_i)^{-1}(f_i^*(\mathfrak{m}_{\mathbb{C}^p,0})\theta(f_i) + tf_i(\theta_{\mathbb{C}^n,x_i}))\right),$$

where $\mathrm{ev} : \theta_{\mathbb{C}^p,0} \to T_0\mathbb{C}^p$ is the evaluation map given by $\mathrm{ev}(Y) = Y_0$.

The definition of $\tau(f_i)$ makes sense because $f_i^*(\mathfrak{m}_{\mathbb{C}^p,0})\theta(f_i) + tf_i(\theta_{\mathbb{C}^n,x_i})$ is contained in $\theta(f_i)$ and ωf_i maps $\theta_{\mathbb{C}^p,0}$ into $\theta(f_i)$. In fact, $\tau(f_i)$ is a $\mathbb{C}$-vector subspace of $T_0\mathbb{C}^p$.

In the case that f_i is stable, $\tau(f_i)$ has a very nice geometrical interpretation as follows. Let us fix a small enough representative of the germ $f_i : U_i \to V$, where U_i, V are open sets in $\mathbb{C}^n, \mathbb{C}^p$ respectively. Denote by A_i the subset of points $x \in U_i$ such that the germ of f_i at x is $\mathscr{A}$-equivalent to the germ of f_i at x_i. This subset is called the analytic stratum of f_i in the source and has two important properties:

1. A_i is a submanifold of U_i, and
2. the restriction $f_i|_{A_i} : A_i \to V$ is an immersion.

Then $\tau(f_i)$ is equal to $d_{x_i}f_i(T_{x_i}A_i)$, i.e., the tangent space to the image $f_i(A_i) \subset V$ at the origin. We will not prove this property at this moment, but it may give some help to understand the arguments.

We recall the definition of regular intersection of subspaces $E_1, \dots, E_r$ of a vector space F of finite dimension.

Definition 3.3.6. We say that $E_1, \dots, E_r$ have *regular intersection* (or *meet in general position*) if

$$\mathrm{codim}\,(E_1 \cap \dots \cap E_r) = \mathrm{codim}\,(E_1) + \dots + \mathrm{codim}\,(E_r).$$

An equivalent condition to $E_1, \ldots, E_r$ having regular intersection is that the canonical mapping

$$F \longrightarrow (F/E_1) \oplus \cdots \oplus (F/E_r),$$

is surjective. This follows easily from the fact that the kernel is precisely $E_1 \cap \cdots \cap E_r$.

Theorem 3.3.7. *A multi-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is stable if and only if each branch $f_i : (\mathbb{C}^n, x_i) \to (\mathbb{C}^p, 0)$ is stable and $\tau(f_1), \ldots, \tau(f_r)$ have regular intersection.*

Proof. In order to simplify the notation, we set:

$$R = \mathcal{O}_{\mathbb{C}^p,0},\ \mathfrak{m} = \mathfrak{m}_{\mathbb{C}^p,0},\ S = \mathcal{O}_{\mathbb{C}^n,S},\ N = \theta_{\mathbb{C}^p,0},\ M = \theta_{\mathbb{C}^n,S},\ L = \theta(f).$$

Note that $\omega f(\mathfrak{m}N) \subset (f^*\mathfrak{m})L$, hence ωf induces the homomorphism:

$$\overline{\omega} f : T_0\mathbb{C}^p \cong \frac{N}{\mathfrak{m}N} \longrightarrow \frac{L}{(f^*\mathfrak{m})L + tf(M)}.$$

We first show

$$f \text{ is stable if and only if } \overline{\omega} f \text{ is surjective.} \tag{3.3.5}$$

In fact, if f is stable then $T^1(f) = 0$, that is, $\omega f(N) + tf(M) = L$. For any $Z \in L$, there exist $X \in M$ and $Y \in N$ such that $Z = tf(X) + \omega f(Y)$. This gives $\overline{\omega} f([Y]) = [Z]$ and thus, $\overline{\omega} f$ is surjective.

Conversely, suppose now that $\overline{\omega} f$ is surjective. This implies that

$$\omega f(N) + tf(M) + (f^*\mathfrak{m})L = L. \tag{3.3.6}$$

We define $L' = L/tf(M)$ and denote by $\pi : L \to L'$ the canonical projection. Note that L' is a finitely generated S-module (since in fact $L \cong S^p$). Then (3.3.6) may be rewritten as

$$\pi \circ \omega f(N) + (f^*\mathfrak{m})L' = L'. \tag{3.3.7}$$

Considering L' as an R-module via f, (3.3.7) becomes

$$\pi \circ \omega f(N) + \mathfrak{m}L' = L'. \tag{3.3.8}$$

Since $\pi \circ \omega f(N)$ is finitely generated over R, we have that $L'/\mathfrak{m}L'$ is also finitely generated over R, and hence, finitely generated over $\mathbb{C}$. Therefore by the Preparation Theorem 1.4.2, L' is finitely generated over R. Now from (3.3.8) it follows by Nakayama's Lemma that $\pi \circ \omega f(N) = L'$ and hence, $\omega f(N) + tf(M) = L$.

The same argument, applied to each branch f_i, shows that

$$f_i \text{ is stable if and only if } \overline{\omega} f_i : T_0\mathbb{C}^p \longrightarrow \frac{L_i}{(f_i^*\mathfrak{m})L_i + tf_i(M_i)} \text{ is surjective} \tag{3.3.9}$$

where now $M_i = \theta_{\mathbb{C}^n,x_i},\ L_i = \theta(f_i)$. Note that the kernel of $\overline{\omega} f_i$ is $\tau(f_i)$. Thus, if f_i is stable we have

$$\frac{T_0\mathbb{C}^p}{\tau(f_i)} \cong \frac{L_i}{(f_i^*\mathfrak{m})L_i + tf_i(M_i)}. \tag{3.3.10}$$

On the other hand, we also have an isomorphism

$$\frac{L}{(f^*\mathfrak{m})L + tf(M)} \cong \bigoplus_{i=1}^{r} \frac{L_i}{(f_i^*\mathfrak{m})L_i + tf_i(M_i)}, \tag{3.3.11}$$

from which it follows, by (3.3.10), that we can write $\overline{\omega}f$ in the form

$$\overline{\omega}f : T_0\mathbb{C}^p \longrightarrow (T_0\mathbb{C}^p/\tau(f_1)) \oplus \cdots \oplus (T_0\mathbb{C}^p/\tau(f_r)). \tag{3.3.12}$$

It is now immediate that $\overline{\omega}f$ is surjective if and only if each $\overline{\omega}f_i$ is surjective and the $\tau(f_i)$ have regular intersection. By (3.3.5) and (3.3.9), this proves the theorem. $\square$

3.4 The contact tangent space

The examples dealt with so far are somewhat atypical. Calculating $T\mathcal{A}_e f$ is generally rather complicated. Checking that a given map-germ is it stable, however, is made much easier by a theorem of John Mather, which makes use of an auxiliary module known as the *contact tangent space*, and denoted $T\mathscr{K}_e f$, defined by

$$T\mathscr{K}_e f = tf(\theta_{\mathbb{C}^n,0}) + f^* m_{\mathbb{C}^p,0}\theta(f).$$

Note that $f^*m_{\mathbb{C}^p,0}$ is simply the ideal in $\mathcal{O}_{\mathbb{C}^n,0}$ generated by the component functions of f. In fact we have already met $T\mathscr{K}_e f$, though not by that name, in the proof of Theorem 3.3.7 and, implicitly, in the definition of $\tau(f)$ in 3.3.5.

When $p = 1$, $T\mathscr{K}_e f$ is just the ideal $(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ of $\mathcal{O}_{\mathbb{C}^n,0}$. In any case it is always an $\mathcal{O}_{\mathbb{C}^n,0}$-module, which makes calculating with it very much easier than calculating $T\mathscr{A}_e f$. Like $T\mathscr{A}_e f$, $T\mathscr{K}_e f$ is the 'extended' tangent space to the orbit of f under the action of a group, in this case the contact group $\mathscr{K}$, which we will not say anything about yet (the true tangent space here is $T\mathscr{K} f = tf(m_n\theta_n) + f^*(m_p)\theta(f)$.) The role of $T\mathscr{K}_e f$ here does not involve this geometrical interpretation. We will discuss $\mathscr{K}$ in Section 4.

Let $v_1, \ldots, v_p$ be members of a vector space V over a field k. We denote the subspace spanned over k by $v_1, \ldots, v_p$ by $\mathrm{Sp}_k\{v_1, \ldots, v_p\}$.

Mather's theorem is

Theorem 3.4.1. *The following are equivalent:*

1. $T\mathscr{A}_e f = \theta(f)$ *(i.e.* $T^1(f) = 0$*, so* f *is stable).*
2. $T\mathscr{K}_e f + Sp_{\mathbb{C}}\{\partial/\partial y_1, \ldots, \partial/\partial y_t\} = \theta(f)$
3. $T\mathscr{K}_e f + Sp_{\mathbb{C}}\{\partial/\partial y_1, \ldots, \partial/\partial y_t\} + \mathfrak{m}_s^{t+1}\,\theta(f) = \theta(f)$.

Proof. (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3) are trivial, since the left hand sides of the equalities increase from each statement to the next.

To see that (3) $\Longrightarrow$ (2), suppose that (3) holds and let $\alpha_1, \ldots, \alpha_t \in \mathfrak{m}_S$. We will show that $\alpha_1 \cdots \alpha_t \partial/\partial y_i \in T\mathscr{K}_e f + \mathfrak{m}_S^{t+1}\,\theta(f)$. Because every member of $\mathfrak{m}_S^t\,\theta(f)$ is a sum of such elements, it will follow that

$$\mathfrak{m}_S^t\,\theta(f) \subset T\mathscr{K}_e f + \mathfrak{m}_S^{t+1}\,\theta(f),$$

and therefore, by Nakayama's Lemma, that

$$\mathfrak{m}_S^t\,\theta(f) \subset T\mathscr{K}_e f.$$

To see that $\alpha_1\cdots\alpha_t\partial/\partial y_i \in T\mathscr{K}_e f + \mathfrak{m}_S^{t+1}\,\theta(f)$, observe that because, by (3),

$$\dim_{\mathbb{C}}\theta(f)/T\mathscr{K}_e f + \mathfrak{m}_S^{t+1}\,\theta(f) \leq t,$$

the $t+1$ elements

$$\partial/\partial y_i, \alpha_1\partial/\partial y_i, \ldots, \alpha_1\cdots\alpha_t\partial/\partial y_i$$

cannot be linearly independent. Thus there exist $c_0, \ldots, c_t \in \mathbb{C}$, not all zero, such that

$$c_0\partial/\partial y_i + c_1\alpha_1\partial/\partial y_i + \cdots + c_t\alpha_1\cdots\alpha_t\partial/\partial y_i = 0 \tag{3.4.1}$$

in $\theta(f)/T\mathscr{K}_e f + \mathfrak{m}_S^{t+1}\,\theta(f)$. Let c_j be the first of the c_i to be non-zero. Then (3.4.1) can be rewritten as

$$(c_j\alpha_1\cdots\alpha_j + \cdots c_t\alpha_1\cdots\alpha_t)\partial/\partial y_i \in T\mathscr{K}_e f + \mathfrak{m}_S^{t+1}\,\theta(f).$$

The left hand side here is an $\mathcal{O}_S$-unit times $\alpha_1\cdots\alpha_j\partial/\partial y_i$, and thus $\alpha_1\cdots\alpha_j\partial/\partial y_i$, and hence $\alpha_1\cdots\alpha_t\partial/\partial Y_i$, are members of $T\mathscr{K}_e f + \mathfrak{m}_S^{t+1}\,\theta(f)$.

To see that (2) $\Longrightarrow$ (1), consider the $\mathcal{O}_{\mathbb{C}^t,0}$ module $M := \theta(f)/tf(\theta_{\mathbb{C}^n,0})$. Now

$$M/\,\mathfrak{m}_t\cdot M = M/f^*\,\mathfrak{m}_t\,M = \frac{\theta(f)}{tf(\theta_{\mathbb{C}^n,0}) + f^*\,\mathfrak{m}_t\,\theta(f)},$$

and by hypothesis this is generated as a $\mathbb{C}$-vector space by the classes of $\partial/\partial y_1, \ldots, \partial/\partial y_t$ in $M/\,\mathfrak{m}_t\cdot M$. It follows by the Preparation Theorem that M is generated as $\mathcal{O}_{\mathbb{C}^t,0}$-module by the classes of $\partial/\partial y_1, \ldots, \partial/\partial y_t$ in M. Now by definition the $\mathcal{O}_{\mathbb{C}^t,0}$ submodule of $\theta(f)$ generated by $\partial/\partial y_1, \ldots, \partial/\partial y_t$ is just $\omega f(\theta_{\mathbb{C}^t,0})$; so from the fact that M is generated over $\mathcal{O}_{\mathbb{C}^t,0}$ by the classes of $\partial/\partial y_1, \ldots, \partial/\partial y_t$, we deduce simply that $\theta(f) = tf(\theta_{\mathbb{C}^n,0}) + \omega f(\theta_{\mathbb{C}^t,0})$ – i.e. that $T^1(f) = 0$. □

Corollary 3.4.2. *Whether or not $f : (\mathbb{C}^s, 0) \to (\mathbb{C}^t, 0)$ is stable is determined by its $t+1$-jet.*

Proof. If $j^{t+1}f = j^{t+1}g$ then

$$T\mathscr{K}_e f + \mathfrak{m}_S^{t+1}\,\theta(f) = T\mathscr{K}_e g + \mathfrak{m}_S^{t+1}\,\theta(g).$$

So (3) holds for f if and only if it holds for g. □

Example 3.4.3. (1) We apply Theorem 3.4.1 to the map-germ f of Example 3.1.1(1). We have

$$\begin{aligned} T\mathscr{K}_e f &= tf(\theta_{\mathbb{C}^2,0}) + f^*m_{\mathbb{C}^3,0}\theta(f) \\ &= \mathcal{O}_{\mathbb{C}^2,0}\cdot\{\partial f/\partial x, \partial f/\partial y\} + (x, y^2)\theta(f) \\ &= \mathcal{O}_{\mathbb{C}^2,0}\cdot\left\{\begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ 2y \\ x \end{pmatrix}\right\} + \begin{pmatrix} (x, y^2) \\ (x, y^2) \\ (x, y^2) \end{pmatrix} \end{aligned}$$

You can easily show that $T\mathscr{K}_e f + \mathrm{Sp}_{\mathbb{C}}\{\partial/\partial y_1, \partial/\partial y_2, \partial/\partial y_3\} = \theta(f)$; in particular, since (x, y^2) contains the square of the maximal ideal of $\mathcal{O}_{\mathbb{C}^2,0}$, it's necessary only to check for terms of degree 0 and 1.

(2) The same theorem can be used to show that the map-germs

1. $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ defined by

$$f(x_1, x_2, x_3) = (x_1, x_2, x_3^4 + x_1 x_3^2 + x_2 x_3)$$

2. $f : (\mathbb{C}^4, 0) \to (\mathbb{C}^5, 0)$ defined by

$$f(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4^3 + x_1 x_4, x_2 x_4^2 + x_3 x_4)$$

3. $f : (\mathbb{C}^5, 0) \to (\mathbb{C}^6, 0)$ defined by

$$f(x, y, a, b, c, d) = (x^2 + ay, xy + bx + cy, y^2 + dx, a, b, c, d)$$

are stable. These are left as **Exercises**.

3.5 Construction of stable germs as unfoldings

The reader will note that each of the germs listed in Example 3.4.3(2) is itself an unfolding of a germ of rank 0 (i.e. whose derivative at 0 vanishes). Of course, thanks to the inverse function theorem *any* germ can be put in this form, in suitable coordinates. But in fact there is a general procedure for finding *all* stable map-germs as unfoldings of lower-dimensional germs of rank zero, due to Mather in [Mat69b]. The procedure is the following:

1. Given $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ of rank 0, calculate $T\mathscr{K}_e f$. Because f_0 has rank 0 at 0, $T\mathscr{K}_e f_0 \subset \mathfrak{m}_n\, \theta(f_0)$.

2. Find a basis for the quotient $\mathfrak{m}_n\, \theta(f)/T\mathscr{K}_e f$.

3. If $g_1, \ldots, g_d \in \theta(f)$ project to a basis for the quotient $\mathfrak{m}_n\, \theta(f)/T\mathscr{K}_e f$ then the unfolding $F : (\mathbb{C}^n \times \mathbb{C}^d, (0,0)) \to (\mathbb{C}^p \times \mathbb{C}^d, (0,0))$ defined by

$$F(x, u_1, \ldots, u_d) = (f(x) + \sum_j u_j g_j(x), u_1, \ldots, u_d) \tag{3.5.1}$$

   is stable.

Exercise 3.5.1. *Apply this procedure starting with* $f(x, y) = (x^2, y^2)$.

In Chapter 4 we explain this construction.

3.6 Gaffney's calculation of $\mathcal{A}_e$ tangent spaces

An ingenious result, due to Terry Gaffney, and extending Mather's, allows one to transform a guess for $T\mathscr{A}_e f$, (based perhaps on a calculation modulo some power of the maximal ideal (i.e. ignoring all terms of degree higher than some fixed k)) into a rigorous calculation.

Theorem 3.6.1. *Suppose that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is a map-germ such that*

$$T\mathscr{K}_e f \supset \mathfrak{m}^{\ell}_{\mathbb{C}^n,0}\,\theta(f)$$

and $C \subset \theta(f)$ is an $\mathcal{O}_{\mathbb{C}^p,0}$-submodule such that

$$C \supset \mathfrak{m}^{k}_{\mathbb{C}^n,0}\,\theta(f)$$

(where $k > 0$). Then

$$C = T\mathscr{A}_e f \qquad \Longleftrightarrow \qquad C = T\mathscr{A}_e f + f^*\,\mathfrak{m}_{\mathbb{C}^p,0}\,C + \mathfrak{m}^{k+\ell}_{\mathbb{C}^n,0}\,\theta(f).$$

A proof can be found in [Mon85, 3:2].

Exercise 3.6.2. Find the smallest integer ℓ such that $T\mathscr{K}_e f \supset \mathfrak{m}^{\ell}_2\,\theta(f)$ when f is the map germ of Example 3.1.1(1); ditto when f is the germ of Example 3.1.1(3).

Remark 3.6.3. Theorem 3.6.1 can be used to justify calculating $T\mathscr{A}_e f$ at the level of formal power series, as we did in Example 3.1.1(1), provided this calculation yields an estimate C for (the formal version of) $T\mathscr{A}_e f$, which contains the formal version of $\mathfrak{m}^{k}_{\mathbb{C}^n,S}\,\theta(f)$ for some finite k. For since $T\mathscr{A}_e f \subseteq T\mathscr{K}_e f + \mathbb{R}\cdot\{\partial/\partial y_1, \ldots, \partial/\partial y_p\}$, it follows that $T\mathscr{K}_e f + \mathfrak{m}^{\ell+1}\,\theta(f)$ has finite codimension (for every $\ell \in \mathbb{N}$) and therefore $T\mathscr{K}_e f + \mathfrak{m}^{\ell+1}\,\theta(f) \supseteq \mathfrak{m}^{\ell}\,\theta(f)$ for some finite ℓ. Because $T\mathscr{K}_e f$ is an $\mathcal{O}_{\mathbb{C}^n,S}$-module (unlike $T\mathscr{A}_e f$), it follows by Nakayama's Lemma that $T\mathscr{K}_e f \supseteq \mathfrak{m}^{\ell}\,\theta(f)$. Therefore by 3.6.1, the calculation of $T\mathscr{A}_e f$ can be carried out modulo $\mathfrak{m}^{k+\ell}\,\theta(f)$, so that we may neglect the difference between formal and convegent power series, and between formal power series and C^∞ germs.

3.7 Geometric criterion for finite $\mathscr{A}_e$-codimension

We give here a very nice and easy geometric criterion due to Mather and Gaffney for a multi-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ to have finite $\mathscr{A}_e$-codimension. Roughly speaking, a multi-germ has finite $\mathscr{A}_e$-codimension if and only if it has an isolated instability. In order to simplify the rigorous version of this property, we recall first the notion of locally stable mapping.

Definition 3.7.1. Let $f : M \to N$ be a holomorphic map between complex manifolds and denote by $C(f)$ its critical set. We say that f is *locally stable* if for any $y \in N$, the set $S = f^{-1}(y) \cap C(f)$ is finite and the multi-germ $f : (M, S) \to (N, y)$ is stable.

By abuse of language we just call a mapping stable instead of locally stable, whenever there is no possible confusion. We recall that the critical set $C(f)$ is by definition the set of points $x \in M$ such that $d_x f$ is not surjective, so that $C(f) = M$ when $\dim M < \dim N$, but $C(f)$ coincides with the singular set $\Sigma(f)$ otherwise. The fact that we only need to check the stability of the mapping at the critical points is based on the following easy exercise.

Exercise 3.7.2. Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a holomorphic multi-germ. Show that f is stable if and only if the germ at the smaller set, $S \cap \mathcal{C}(f)$, namely $f : (\mathbb{C}^n, S \cap C(f)) \to (\mathbb{C}^p, 0)$, is stable.

The proof of the geometric criterion is based on the Nullstellensatz for coherent sheaves 3.7.5. In order to use these techniques, we need to set some notation. Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a holomorphic multi-germ and fix a representative $f : U \to V$, where U, V are open sets in $\mathbb{C}^n, \mathbb{C}^p$ respectively. We consider the following sheaves and morphisms:

- $\theta_U, \theta(f)$ are the sheaves of $\mathcal{O}_U$-modules of vector fields on U and of vector fields along f, respectively and $tf : \theta_U \to \theta(f)$ is the morphism defined by composition with df.

- θ_V is the sheaf of $\mathcal{O}_V$-modules of vector fields on V and $\omega f : \theta_V \to f_*\theta(f)$ is the induced morphism defined by composition with f.

Lemma 3.7.3. *If $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ has finite $\mathscr{A}_e$-codimension, then there is a small enough representative $f : U \to V$ such that $f^{-1}(0) \cap C(f) \subset S$.*

Proof. By the hypothesis that $\dim_{\mathbb{C}} T^1(f) < \infty$, there exist $h_1, \ldots, h_r \in \mathcal{O}_{\mathbb{C}^n,S}$ such that

$$T\mathscr{A}_e(f) + \mathrm{Sp}_{\mathbb{C}}\{h_1, \ldots, h_r\} = \theta(f).$$

But since $\omega f(\theta_{\mathbb{C}^p,0}) \subset (f^*\mathfrak{m}_{\mathbb{C}^p,0})\theta(f) + \{\partial/\partial y_1, \ldots, \partial/\partial y_p\}$, then we also deduce

$$T\mathscr{K}_e(f) + \mathrm{Sp}_{\mathbb{C}}\{h_1, \ldots, h_r, \partial/\partial y_1, \ldots, \partial/\partial y_p\} = \theta(f),$$

and hence, $\dim_{\mathbb{C}}\theta(f)/T\mathscr{K}_e(f) < \infty$. Moreover, for each $x \in S$ we also have $\dim_{\mathbb{C}}\theta(f_x)/T\mathscr{K}_e(f_x) < \infty$, where f_x denotes the mono-germ of f at x.

Let $f : U \to V$ be any representative of the multi-germ and consider the following sheaf of $\mathcal{O}_U$-modules:

$$\mathscr{S} = \frac{\theta(f)}{tf(\theta_U) + (f^*\,\mathfrak{m}_0)\theta(f)},$$

where $\mathfrak{m}_0$ is the ideal sheaf in $\mathcal{O}_V$ of functions vanishing at 0. Since all sheaves appearing the definition of $\mathscr{S}$ are coherent (in fact, $\theta_U, \theta(f)$ are locally free of rank n, p respectively), $\mathscr{S}$ is also coherent. On the other hand, the stalk of $\mathscr{S}$ at $x \in U$ is

$$\mathscr{S}_x = \frac{\theta(f_x)}{tf_x(\theta_{U,x}) + (f_x^*\mathfrak{m}_{V,0})\theta(f_x)} = \frac{\theta(f_x)}{T\mathscr{K}_e(f_x)}.$$

Hence, for each $x \in S$, $\dim_{\mathbb{C}}\mathscr{S}_x < \infty$, and by the Nullstellensatz 3.7.5, we can find an open neighbourhood U' of S in U such that

$$\mathrm{Supp}\mathscr{S} \cap U' \subset S.$$

To finish the proof, it only remains to show that $\mathrm{Supp}\mathscr{S} = C(f) \cap f^{-1}(0)$. In fact, if $x \in C(f) \cap f^{-1}(0)$, then $f(x) = 0$ and $d_x f$ is not surjective. This implies that $f_x^*\mathfrak{m}_{V,0} \subset \mathfrak{m}_{U,x}$ and that $\partial/\partial y_i \notin tf_x(\theta_{U,x})$ for some $i = 1, \ldots, p$. So, we have $\mathscr{S}_x \neq 0$.

Conversely, if $x \notin C(f) \cap f^{-1}(0)$, then either $f(x) \neq 0$ or $d_x f$ is surjective. Now either $f_x^*\mathfrak{m}_{V,0} = \mathcal{O}_{U,x}$ or $\partial/\partial y_i \in tf_x(\theta_{U,x})$ for all $i = 1, \ldots, p$ and hence, $\mathscr{S}_x = 0$ in both cases. $\square$

Theorem 3.7.4. *[Mather-Gaffney criterion] A holomorphic multi-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ has finite $\mathscr{A}_e$-codimension if and only if there is a small enough representative $f : U \to V$ such that:*

1. *$f^{-1}(0) \cap C(f) \subset S$;*

2. *the restriction $f : U \setminus S \to V \setminus \{0\}$ is stable.*

Proof. Suppose first that f has finite $\mathscr{A}_e$-codimension. By Lemma 3.7.3, we can assume from the beginning that there is a small enough representative satisfying condition (1). Then, by shrinking the neighborhoods U, V if necessary, we can also assume that the restriction to the critical set $f : C(f) \to V$ is finite (i.e., closed and finite-to-1).

We define

$$\mathscr{S} = f_*\left(\left.\frac{\theta(f)}{tf(\theta_U)}\right|_{C(f)}\right),$$

which is now a coherent sheaf on V by the Finite Mapping Theorem 3.7.6. Let $\overline{\omega f} : \theta_V \to \mathscr{S}$ be the morphism induced from ωf and consider its cokernel, i.e.,

$$\mathscr{T} = \frac{\mathscr{S}}{\overline{\omega f}(\theta_V)}.$$

Then $\mathscr{T}$ also is coherent, and for each $y \in V$, the stalk at y is, by definition,

$$\mathscr{T}_y = \frac{\mathscr{S}_y}{\omega f(\theta_{V,y})} = \frac{\left(\bigoplus_{x \in S'} \dfrac{\theta(f_x)}{tf_x(\theta_{U,x})}\right)}{\omega f(\theta_{V,y})} \cong \frac{\theta(f_{S'})}{tf_{S'}(\theta_{U,S'}) + \omega f_{S'}(\theta_{V,y})} = T^1(f_{S'}),$$

where $S' = f^{-1}(y) \cap C(f)$ and $f_{S'}$ denotes the multi-germ of f at S'.

Now we prove the equivalence of finite $\mathscr{A}_e$-codimension with condition (2). If

$$\dim_{\mathbb{C}} \mathscr{T}_0 = \dim_{\mathbb{C}} T^1(f_S) < \infty,$$

then by the Nullstellensatz 3.7.5 there is an open set V', $0 \in V' \subset V$, such that $\mathrm{Supp}\mathscr{T} \cap V' \subset \{0\}$. But this means that $\mathscr{T}_y = 0$ for any $y \in V' \setminus \{0\}$ and hence, the restriction of f to $U' \setminus S$ is stable, where $U' = f^{-1}(V')$.

Conversely, assume there is an open set U', $S \subset U' \subset U$, such that the restriction of f to $U' \setminus S$ is stable. Since $f : C(f) \to V$ is finite, there is an open neighbourhood V' of 0, contained in V, such that $f(U' \cap C(f)) = V' \cap f(C(f))$. Then for each $y \in V' \setminus \{0\}$ we have $\mathscr{T}_y = 0$. Thus, $\mathrm{Supp}\mathscr{T} \cap V' \subset \{0\}$ and again by 3.7.5 we get

$$\dim_{\mathbb{C}} \mathscr{T}_0 = \dim_{\mathbb{C}} T^1(f_S) < \infty.$$

$\square$

We need to add these two theorems on coherent sheaves:

Theorem 3.7.5 (Nullstellensatz)**.** *Let $\mathscr{S}$ be a coherent sheaf on a complex space X. Given $x \in X$, the following statements are equivalent:*

1. $\dim_{\mathbb{C}} \mathscr{S}_x < \infty$,
2. *there exists an open neighbourhood U of x in X such that* $\mathrm{Supp}\mathscr{S} \cap U \subset \{x\}$.

Theorem 3.7.6 (Finite Mapping Theorem)**.** *Let $f : X \to Y$ be a finite mapping between complex spaces. If $\mathscr{S}$ is a coherent sheaf on X, then $f_*\mathscr{S}$ is a coherent sheaf on Y.*

Chapter 4

The contact group $\mathscr{K}$

The contact group $\mathscr{K}$ acting on the set of map-germs $(\mathbb{C}^s, 0) \to (\mathbb{C}^t, 0)$ is defined as follows. As a group, $\mathscr{K}$ is the set of diffeomorphisms of $(\mathbb{C}^s \times \mathbb{C}^t, (0,0))$ of the form

$$\Phi(x, y) = (\varphi(x), \psi(x, y))$$

where $\psi(x, 0) = 0$ for all x. It is obvious that $\mathscr{K}$ is a subgroup of $\mathrm{Diff}(\mathbb{C}^s \times \mathbb{C}^t), (0,0)$. It acts on the set of germs of maps $(\mathbb{C}^s, 0) \to (\mathbb{C}^t, 0)$ via its action on their graphs: if $f : (\mathbb{C}^s, 0) \to (\mathbb{C}^t, 0)$ and $\Phi \in \mathscr{K}$ then $\Phi \cdot f$ is the map-germ $(\mathbb{C}^s, 0) \to (\mathbb{C}^t, 0)$ whose graph is $\Phi(\mathrm{graph}(f))$. Since

$$\mathrm{graph}(f) = \{(x, f(x)) : x \in S\},$$

this means that

$$\mathrm{graph}\big((\Phi \cdot f)\big) = \{(\phi(x), \psi(x, f(x)) : x \in S\}$$

and thus

$$(\Phi \cdot f)(\varphi(x)) = \psi(x, f(x)),$$

so that

$$(\Phi \cdot f)(x) = \psi\big(\varphi^{-1}(x), f(\varphi^{-1}(x))\big). \tag{4.0.1}$$

We will see shortly that germs are contact-equivalent if and only if their fibres are isomorphic, and so contact equivalence has a clear geometric significance. Nevertheless its significance for the theory of singularities of mappings goes much further than this. Theorem 3.4.1 has already given a glimpse of this.

Observe that $\mathscr{R}$ and $\mathscr{L}$ (and therefore $\mathscr{R} \times \mathscr{L} = \mathscr{A}$) are are naturally embedded in $\mathscr{K}$: given $\varphi \in \mathscr{R}$ and $\eta \in \mathscr{L}$, define Φ_φ and Φ_η by $\Phi_\varphi(x, y) = (\varphi(x), y)$, $\quad \Phi_\eta(x, y) = (x, \eta(y))$; then by (4.0.1)

$$(\Phi_\varphi \cdot f)(x) = f(\varphi^{-1}(x)), \quad (\Phi_\eta \cdot f)(x) = \eta \circ f(x).$$

We define another subgroup $\mathscr{C}$ of $\mathscr{K}$ to be the set of all those $\Phi = (\varphi, \psi) \in \mathscr{K}$ such that φ is the identity. Thus by (4.0.1), $\Phi = (\mathrm{id}, \psi) \in \mathscr{C}$ acts by

$$(\Phi \cdot f)(x) = \psi(x, f(x)).$$

Proposition 4.0.7. *$\mathscr{K}$ is the semi-direct product of $\mathscr{R}$ and $\mathscr{C}$.*

Proof. First we show that $\mathscr{K} = \mathscr{C}\mathscr{R}$. Given $\Phi = (\varphi, \psi) \in \mathscr{K}$, define $\Phi_\varphi \in \mathscr{R} \subset \mathscr{K}$ by $\Phi_\varphi(x,y) = (\varphi(x), y)$, and $\Phi_1 \in \mathscr{C} \subset \mathscr{K}$ by $\Phi_1(x,y) = (x, \psi(\varphi^{-1}(x), y))$. Then $\Phi = \Phi_1 \circ \Phi_\varphi$.

In view of this, to show that $\mathscr{C}$ is normal, we need only show that if $\Gamma \in \mathscr{C}$ and $\Phi_\varphi \in \mathscr{R} \subset \mathscr{K}$ then

$$\Phi_{\varphi^{-1}}\Gamma\Phi_\varphi \in \mathscr{C}.$$

This is straightforward:

$$\left(\Phi_{\varphi^{-1}}\Gamma\Phi_\varphi\right)(x,y) = \left(\Phi_{\varphi^{-1}}\Gamma\right)(\phi(x), y) = \Phi_{\varphi^{-1}}(\phi(x), \psi(\varphi(x), y)) = (x, \psi(\varphi(x), y)).$$

□

Let $\mathrm{Gl}_t(\mathcal{O})$ be the group of invertible $t \times t$ matrices over $\mathcal{O} = \mathcal{O}_{\mathbb{C}^s,0}$. It acts on the space of germs of maps $(\mathbb{C}^s, 0) \to (\mathbb{C}^t, 0)$: if $A \in \mathrm{Gl}_t(\mathcal{O})$ and $f : (\mathbb{C}^s, 0) \to (\mathbb{C}^t, 0)$ then $(A \cdot f)(x) = A(x)f(x)$. In fact the map $(x, y) \mapsto (x, A(x)y)$ is a diffeomorphism of $(\mathbb{C}^s \times \mathbb{C}^t, (0,0))$ and maps $\mathbb{C}^s \times \{0\}$ to itself, and as such lies in the group $\mathscr{C}$. We will denote by $\mathscr{C}_L$ the subgroup of $\mathscr{C}$ consisting of all such maps.

Proposition 4.0.8. *Map-germs $f, g : (\mathbb{C}^s, 0) \to (\mathbb{C}^t, 0)$ are $\mathscr{C}$-equivalent only if they are $\mathscr{C}_L$-equivalent.*

Proof. Let $\Gamma \in \mathscr{S}$ with $\Gamma(x,y) = (x, \psi(x,y))$, and let ψ have components $\psi_1, \ldots, \psi_t$. Because $\psi(x, 0) = 0$, for each $i = 1, \ldots, t$ we have

$$\psi_i(x,y) = \sum_{j=1}^{t} y_j \psi_{ij}(x,y)$$

for some functions ψ_{ij}. It follows that for any $f : (\mathbb{C}^s, 0) \to (\mathbb{C}^t, 0)$ with components $f_1, \ldots, f_t$,

$$\begin{aligned} (\Gamma \cdot f)(x) = \psi(x, f(x)) &= \left(\psi_1(x, f(x)), \ldots, \psi_t(x, f(x))\right) \\ &= \left(\sum_{j=1}^{t} \psi_{1j}(x, f(x)) f_j(x), \ldots, \sum_{j=1}^{t} \psi_{tj}(x, f(x)) f_j(x)\right). \end{aligned} \tag{4.0.2}$$

Let $a_{ij}(x) = \psi_{ij}(x, f(x))$, define $A \in \mathrm{Gl}_t(\mathcal{O})$ by $A = (a_{ij})$, and let Γ_A be the corresponding element of $\mathscr{C}$. Then by (4.0.2), we have $\Gamma_A \cdot f = \Gamma \cdot f$. Note that $A \in \mathrm{Gl}_t(\mathcal{O})$, i.e. that the matrix $A(0)$ is invertible; this holds because the matrix of the linear isomorphism $d_0\Gamma$ is equal to

$$\begin{pmatrix} I_s & 0 \\ 0 & A(0) \end{pmatrix}.$$

□

It is an odd feature of this proof that the element $\Gamma_A \in \mathscr{C}_L$ that we construct depends on the map-germ f; we have not defined a retraction $\mathscr{C} \to \mathscr{C}_L$.

Proposition 4.0.9. *Let $f, g : (\mathbb{C}^s, 0) \to (\mathbb{C}^t, 0)$ be map germs and suppose that the ideals $(f_1, \ldots, f_t)$ and $(g_1, \ldots, g_t)$ of $\mathcal{O}$ are equal. Then the map-germs f and g are $\mathscr{C}$-equivalent.*

Proof. Because the two ideals are equal, there exist $a_{ij} \in \mathcal{O}$ and $b_{ij} \in \mathcal{O}$, for $1 \leq i, j \leq t$, such that

$$f_i = \sum_j a_{ij} g_j \quad \text{and } g_i = \sum_j b_{ij} f_j \quad \text{for } 1 \leq i \leq t. \tag{4.0.3}$$

Defining matrices $A = (a_{ij})$ and $B = (b_{ij})$ and writing $\mathbf{f}$ and $\mathbf{g}$ for the column vectors $(f_1, \ldots, f_t)^t$ and $(g_1, \ldots, g_t)^t$, (4.0.3) becomes

$$A\mathbf{f} = \mathbf{g} \text{ and } B\mathbf{g} = \mathbf{f},$$

so that $BA\mathbf{f} = \mathbf{f}$ and $AB\mathbf{g} = \mathbf{g}$. Unfortunately, despite this, A and B need not be invertible (consider for example the case where $f_1 = f_2$ and $g_1 = g_2$; it's easy to find non-invertible A and B such that (4.0.3) holds); to find a suitable element of $\mathscr{C}$ transforming $\mathbf{f}$ to $\mathbf{g}$ we modify A to ensure its invertibility.

Lemma 4.0.10. *Let $A_0, B_0 : \mathbb{C}^t \to \mathbb{C}^t$ be linear maps. There exists a linear map $C_0 : \mathbb{C}^t \to \mathbb{C}^t$ such that $A_0 + C_0(I_t - B_0A_0)$ is invertible.*

Proof of Lemma. Let W be a complement to im A_0 in $\mathbb{C}^t$, and choose $Q_0 : \mathbb{C}^t \to \mathbb{C}^t$ such that $Q_0| : \ker A_0 \to W$ is an isomorphism. Define $C_0 = A_0 + Q_0(I_t - B_0A_0)$, where I_t is the $t \times t$ identity matrix. Then C_0 is injective and therefore an isomorphism. □

We apply the lemma by taking A_0 and B_0 to be $A(0)$ and $B(0)$ respectively. Define the $t \times t$ matrix C by $C = A + Q_0(I_t - BA)$. Then $C(0)$ is the matrix C_0 of the lemma, so C is invertible. Clearly $(I_t - BA)$ annihilates $\mathbf{f}$, so $C \cdot \mathbf{f} = \mathbf{g}$, and f and g are $\mathscr{C}$-equivalent (indeed, $\mathscr{C}_L$-equivalent), as required. □

Theorem 4.0.11. *Let $f, g : (\mathbb{C}^s, 0) \to (\mathbb{C}^t, 0)$ be map germs. The following are equivalent:*

1. *the germs $(f^{-1}(0), 0)$ and $(g^{-1}(0), 0)$, with their possibly non-reduced stucture, are isomorphic.*
2. *the map-germs f and g are $\mathscr{K}$-equivalent.*

Proof. (1) $\Longrightarrow$ (2): Let $\varphi : (\mathbb{C}^\ell) \to (\mathbb{C}^\ell, 0)$ induce an isomorphism $(f^{-1}(0), 0) \simeq (g^{-1}(0), 0)$. Then the ideals $\big(f_1, \ldots, f_t\big)$ and $\big((g \circ \varphi)_1, \ldots, (g \circ \varphi)_t\big)$ of $\mathcal{O}_{\mathbb{C}^s,0}$ are equal, and therefore by 4.0.9 the germs f and $g \circ \varphi$ are $\mathscr{C}$-equivalent. It follows that f and g are $\mathscr{K}$-equivalent.

(2) $\Longrightarrow$ (1): Suppose that $\Phi = (\varphi, \psi) \in \mathscr{K}$ transforms the graph of f to that of g. Then $g \circ \varphi$ and f are $\mathscr{C}$-equivalent and hence $\mathscr{C}_L$-equivalent. It follows immediately that the ideals $\big((g \circ \varphi)_1, \ldots, (g \circ \varphi)_t\big)$ and $\big(f_1, \ldots, f_t\big)$ are equal, and thus the (possibly non-reduced) germs $(f^{-1}, 0)$ and $((g \circ \varphi)^{-1}, 0)$ are the same. Thus $(f^{-1}(0), 0)$ and $(g^{-1}(0), 0)$ are isomorphic. □

The quotient $\mathcal{O}_{\mathbb{C}^s,0}/f^*\mathfrak{m}_{\mathbb{C}^t,0}$ is the *local algebra* of the germ f, and denoted by $Q(f)$. It is the algebra of germs on the fibre $f^{-1}(0)$. Theorem 4.0.11 says in effect that germs are $\mathscr{K}$-equivalent if and only if their local algebras are isomorphic.

Exercise 4.0.12. *Given $\mathcal{K}$-equivalent germs f and g, find a natural isomorphism $\theta(f) \to \theta(g)$ which passes to the quotient to define an isomorphism $\theta(f)/T\mathcal{K}_e f \to \theta(g)/T\mathcal{K}_e f$. See Exercise 3.0.14 for hints.*

Theorem 4.0.13. ([Mat69b]) *Stable map-germs $(\mathbb{C}^s, 0) \to (\mathbb{C}^t, 0)$ are $\mathscr{A}$-equivalent if and only if they are $\mathscr{K}$-equivalent, and thus stable germs are classified by the isomorphism classes of their local algebras.*

If $f : (\mathbb{C}^s, 0) \to (\mathbb{C}^t, 0)$ and $s > t$ then $Q(f)$ cannot be finite-dimensional, by the Hauptidealsatz. It is therefore of interest that 4.0.14 can be strengthened as follows. Let $\mathfrak{m}$ be the maximal ideal in $Q(f)$, and for each $k \in \mathbb{N}$, let

$$Q_k(f) = Q(f)/\,\mathfrak{m}^{k+1} = \mathcal{O}_S\,/(f^*\,\mathfrak{m}_T + \mathfrak{m}_S^{k+1})\,\mathcal{O}_S\,.$$

Corollary 4.0.14. ([Mat69b, Theorem A]) *Stable map-germs $f, g : (\mathbb{C}^s, 0) \to (\mathbb{C}^t, 0)$ are $\mathscr{A}$-equivalent if and only $Q_{t+1}(f) \simeq Q_{t+1}(g)$.*

The proof of 4.0.14 from 4.0.13 is very different from the proof (3) $\implies$ (2) in 3.4.1. The stability of f does *not* imply that $f^*\,\mathfrak{m}_T\,\mathcal{O}_S \supset \mathfrak{m}_S^{t+1}$. What is obvious is that $Q_{t+1}(f)$ depends only on the $t+1$-jet of f; for if f and g agree up to degree $t+1$ then

$$(f^*\,\mathfrak{m}_T + \mathfrak{m}_S^{t+2})\,\mathcal{O}_S = (g^*\,\mathfrak{m}_T + \mathfrak{m}_S^{t+2})\,\mathcal{O}_S\,.$$

One can deduce 4.0.14 from 4.0.13 as follows:

1. If $Q_{t+1}(f) \simeq Q_{t+1}(g)$ then there exists a diffeomorphism $\varphi : S \to S$ such that $\varphi^*(f^*\,\mathfrak{m}_T\,\mathcal{O}_S)) + \mathfrak{m}_S^{t+1} = g^*\,\mathfrak{m}_T\,\mathcal{O}_S + \mathfrak{m}_S^{t+1}$. By the argument of Proposition 4.0.9, there exists a matrix $C \in \mathrm{Gl}_t(\mathcal{O}_S)$ such that $C \cdot f \circ \varphi = g \mod \mathfrak{m}_S^{t+1}$. Thus f is $\mathscr{K}$ equivalent to a germ g_1 which agrees with g up to degree $t+1$. Because $j^{t+1}g_1 = j^{t+1}g$, g_1 is stable, by 3.4.2.
2. Stable germs $(\mathbb{C}^s, 0) \to (\mathbb{C}^t, 0)$ are $t+1$-determined for $\mathscr{A}$-equivalence. We will shortly prove this as Theorem 4.0.16. From this, it follows that g and g_1 are $\mathscr{A}$-equivalent. Now by Theorem 4.0.13, g_1 and f are $\mathscr{A}$-equivalent, and the $\mathscr{A}$-equivalence of f and g follows.

In fact Theorem 4.0.16 is used in the proof of Theorem 4.0.13.

In preparation for the proof of Theorem 4.0.16, we need the following result.

Proposition 4.0.15. *If $f : (\mathbb{C}^s, 0) \to (\mathbb{C}^t, 0)$ is stable then $T\mathscr{A}f = T\mathscr{K}f$ and consequently $T\mathscr{A}f \supset \mathfrak{m}_S^{t+1}\,\theta(f)$.*

Proof. To show that

$$tf(\mathfrak{m}_S\,\theta_S) + \omega f(\mathfrak{m}_t\,\theta_T) = tf(\mathfrak{m}_S\,\theta_S) + f^*\,\mathfrak{m}_T\,\theta(f),$$

it is necessary only to show that $f^*\,\mathfrak{m}_T\,\theta(f)$ is contained in the left hand side of this equality. This is easy: because f is stable,

$$f^*\,\mathfrak{m}_T\,\theta(f) = f^*\,\mathfrak{m}_T(tf(\theta_S) + \omega f(\theta_T)) = tf(\mathfrak{m}_T\,\theta_S) + \omega f(\mathfrak{m}_T\,\theta_T) \subset tf(\mathfrak{m}_S\,\theta_S) + \omega f(\mathfrak{m}_T\,\theta_T).$$

$\square$

Theorem 4.0.16. *Suppose $f : (\mathbb{C}^s, 0) \to (\mathbb{C}^t, 0)$ is infinitesimally stable and let $g : (\mathbb{C}^s, 0) \to (\mathbb{C}^t, 0)$ be any germ such that $j^{p+1}f = j^{p+1}g$. Then f and g are $\mathscr{A}$-equivalent.*

Proof. Write $f_u(x) = f(x) + u(g - f)(x)$. Since $j^{t+1}f_u = j^{t+1}f$ for all u, we know from 3.4.2 that the germ at 0 of f_u is infinitesimally stable for all u. Using this, we show that for any fixed representatives of f and g, for each value u_0 of u, there is a neighbourhood U of u_0 in the parameter space $\mathbb{C}$ such that the germs of f_u and f_{u_0} are $\mathscr{A}$-equivalent for all $u \in U$. We refer to this property as the *local $\mathscr{A}$-triviality* of the deformation f_u.

A finite number of such neighbourhoods cover the compact interval $[0,1]$, and it follows by transitivity that $f = f_0 \simeq_{\mathscr{A}} f_1 = g$.

For simplicity of notation, we assume in the following proof that $u_0 = 0$. This does not sacrifice any generality; indeed, by re-baptising f_{u_0} as f, we are able to deduce the general statement from this apparently special case.

Proof of local $\mathscr{A}$-triviality for $u_0 = 0$:

Consider the unfolding $F : (\mathbb{C} \times \mathbb{C}^s, (0,0)) \to (\mathbb{C} \times \mathbb{C}^t, (0,0))$ defined by $F(u,x) = (u, f(x) + u(g(x) - f(x)))$. Denote by $\bar{t}F$ and $\bar{\omega}F$ the obvious homomorphisms $\theta_{P\times S/P} \to \theta(F/P)$ and $\theta_{P\times T/P} \to \theta(F/P)$ obtained from tF and ωF by suppressing mention of the (null) $\partial/\partial u$ component. We extend the elements of θ_S, θ_T and $\theta(f)$ to elements, of the same name, of $\theta_{P\times S/P}$, $\theta_{P\times T/P}$ and $\theta(F/P)$ respectively, whose values at (u,x) and (u,y) are independent of u.

Since $g - f \in \mathfrak{m}_S^{t+2}\,\theta(f)$, we have $\partial F/\partial u \in \mathfrak{m}_S^{t+2}\,\theta(F/P)$. It follows, by the Thom-Levine Theorem, 2.1.3, that we need only show that

$$\mathfrak{m}_S^{p+2}\,\theta(F/P) \;\subseteq\; \bar{t}F(\mathfrak{m}_S\,\theta_{P\times S/P}) + \bar{\omega}F(\mathfrak{m}_T\,\theta_{P\times T/P}) \tag{4.0.4}$$

For suppose that

$$\frac{\partial F}{\partial u} = \bar{t}F(\xi) + \bar{\omega}F(\chi).$$

Then writing

$$\chi = \eta + \partial/\partial u \quad\text{and } \tilde{\chi} = \partial/\partial u - \xi,$$

we obtain

$$tF(\tilde{\chi}) = \omega F(\chi).$$

The integral flow Φ_t of χ has the form

$$\Phi_t(u,y) = (u, \phi_t(u,y))$$

and moreover $\phi_t(u,0) = 0$ for all t, u, since $\eta \in \mathfrak{m}_T\,\theta_{P\times T/P}$. Similarly, the integral flow $\tilde{\Phi}$ of $\tilde{\chi}$ has the form

$$\tilde{\Phi}_t(u,x) = (u, \tilde{\varphi}_t(u,x))$$

with $\tilde{\varphi}_t(u,0) = 0$ for all t, u. By Thom-Levine, we have

$$F \circ \tilde{\Phi}_t(u,x) = \Phi_t \circ F$$

and in particular

$$F \circ \tilde{\Phi}_u(0,x) = \Phi_u(F(0,x)). \tag{4.0.5}$$

Write

$$\Phi_u(0,y) = (u, \varphi_u(y)), \quad \text{and} \;\; \tilde{\Phi}_u(0,x) = (u, \tilde{\varphi}_u(x)).$$

Then from (4.0.5) we get

$$(u, f_u(\tilde{\varphi}_u(x)))) = (u, \varphi_u(f(x)));$$

in other words,

$$f_u \circ \tilde{\varphi}_u = \varphi_u \circ f. \tag{4.0.6}$$

Since $\phi_u(0) = 0$ in T and $\tilde{\varphi}_u(0) = 0$ in S, this means that f_u and f are $\mathscr{A}$-equivalent.

Note that the diffeomorphisms we have constructed are merely germs at $(0,0)$ in $P \times S$ and $P \times T$. By choosing representatives, we obtain a neighbouhood U of 0 in P such that the equation (4.0.6) holds for all $u \in U$.

Now we proceed to prove (4.0.4). Let $\alpha \in \mathfrak{m}_S^{t+2}\,\theta(F/P)$ and let α_0 be the restriction of α to $\{u = 0\}$. Thus $\alpha_0 \in \mathfrak{m}_S^{t+2}\,\theta(f)$ and so there exist $\xi \in \mathfrak{m}_S\,\theta_S$ and $\eta \in \mathfrak{m}_T\,\theta_T$ such that $\alpha_0 = tf(\xi) + \omega f(\eta)$. Note that $\alpha - \alpha_0 = u\alpha_1$ for some $\alpha_1 \in \mathfrak{m}_S^{t+2}\,\theta(F/P)$, so $\alpha - \alpha_0 \in \mathfrak{m}_{P\times T}\,\mathfrak{m}_S^{t+2}\,\theta(F/P)$. Now $\bar{t}F(\xi) - tf(\xi)$ lies in $\mathfrak{m}_{P\times T}\,\mathfrak{m}_S^{t+2}\,\theta(F/P)$ also, for $\partial F/\partial x_i - \partial f/\partial x_i = u\partial(g-f)/\partial x_i \in \mathfrak{m}_{P\times T}\,\mathfrak{m}^{t+1}\,\theta(F/P)$, and the components of ξ lie in $\mathfrak{m}_S$. It is easy to see that $\bar{\omega}F(\eta) - \omega f(\eta) \in \mathfrak{m}_{P\times T}\,\mathfrak{m}_S^{t+2}\,\theta(F/P)$.

It follows that

$$\alpha = \alpha_0 + u\alpha_1 = \bar{t}F(\xi) + \bar{\omega}F(\eta) + u\alpha_1 \in \bar{t}F(\mathfrak{m}_S\,\theta_{P\times S/P}) + \bar{\omega}F(\theta_{P\times T/P}) + \mathfrak{m}_{P\times T}\,\mathfrak{m}_S^{t+2}\,\theta(F/P);$$

thus

$$\mathfrak{m}_S^{t+2}\,\theta(F/P) \subseteq \bar{t}F(\mathfrak{m}_S\,\theta_{P\times S/P}) + \omega F(\mathfrak{m}_T\,\theta_{P\times T/P}) + \mathfrak{m}_{P\times T}\,\mathfrak{m}_S^{t+2}\,\theta(F/P). \tag{4.0.7}$$

The last line invites the application of Nakayama's Lemma in the form 1.2.2, except that if $s > t$ then $\mathfrak{m}_S^{p+2}\,\theta(F/P)$ is not a finitely generated $\mathcal{O}_{P\times T}$-module.

To circumvent this difficulty we project the inclusion (4.0.7) into $M := \theta(F/P)/\bar{t}F(\mathfrak{m}_S\,\theta_{P\times S/P})$. To spare the notation, write $Q := \bar{t}F(\mathfrak{m}_S\,\theta_{P\times S/P})$. From (4.0.7) we obtain

$$\frac{\mathfrak{m}_S^{t+2}\,\theta(F/P) + Q}{Q} \subseteq \frac{Q + \omega F(\mathfrak{m}_T\,\theta_{P\times T/P})}{Q} + \mathfrak{m}_{P\times T}\,\frac{\mathfrak{m}_S^{t+2}\,\theta(F/P) + Q}{Q}. \tag{4.0.8}$$

Now M is a finitely generated $\mathcal{O}_{P\times T}$ module. For it is a finitely generated $\mathcal{O}_{P\times S}$-module, and moreover

$$\begin{aligned} \dim_{\mathbb{C}} \frac{M}{F^*(\mathfrak{m}_{P\times T})M} &= \frac{\theta(F/P)}{\bar{t}F(\mathfrak{m}_S\,\theta_{P\times S/P}) + F^*\,\mathfrak{m}_{P\times T}\,\theta(F/P)} \\ &\simeq \frac{\theta(f)}{tf(\mathfrak{m}_S\,\theta_S) + \omega f(\mathfrak{m}_T\,\theta_T)}, \end{aligned}$$

and by Proposition 4.0.15, the dimension of the last quotient is less than or equal to the dimension of $\theta(f)/\,\mathfrak{m}_S^{t+1}\,\theta(F)$, and is therefore finite. By the Preparation Theorem 1.4.1, this implies that M is finitely generated over $\mathcal{O}_{P\times T}$.

It follows that the left hand side of the inclusion (4.0.8), is finitely generated over $\mathcal{O}_{P\times T}$; for if $m_1, \ldots, m_n$ generate M, then the left hand side of (4.0.8) is generated by elements $x^c m_i$, where $i = 1, \ldots, n$ and x^c runs over all monomials of degree $t+2$ in $x_1, \ldots, x_s$. We can now conclude, by Nakayama's Lemma, that

$$\frac{\mathfrak{m}_S^{t+2}\,\theta(F/P) + Q}{Q} \subseteq \frac{Q + \omega F(\mathfrak{m}_T\,\theta_{P\times T/P})}{Q}$$

and therefore that (4.0.4) holds. □

Remarks on the proof Theorem 4.0.16 uses the "small increment" method introduced in the proof of Theorem 2.1.5. One starts with a statement concerning the tangent space $T\mathscr{G}f$ (where $\mathscr{G} = \mathscr{R}, \mathscr{A}$ or $\mathscr{K}$), of the form

$$T\mathscr{G} \supset \mathfrak{m}_S^k\,\theta(f) \tag{4.0.9}$$

for some k, and then shows using Nakayama's Lemma that if F is an unfolding or deformation of f on a single parameter, u, for which $\partial F/\partial u|_{\{u=0\}} \in T\mathscr{G}f$, then $\partial F/\partial u$ is contained in the parametrised version of $T\mathscr{G}f$, $\mathfrak{m}_S(\partial F/\partial x_1, \ldots, \partial F/\partial x_s)$ in the case of Theorem 2.1.5, and $\bar{t}F(\theta_{P\times S/P}) + \bar{\omega}F(\mathfrak{m}_T\,\theta_{P\times T/P})$ in the case of Theorem 4.0.16.

From this it follows by the Thom-Levine Theorem that in any representative, f_u is $\mathscr{G}$-equivalent to f for sufficiently small u. This step works in many different circumstances. To prove the stronger result, that f is not merely equivalent to f_u for u sufficiently close to zero, but to f_1, we have to show that the first step can be applied for each fixed value of $u_0 \in [0,1]$ – that f_u is $\mathscr{G}$-equivalent to f_{u_0} for all u sufficiently close to u_0. This requires showing that for any u, the original estimate (4.0.9) holds with f_u in place of f. In the case of Theorem 2.1.5, this had to be done by an additional argument, which we left to the reader, as Exercise 2.1.7. In the proof we have just finished, the extra step was not needed, or, rather, had been taken care of before the proof began. The estimate (4.0.9) in this case was that $T\mathscr{A} \supset \mathfrak{m}_S^{t+1}\,\theta(f)$, which follows from the stability of f (Proposition4.0.15). If $j^{t+2}g = j^{t+2}f$ and $f_u = f + t(g-f)$ then f_u is infinitesimally stable for all u, by Corollary 3.4.2, so that (4.0.9) holds for f_u for all u.

Exercises 4.0.17. Since the isomorphism type of the local algebra Q_f determines f up to contact equivalence, algebraic properties of $Q(f)$ must reflect contact-invariant properties of f, and one should be able to determine invariants of f from $Q(f)$ alone.

1. Let $f : (\mathbb{C}^s, 0) \to (\mathbb{C}^t, 0)$. Show that the rank of $d_0 f$ is contact-invariant.

2. Characterise the local algebra $Q(f)$ when f is an immersion, and when f is a submersion.

3. How can one determine the rank of $d_0 f$ from $Q(f)$? There are several correct answers here; find one involving the dimension of a certain quotient.

4.1 Proof that stable map-germs are classified by their local algebra

The $p+1$-determinacy of stable germs reduces questions about their classification under the group $\mathscr{A}$ to questions about the classification of $(p+1)$-jets in $J^{p+1}(n,p)$ under the action of $\mathscr{A}^{(p+1)}$. Call a k-jet *stable* if it is the k-jet of a stable map-germ. From Proposition 4.0.15 we know that the tangent space to the $\mathscr{A}^{(p+1)}$ orbit of a stable jet is equal to the tangent space to its $\mathscr{K}^{(p+1)}$-orbit. Both are the orbits of algebraic groups acting algebraically, and it follows that both are manifolds. Since $\mathscr{A}$-equivalence implies $\mathscr{K}$-equivalence, for any jet z we have

$$\mathscr{A}^{(p+1)}z \subset \mathscr{K}^{(p+1)}z. \tag{4.1.1}$$

and the equality of tangent spaces shows that in fact $\mathscr{A}^{(p+1)}z$ is open in $\mathscr{K}^{(p+1)}z$. The two orbits are not in general equal; indeed, a stable germ (or jet) may be $\mathscr{K}$-equivalent, but cannot be $\mathscr{A}$-equivalent, to an unstable one. We will show that, denoting the set of stable jets by $\mathrm{St}^{(p+1)}$, for any stable jet we have

$$\mathscr{A}^{(p+1)}z = \mathscr{K}^{(p+1)}z \cap \mathrm{St}^{(p+1)}. \tag{4.1.2}$$

In the complex case, this will follow easily from the following lemma.

Lemma 4.1.1. *If $z \in St^{(p+1)}$ then $\mathscr{A}^{(p+1)}z$ is open and closed in $\mathscr{K}^{(p+1)}z \cap St^{(p+1)}$.*

Proof. $\mathscr{K}^{(p+1)}z \cap \mathrm{St}^{(p+1)}$ is the union of the $\mathscr{A}^{(p+1)}$-orbits of its members. Each is open in $\mathscr{K}^{(p+1)}z \cap \mathrm{St}^{(p+1)}$, so each, as the complement of the union of the others, is also closed. □

Proof of Theorem 4.0.13. By the $p+1$ determinacy of stable germs, and the preceding lemma, it is enough to show that if $z \in \mathrm{St}^{(p+1)}$ then $\mathscr{K}^{(p+1)}z \cap \mathrm{St}^{(p+1)}$ is connected. Now the complex algebraic group $\mathscr{K}^{(p+1)}$ itself is connected, and hence its continuous image $\mathscr{K}^{(p+1)}z$ is also connected. The complement of $\mathrm{St}^{(p+1)}$ in $J^{p+1}(n,p)$ is a closed algebraic subset, and its intersection with the connected complex manifold $\mathscr{K}^{(p+1)}z$ is therefore a closed complex algebraic subset of $\mathscr{K}^{(p+1)}z$. It is a proper subset, since z itself lies in $\mathrm{St}^{(p+1)}$. Hence it does not separate $\mathscr{K}^{(p+1)}z$. We have shown that $\mathscr{K}^{(p+1)}z \cap \mathrm{St}^{(p+1)}$ is connected. □

Theorem 4.0.13 also holds for real C^∞ germs. The proof involves extra complications, since $\mathscr{K}^{(p+1)}z$ may no longer be connected and the complement of $\mathrm{St}^{(p+1)}$ may separate it further. We refer the reader to Mather's paper [Mat69b] for the details.

Example 4.1.2. Suppose that $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ has corank 1. In linearly adapted coordinates, f takes the form

$$f(x,y) = (x, a(x,y), b(x,y)),$$

where $\partial a/\partial y$ and $\partial b/\partial y$ both vanish at $(0,0)$. By a coordinate change in the target we can remove from a and b any pure power of x. So we may assume that the 2-jet of f takes the form

$$j^2 f = (x, a_{11}xy + a_{02}y^2, b_{11}xy + b_{02}y^2).$$

The non-immersive locus of f is defined by the two functions $\partial a/\partial y$ and $\partial b/\partial y$. We denote by $\mathscr{R}_f$ the ideal they generate - the *ramification ideal*. By an easy application of Nakayama's Lemma,

$$\mathscr{R}_f = \mathfrak{m} \quad\Leftrightarrow\quad \mathscr{R}_f + \mathfrak{m}^2 = \mathfrak{m} \quad\Leftrightarrow\quad (a_{11}x + 2a_{02}y, b_{11}x + 2b_{02}y) + \mathfrak{m}^2 = \mathfrak{m} \quad\Leftrightarrow\quad \begin{vmatrix} a_{11} & a_{02} \\ b_{11} & b_{02} \end{vmatrix} \neq 0.$$

It follows that if $\mathscr{R}_f = \mathfrak{m}$ then after a linear change of coordinates in the target, involving only the second and third coordinates, we have

$$j^2 f = (x, y^2, xy). \tag{4.1.3}$$

For the polynomial germ defined by the right hand side of this equality it is obvious that

$$: f^*\,\mathfrak{m}_2\,\mathcal{O}_2 = (x, y^2) \tag{4.1.4}$$

In fact, by Nakayama's Lemma, this holds for any germ with this 2-jet. It is now easy to show that condition (2) of Theorem 3.4.1 holds, so that f is stable. Because of (4.1.4), the local algebra of f is isomorphic to that of the cross cap. Hence by Theorem 4.0.13, f is equivalent to the cross cap.

4.2 Consequences of Finite Codimension

Let $f : \mathbb{C}^n \to \mathbb{C}^p$ (or $\mathbb{R}^n \to \mathbb{R}^p$) be an analytic (or C^∞) map. Its k-jet at a point x is the p-tuple consisting of the Taylor polynomials of degree k of its component functions. The k-jet of f at x is denoted by $j^k f(x)$. We say that a map-germ $f : (\mathbb{C}^n, x) \to (\mathbb{C}^p, y)$ is *k-determined for $\mathscr{A}$-equivalence* if any other map-germ having the same k-jet at x is $\mathscr{A}$-equivalent to f, and *finitely determined for $\mathscr{A}$-equivalence* if this holds for some finite value of k.

Theorem 4.2.1. (J.Mather [Mat68b]) *f is finitely determined if and only if* $\dim_{\mathbb{C}} T^1(f) < \infty$.

The smallest value of k for which this holds is the *determinacy degree* of f. Finding good estimates for the determinacy degree of f in terms of easily calculable data was once a major endeavour. Mather's original estimates (in [Mat68b]) were impractically large. They were greatly improved by Terry Gaffney and Andrew du Plessis ([Gaf79], [dP80]). In particular the following estimate due to Gaffney is useful:

Theorem 4.2.2. ([Gaf79]) *If* $T\mathscr{A}_e f \supset \mathfrak{m}^k_{\mathbb{C}^n,0}\,\theta(f)$ *and* $T\mathscr{K}_e f \supset \mathfrak{m}^\ell_{\mathbb{C}^n,0}\,\theta(f)$ *then f is $k+\ell$-determined.*

Since we are reaching conclusions about the $\mathscr{A}$-orbit of f, it is slightly curious that our hypotheses are framed in terms of $T\mathscr{A}_e f$ and not $T\mathscr{A} f$. Indeed it is (almost) obvious that if f is k-determined then

$$T\mathscr{A} f \supset \mathfrak{m}_n^{k+1}\,\theta(f) \tag{4.2.1}$$

To make this clear, we introduce the jet spaces $J^k(n,p)$.

Definition 4.2.3.
1. $\mathfrak{m}(n,p)$ is the vector space of all germs $(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$. It can be identified with $\mathfrak{m}_n\,\theta(f)$ for any $f\in\mathcal{O}(n,p)$.
2. $J^k(n,p)$ is the set of k-jets of germs $(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$.
3. $j^k:\mathcal{O}(n,p)\to J^k(n,p)$ is the operation "take the k-jet". The map $j^k:\mathcal{O}(n,p)\to J^k(n,p)$ is surjective. Its kernel is $\mathfrak{m}_n^k\,\mathfrak{m}(n,p)$, so we can view $J^k(n,p)$ as $\mathfrak{m}(n,p)/\,\mathfrak{m}_n^k\,\mathfrak{m}(n,p)$.
4. For $k\le\ell$, $\pi^\ell_k : J^\ell(n,p)\to J^k(n,p)$ is the projection ("truncate at degree k")
5. $\mathscr{A}^k = j^k(\mathscr{A})\subset J^k(n,n)\times J^k(p,p)$ is the quotient of $\mathscr{A}$ acting naturally on $J^k(n,p)$.

The diagram (in which the rows are group actions)

$$\begin{tikzcd}
\mathscr{A}\times\mathfrak{m}_n\,\mathfrak{m}(n,p) \arrow[r] \arrow[d, "j^k\times j^k"'] & \mathfrak{m}_n\,\mathcal{O}(n,p) \arrow[d, "j^k"] \\
\mathscr{A}^{(k)}\times J^k(n,p) \arrow[r] & J^k(n,p)
\end{tikzcd} \tag{4.2.2}$$

is commutative. The lower row is a finite-dimensional model of the upper row. In the lower row we really do have an algebraic group acting algebraically on an algebraic variety - indeed, on a finite dimensional complex vector space. This model provides motivation for many assertions, such as the statement that if f is k-determined then $T\mathscr{A} f\supset\mathfrak{m}_n^{k+1}\,\theta(f)$. What is clear is that if f is k-determined then

$$\mathscr{A}^{(\ell)} j^\ell f(0) = (\pi^\ell_k)^{-1}\big(\mathscr{A}^{(k)} j^k f(0)\big).$$

Now π^ℓ_k is linear, and its kernel is $j^\ell\big(\mathfrak{m}^{k+1}\,\theta(f)\big)$. So if f is k-determined,

$$T\mathscr{A}^{(\ell)} j^\ell f(0)\supset j^\ell\big(m^{k+1}\theta(f)\big)$$

Since

$$J^\ell(n,p) = \mathfrak{m}_n\,\theta(f)/\,\mathfrak{m}_n^{\ell+1}\,\theta(f),$$

this can be rewritten

$$T\mathscr{A}f + \mathfrak{m}_n^{\ell+1}\,\theta(f) \supset \mathfrak{m}_n^{k+1}\,\theta(f), \tag{4.2.3}$$

almost the statement (4.2.1) described as obvious above. If we knew that $\mathfrak{m}_n^{k+1}\,\theta(f)$ were a finitely generated module over $\mathcal{O}_{\mathbb{C}^p,0}$ then an application of Nakayama's Lemma would prove (4.2.1). But we don't know it, and in fact if $n > p$ it can't be true. Neverthless, it is possible to deduce (4.2.1) from (4.2.3) using some algebraic/analytic geometry:

1. $T\mathscr{K}_e f \supset T\mathscr{A}f$, so (4.2.3) implies

$$T\mathscr{K}_e f + \mathfrak{m}_n^{\ell+1}\,\theta(f) \supset \mathfrak{m}_n^{k+1}\,\theta(f). \tag{4.2.4}$$

2. Because (4.2.4) involves only $\mathcal{O}_{\mathbb{C}^n,0}$-modules, by Nakayama's Lemma we deduce that $T\mathscr{K}_e f \supset \mathfrak{m}_n^{k+1}\,\theta(f)$. This implies that $\dim_{\mathbb{C}}\big(\theta(f)/T\mathscr{K}_e f\big) < \infty$ (f is "$\mathscr{K}$-finite", or has "finite singularity type".)

3. Let J_f be the ideal in $\mathcal{O}_{\mathbb{C}^n,0}$ generated by the $p \times p$ minors of the matrix of df. Its locus of zeros is the critical set $\sum_f$, the set of points where f is not a submersion. By taking the determinants of p-tuples of elements of $\theta(f)$, from the fact that f is $\mathscr{K}$ finite we deduce that $\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0}\,/J_f + f^*\,\mathfrak{m}_p\,\mathcal{O}_{\mathbb{C}^n,0}) < \infty$. This condition has a clear geometrical significance (over the complex numbers!):

$$V(J_f + f^*\,\mathfrak{m}_p\,\mathcal{O}_{\mathbb{C}^n,0}) = \sum\nolimits_f \cap f^{-1}(0),$$

   so f *is finite-to-one on its critical locus.*

4. From this it follows that every coherent sheaf of $\mathcal{O}_{\mathbb{C}^n,0}$ modules supported on $\sum_f$ is finite over $\mathcal{O}_{\mathbb{C}^p,0}$. In particular

$$\big(\mathfrak{m}^{\ell+1}\,\theta(f) + tf(\theta_n)\big)/tf(\theta_n)$$

   is a finite $\mathcal{O}_{\mathbb{C}^p,0}$-module! So now we can apply Nakayama's Lemma to deduce (4.2.1) from (4.2.3): simply take the quotient on both sides by $tf(\theta_n)$.

It took some quite non-elementary steps to get to the "obvious" statement (4.2.1) from the truly obvious statement (4.2.3)!

Exercise 4.2.4. Use the techniques just introduced to prove Theorem 3.4.1. Note that the hypothesis of 3.4.1 is equivalent to

$$\theta(f) = T\mathscr{A}_e f + T\mathscr{K}_e f = T\mathscr{A}_e f + f^*\,\mathfrak{m}_p\,\theta(f).$$

In view of the fact that (4.2.1) *is* true, one might hope that its converse, which also seems reasonable, should also be true. But things are not so simple. They become simpler if we replace the group $\mathscr{A}$ by its subgroup $\mathscr{A}_1$ consisting of pairs of germs of diffeomorphisms whose derivative at 0 is the identity. This observation by Bill Bruce led to what was probably the final paper on finite determinacy, [BdPW87], in which *unipotent groups* $\mathscr{G}$ are identified as those for which the determinacy degree is equal to one less than the smallest power k such that $m_n^k\theta(f) \subseteq T\mathscr{G}_e f$. The group $\mathcal{A}$ itself is not unipotent.

To prove a statement of the kind

$$T\mathcal{A}f \supset \mathfrak{m}_n^k\,\theta(f) \qquad \Longrightarrow \qquad f \text{ is } d(k)\text{-determined}$$

one has to show that if g and f differ by terms in $\mathfrak{m}_n^{d(r)+1}$ then two things happen:

1. first, the germ of deformation $f + t(g - f)$ is trivial – so that for all t is some neighbourhood of 0, $f + t(g - f)$ is equivalent to f.

2. Second, that for *any* value t_0 of t, we also have $T\mathcal{A}(f + t_0(g - f)) \supset \mathfrak{m}_n^k\,\theta(f)$ – so that by the first assertion, the deformation $f + tg$ is trivial also in the neighbourhood of t_0.

In practice, one should not expect to obtain the precise determinacy degree of a map-germ from a general theorem like 4.2.2. Instead, one can often significantly improve an estimate by using another result due to Mather (in [Mat68b, Lemma 3.1]) and known as "Mather's Lemma".

Proposition 4.2.5. *Suppose the Lie group G acts smoothly on the manifold M, and that $W \subset M$ is a smooth connected submanifold. Then a necessary and sufficient condition for W to be contained in a single orbit is that*

1. *for all $x \in W$, $T_xW \subset T_xGx$, and*

2. *the dimension of T_xGx is the same for all $x \in W$.*

One uses the lemma as follows: suppose that it is possible to show, e.g. by applying a general theorem, that f is ℓ-determined, and one wants to show that it is k-determined for some $k < \ell$. Let $M = J^\ell(n, p_)$, $G = \mathcal{A}^{(\ell)}$ and

$$W = \{j^\ell g : j^k g = j^k f\}.$$

Exercise 4.2.6. *Assume that f is ℓ-determined. If the set W just defined lies in a single $\mathcal{A}^{(\ell)}$-orbit then f is k-determined.*

Because we are working modulo $\mathfrak{m}^{\ell+1}$, terms of degree $\ell + 1$ and higher can be ignored in calculating $T\mathcal{A}^{(\ell)}g$, and this may make it relatively straightforward to show that the conditions of Mather's Lemma hold.

4.3 Versal Unfoldings

An unfolding of a map-germ f_0 is $\mathcal{A}_e$-*versal* if it contains, up to parametrised $\mathcal{A}$-equivalence, every possible unfolding of the germ. In this section we make precise sense of this idea, and study some examples.

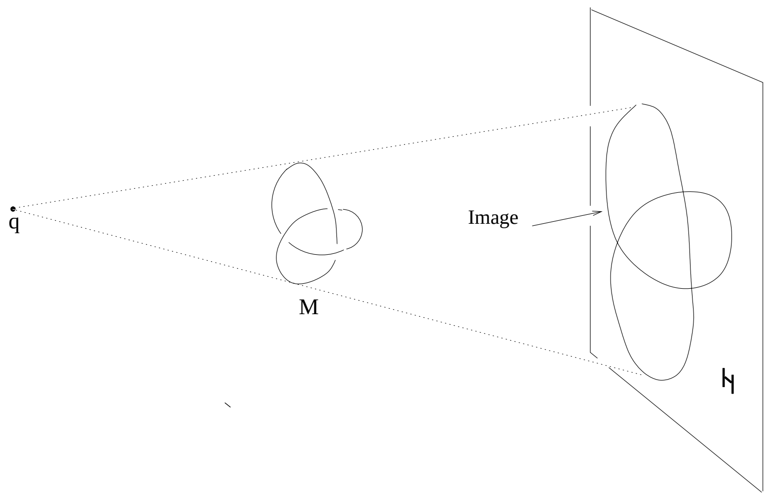
Definition 4.3.1. (1) Let $F, G : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$ be unfoldings of the same map germ $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. They are *equivalent* if there exist germs of diffeomorphisms

$$\Phi : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^n \times \mathbb{C}^d, 0)$$

and

$$\Psi : (\mathbb{C}^p \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$$

such that



1. $\Phi(x,u) = (\varphi(x,u),u)$ and $\varphi(x,0) = x$
2. $\Psi(y,h) = (\psi(y,u),u)$ and $\psi(y,0) = y$
3. $F = \Psi \circ G \circ \Phi$

Note that an unfolding is trivial (Definition 3.0.9) if it is equivalent to the constant unfolding.

(2) With $F(x,u) = (f(x,u),u)$ as in (1), let $h : (\mathbb{C}^e,0) \to (\mathbb{C}^d,0)$ be a map germ. The unfolding $(\mathbb{C}^n \times \mathbb{C}^e,0) \to (\mathbb{C}^p \times \mathbb{C}^e,0)$ defined by

$$(x,v) \mapsto (f(x,h(v)),v)$$

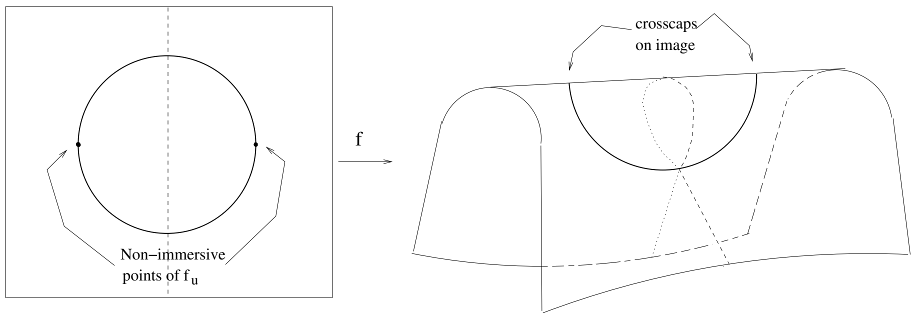
is called the *pull-back* of F by h, and denoted by h^*F. The map-germ h in this context is often called the 'base-change' map, and we say that h^*F is the unfolding *induced from* F by h.

(3) The unfolding F of f_0 is $\mathcal{A}_e$*-versal* if for every other unfolding $G : (\mathbb{C}^n \times \mathbb{C}^e,0) \to (\mathbb{C}^p \times \mathbb{C}^e,0)$ of f_0, there is a base-change map $h : (\mathbb{C}^e,0) \to (\mathbb{C}^d,0)$ such that G is equivalent (in the sense of (1)) to the unfolding h^*F (as defined in (2)).

The term 'versal' if the intersection of the words 'universal' and 'transversal'. Versal unfoldings were once upon a time called universal, but later it was decided that they did not deserve this term, because the base-change map h of part (3) of the definition is not in general unique. Uniqueness is an important ingredient in the "universal properties" which characterise many mathematical objects, and so universal unfoldings were stripped of their title. However the intersection with the word 'transversal' is serendipitous, as we will see.

Example 4.3.2. Some light relief Consider a manifold $M \subset \mathbb{C}^N$. Radial projection from a point q into a hyperplane H is defined by the following picture: It defines a map $P_q : M \to H$. If the hyperplane H is replaced by another hyperplane H', then the corresponding projection $P'_q : M \to H'$ is left-equivalent to P_q; composing P'_q with the restriction of P_q to H', we get P_q. On the other hand, if we vary the point q then we may well deform the projection P_q non-trivially. So we consider the unfolding

$$P : M \times \mathbb{C}^N \to H \times \mathbb{C}^N.$$



It's instructive to look at this over $\mathbb{R}$ with the help of a piece of bent wire and an overhead projector. Are the unstable map-germs one sees versally unfolded in the family of all projections? This is discussed in [Wal77] and again in [Mon95].

Like stability, versality can be checked by means of an infinitesimal criterion. Let $F(x,u) = (f(x,u),u)$ be an unfolding of f_0. Write $\partial f/\partial u_j|_{u=0}$ as $\dot{F}_j$.

Theorem 4.3.3. (Infinitesimal versality is equivalent to versality) *The unfolding F of f_0 is versal if and only if*

$$T\mathscr{A}_e f_0 + Sp_{\mathbb{C}}\{\dot{F}_1,\ldots,\dot{F}_d\} = \theta(f_0)$$

– in other words, if the images of $\dot{F}_1,\ldots,\dot{F}_d$ in $T^1(f_0)$ generate it as (complex) vector space. □

For a proof, see Chapter X of Martinet's book [Mar82]. Martinet proves the theorem for C^∞ map-germs; the proof in the analytic category is the same. Both use the Preparation Theorem, 1.4.1. □

Exercise 4.3.4. Prove 'only if' in Theorem 4.3.3. It follows in a straightforward way from the definitions: let g be an arbitrary element of $\theta(f_0)$ and take, as G, the 1-parameter unfolding $G(x,t) = (f(x) + tg(x), t)$. Show that if G is equivalent to an unfolding induced from F then $g \in T\mathscr{A}_e f_0 + \mathrm{Sp}_{\mathbb{C}}\{\dot{F}_1,\ldots,\dot{F}_d\}$

Example 4.3.5. Consider the map-germ $f_0(x,y) = (x, y^2, y^3 + x^2y)$ of Example 3.1.1. We saw that $y\partial/\partial Z$ projects to a basis for $T^1(f_0)$. So

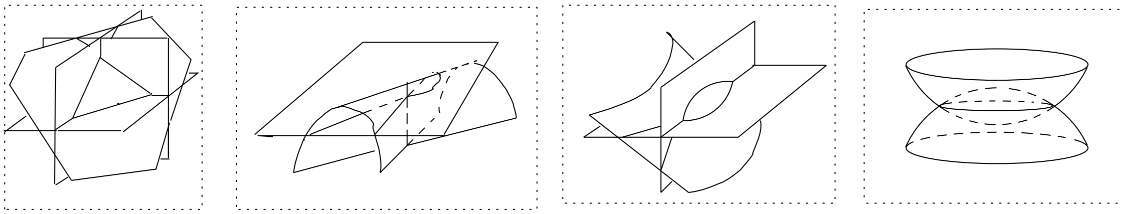
$$F(x,y,u) = (x, y^2, y^3 + x^2y + uy, u)$$

is a versal deformation. What is the geometry here? Think of F as a family of mappings,

$$f_u(x,y) = (x, y^2, y^3 + x^2y + uy).$$

The *ramification ideal* $\mathscr{R}_{f_u} \subset \mathcal{O}_{\mathbb{C}^2}$ generated by the 2×2 minors of the matrix $[df_u]$ defines the set of points where f_u fails to be an immersion. Here $\mathscr{R}_{f_u} = (y, x^2 + u)$. So for $u \neq 0$, f_u has two non-immersive points. They are only visible over $\mathbb{R}$ when $u < 0$. How does f_u behave in the neighbourhood of each of these points? At each, $\mathscr{R}_{f_u}$ is equal to the maximal ideal; it follows that df_u is transverse to the submanifold $\sum^1 \subset L(\mathbb{C}^2,\mathbb{C}^3)$ consisting of linear maps of rank 1. In fact this transversality *characterises* the map-germ f paremeterising the cross-cap (described in 3.1.1(2)) up to $\mathscr{A}$-equivalence, though here we are not yet able to show that. Using this characterisation, we see

Figure 4.1: Images of stable perturbations of codimension 1 germs of maps from the plane to 3-space



that in a neighbourhood of the image of each of the two points $(\pm\sqrt{-u}, 0)$, the image of f_u looks like the drawing in Example 3.1.1. The key to assembling the image of f_u from its constituent parts is the curve of self-intersection. The only points mapped 2-1 by f_u are the points of the curve $\{x^2 + y^2 + u = 0\}$; for $u < 0$ this is a circle when viewed over $\mathbb{R}$. Here points $(x, \pm y)$ share the same image. The two non-immersive points of f_u are the fixed points of the involution $(x, y) \mapsto (x, -y)$ which interchanges pairs of points sharing the same image.

The image contains a chamber; indeed it is homotopy-equivalent to a 2-sphere. This is no coincidence. The next figure shows images of stable perturbations of each of the remaining codimension 1 singularities of maps from surfaces into 3-space. Each is homotopy-equivalent to a 2-sphere. Some choices have been made regarding the real form: sometimes a change of sign which makes no difference over $\mathbb{C}$ does make a difference over $\mathbb{R}$. Nevertheless in all of these cases it is possible to choose a suitable real form whose perturbation is a homotopy 2-sphere.

Exercise 4.3.6. Find versal unfoldings of the following germs:

1. $f : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0),\ f(t) = (t^3, t^4)$.
2. $f : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0),\ f(t) = (t^2, t^5)$.
3. $f : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0),\ f(t) = (t^2, t^{2k+1})$.
4. $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0),\ f(x, y) = (x, y^2, y^3 + x^{k+1}y)$.
5. $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0),\ f(x, y) = (x, y^3 + x^2y)$.

4.4 Stable perturbations

We have looked at examples of mappings from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ for $n = 1, 2$. By inspection, we can see that the perturbations of the unstable maps we considered were at least locally stable: every (mono- and multi-) germ they contain is stable. In the dimension range we have looked at, every germ of finite codimension can be perturbed so that it becomes stable. These are "nice dimensions", to use a term due to John Mather. These dimension-pairs may be characterised by the following property: in the base of a versal deformation, the set of parameter-values u such that f_u has an unstable multi-germ is a proper analytic subvariety. It is known as the *bifurcation set*.

Mather carried out long calculations to determine the nice dimensions, published in [Mat71]. Curiously, the nice dimensions are also characterised by the fact that every stable germ in these dimensions is weighted homogeneous, in appropriate coordinates.

When the bifurcation set B is a proper analytic subvariety of a smooth space, it does not separate it topologically (remember we're working in $\mathbb{C}^d$). That is, any two points u_1 and u_2 in its complement can be joined by a path $\gamma(t)$ which does not meet B. Because f_{u_1} and f_{u_2} are locally stable, each germ of the unfolding

$$(x,t) \mapsto (f_{\gamma(t)}(x), t)$$

is trivial; so f_{u_1} and f_{u_2} are locally isomorphic and globally C^∞-equivalent. Thus, to each complex germ of finite codimension we can associate a *stable perturbation* (any one of the mappings f_u for $u \notin B$) which is independent of the choice of u, at least up to diffeomorphism. Some care must be taken to define the domain of f_u; it is more than a germ, but not a global mapping $\mathbb{C}^n \to \mathbb{C}^p$. The situation is analogous to the construction of the Milnor fibre, in which several choices of neighbourhoods must be made, but in which the final result is nevertheless independent of the choices. Details may be found in [Mar93].

Chapter 5

Stable Images and Discriminants

5.1 Review of the Milnor fibre

In the theory of isolated hypersurface singularities a key role is played by the Milnor fibre. Here is a very brief account.

1. Let f be a complex analytic function defined on some neighbourhood of 0 in $\mathbb{C}^{n+1}$, and suppose it has isolated singularity at 0. Then by the curve selection lemma, there exists $\varepsilon > 0$ such that for ε' with $0 < \varepsilon' \leq \varepsilon$, the sphere of radius ε' centred at 0 is transverse to $f^{-1}(0)$. Let B_ε be the closed ball centred at 0 and with radius ε. Then from the transversality it follows that $f^{-1}(0) \cap B_\varepsilon$ is homeomorphic (indeed, diffeomorphic except at 0) to the cone on its boundary $f^{-1}(0) \cap S_\varepsilon$. The ball B_ε is a *Milnor ball* for the singularity.

2. By an argument involving properness, one can show that for suitably small $\eta > 0$, all fibres $f^{-1}(t)$ with $|t| < \eta$ are transverse to S_ε. Let D_η be the closed ball in $\mathbb{C}$ with radius η and centre 0, and let $D_\eta^* = D_\eta \smallsetminus \{0\}$. By the Ehresmann fibration theorem,
$$f| : B_\varepsilon \cap f^{-1}(D_\eta^*) \to D_\eta^*$$
is a C^∞-locally trivial fibration. It is known as the Milnor fibration. Up to fibre-preserving homeomorphism, it is independent of the choice of ε.

3. Its fibre is called the *Milnor fibre* of f. It has the homotopy type of a wedge of n-spheres, whose number μ, the *Milnor number of* f, is equal to the dimension of the Jacobian algebra of f,
$$\mathcal{O}_{\mathbb{C}^{n+1},0} / J_f.$$

The argument for the last statement is based on two facts:

1. if $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{n+1},0} / J_f = 1$ (in which case f is said to have a 'non-degenerate" critical point), then by the holomorphic Morse lemma, f is right-equivalent to $x \mapsto x_1^2 + \cdots + x_{n+1}^2$. An explicit calculation now shows that the Milnor fibre is diffeomorphic to the unit ball sub-bundle of the tangent bundle of S^n. This has S^n as a deformation-retract.

2. f can be perturbed so that the critical point at 0 splits into non- degenerate critical points. There are exactly μ of them – see Corollary **??** – and each contributes one sphere to the wedge.

The dimension of the Jacobian algebra plays a second, completely different, role in the theory. The quotient by which we measure instability,

$$\frac{\{\frac{d}{dt}f_t|_{t=0} : f_0 = f\}}{\{\frac{d}{dt}f \circ \varphi_t|_{t=0}\}}$$

is the self-same Jacobian algebra, and indeed the Jacobian ideal itself is the extended tangent space for right-equivalence. The analogue of Theorem 4.3.3 shows that one can construct a versal deformation of f (versal for right-equivalence, that is) by taking $g_1, \ldots, g_\mu \in \mathcal{O}_{\mathbb{C}^{n+1},0}$ whose images in the Jacobian algebra span it as vector space, and defining

$$F(x, u_1, \ldots, u_\mu) = f(x) + \sum_j u_j g_j.$$

The Milnor fibration extends to a fibration over the complement of the discriminant Δ in the base-space $S = \mathbb{C}^\mu$; taking its associated cohomology bundle we obtain a holomorphic vector bundle of rank μ over the μ-dimensional space S. It is equipped with a canonical flat connection, the *Gauss-Manin connection.*

The objective now is to show that many of these same ingredients can be found in the theory of singularities of mappings.

5.2 Image Milnor Number and Discriminant Milnor Number

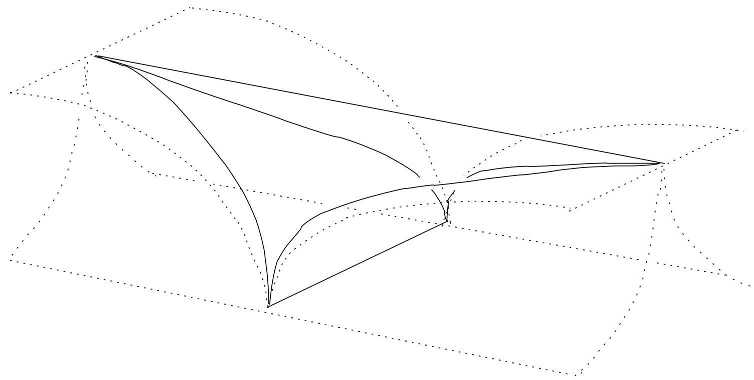
We have already seen, in Example 4.3.5, that the *real* image of each codimension 1 germ f of mappings from surfaces to 3-space grows a 2-dimensional homotopy-sphere when f is suitably perturbed.

Proposition 5.2.1. *(1) Suppose that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is a map-germ of finite codimension. Then the image of a stable perturbation of f has the homotopy type of a wedge of n-spheres.*

(2) Suppose that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a map-germ of finite codimension, with $n \geq p$. Then the discriminant (= set of critical values) of a stable pertubation of f has the homotopy-type of a wedge of $(p-1)$-spheres.

Terminology The number of spheres in the wedge is called the *image Milnor number*, μ_I, in case (1), and the *discriminant Milnor number*, μ_Δ, in case (2).

Proof of 5.2.1 Both statements are consequences of a fibration theorem of Lê Dung Trang ([Trá87]), that says, in effect, that if (X, x_0) is a p-dimensional complete intersection singularity and $\pi : (X, x_0) \to (\mathbb{C}, 0)$ is a function with isolated singularity, in a suitable sense, then the analogue of the Milnor fibre of π (i.e. the intersection of a non-zero level set with a Milnor ball around x_0) has the homotopy-type of a wedge of spheres of dimension $p-1$. To apply this theorem here, we take, as X, the germ of the image in case (1), or discriminant, in case (2), of a 1-parameter stabilisation of f: that is, an unfolding $F : (\mathbb{C}^n \times \mathbb{C}, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}, 0)$ with $F(x, u) = (\tilde{f}(x, u), u) = (f_u(x), u)$ such that f_u is stable for $u \neq 0$. Then $(X, 0)$ is a hypersurface singularity, and thus a complete intersection. We take, as π, the projection to the parameter space. Thus $\pi^{-1}(u)$ is the image (or discriminant) of f_u. The fact that π has isolated singularity is a consequence of the fact that f_u is stable for $u \neq 0$. For this implies that the unfolding is trivial away from $u = 0$, so that the vector field $\partial/\partial u$ in the target of π lifts to a vector field tangent to X. $\square$



Discriminant of stable perturbation of the bi-germ

$$\begin{cases} (u,v,w) \mapsto (u,v,w^3 - uw) \\ (x,y,z) \mapsto (x, y^3 + xy, z) \end{cases}$$

In [Sie91] Dirk Siersma proves a similar theorem in the case where X is a hypersurface, and goes on to show that the number of spheres in the wedge is equal to the sum of the Milnor numbers of the isolated critical points of the defining equation g of the image/discriminant which move off the image/discriminant as f (and with it g) is deformed. More precisely, suppose that

$$G : (\mathbb{C}^p \times \mathbb{C}, (0,0)) \to (\mathbb{C} \times \mathbb{C}, (0,0))$$

is an unfolding of $g : (\mathbb{C}^p, 0) \to (\mathbb{C}, 0)$, with $G(x,t) = (g(x,t), t)$, and suppose that $\varepsilon > 0$ and $\eta > 0$ are such that the restriction

$$g| : (B_\varepsilon \times D_\eta) \cap g^{-1}(D_\eta \smallsetminus \{0\}) \to D_\eta \smallsetminus \{0\}$$

is a Milnor fibration for g, i.e. a locally trivial fibre bundle.

Let $\rho > 0$ be chosen so that $G| : (\partial B_\varepsilon \times D_\rho) \cap G^{-1}(D_\eta \times D_\rho) \to D_\eta \times D_\rho$ is a stratified submersion, with respect to strata $\{0\} \times D_\rho$ and $D_\eta \smallsetminus \{0\} \times D_\rho$ on $D_\eta \times D_\rho$, and some corresponding stratification on $(\partial B_\varepsilon \times D_\rho) \cap G^{-1}(D_\eta \times D_\rho)$. This means that

1. $g_u^{-1}(t)$ is stratified transverse to ∂B_ε for $u \in D_\rho, t \in D_\eta$;
2. $g_u^{-1}(D_\eta) \cap \partial B_\varepsilon$ is homeomorphic to $g^{-1}(D_\eta) \cap \partial B_\varepsilon$ for $u \in D_\rho$;
3. the Milnor fibres of g_u and g are homeomorphic.

Theorem 5.2.2. *Let $G, \varepsilon, \eta, \rho$ be as just described, and let $u \in D\rho$ be such that all fibres of g_u are smooth, except for some which have isolated singularities, and $X_u = g_u^{-1}(0) \cap B_\varepsilon$ which may have non-isolated singularities. Then X_u has the homotopy type of a wedge of spheres of dimension $p-1$, and the number of these spheres is equal to the sum of the Milnor numbers of the singularities of g_u in the fibres different from X_u.*

Proof. The proof is based on Morse theory. Up to homotopy, $B_\varepsilon \cap g_u^{-1}(D_\eta)$ is obtained from $X_u = B_\varepsilon \cap g_u^{-1}(0)$ by progressively thickening it, i.e. by considering

$$|g_u|^{-1}([0,\delta])$$

and increasing δ. For small enough δ, $|g_u|^{-1}([0,\delta])$ has X_u as deformation-retract. Except when δ passes through a critical value of $|g_u|$, the thickening does not change the homotopy type. The critical points of $|g_u|$ off $g_u^{-1}(0)$ are the same as those of g_u, and each has index equal to the ambient dimension, because of the complex structure (see Lemma 5.2.3 below). Thus, the space $B_\varepsilon \cap g_u^{-1}(D_\eta)$ is obtained from $g_u^{-1}(0)$ by gluing in cells of dimension p. Now the assumptions about transversality to the boundary imply that $B_\varepsilon \cap g_u^{-1}(D_\eta)$ is homeomorphic to $B_\varepsilon \cap g^{-1}(D_\eta)$, and therefore contractible. It follows by a standard Mayer-Vietoris type argument that $g^{-1}(0)$ is homotopy-equivalent to the wedge of the boundaries of these cells. We can assume that g_u has only non-degenerate critical points off $g_u^{-1}(0)$; so the number of cells is the sum of their Milnor numbers. □

Lemma 5.2.3. *Suppose $g : (\mathbb{C}^p, P) \to \mathbb{C}$ is a complex analytic function with a non-degenerate critical point at P, and $g(P) \neq 0$. Then*

1. *if $v \in T_P\mathbb{R}^{2p}$ is an eigenvector of $d^2|g|(P)$ with eigenvalue λ, then iv is an eigenvector with eigenvalue $-\lambda$,*
2. *the index of the Hessian of $|g|$ at P is equal to p, and*
3. *the negative eigenspace of $d^2|g|(P)$ spans $\mathbb{C}^p$ over $\mathbb{C}$.*

Proof. Modulo higher-order terms, we have $g(P+z) = a + ib + z^t(A+iB)z$, where $g(P) = a+ib$ and $A+iB$ is the matrix of $(1/2)d^2g(P)$, written as a sum of real and imaginary matrices. With x and y in $\mathbb{R}^p$, we therefore have

$$|g|^2(P+x+iy) = a^2 + 2a\{x^tAx - y^tAy - 2x^tBy\} + b^2 + 2b\{x^tBx - y^tBy + 2x^tAy\},$$

again modulo higher-order terms. It follows that

$$d^2|g|^2(P)(x+iy, x+iy) = 4a\{x^tAx - y^tAy - 2x^tBy\} + 4b\{x^tBx - y^tBy + 2x^tAy\}.$$

From this it is easy to check that $d^2|g|^2(P)(iz,iz) = -d^2|g|^2(P)(z,z)$. Since $d^2|g|^2(P) = 2|g(P)|d^2|g|(P)$, 1. follows.

The remaining two statements are obvious consequences of the first. □

This counting procedure is essential for the proofs of the following theorems.

Theorem 5.2.4. ([DM91]) *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ have finite $\mathcal{A}_e$-codimension, with $n \geq p$ and (n,p) nice dimensions. Then*

$$\mu_\Delta(f) \geq \mathcal{A}_e - codim(f)$$

with equality if f is weighted homogeneous.

Theorem 5.2.5. ([dJvS91], [Mon95]), *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ ($n = 1$ or 2) have finite $\mathcal{A}_e$-codimension. Then*

$$\mu_I(f) \geq \mathcal{A}_e - codim(f) \tag{5.2.1}$$

with equality if f is weighted homogeneous.

Theorem 5.2.5 was proved for $n = 2$ by de Jong and van Straten in [dJvS91]; another proof, also inspired by de Jong and van Straten, was given in [Mon91], and an analogous proof for the case $n = 1$ was given in [Mon95].

A number of examples ([CMWA02],[HK99],[Hou02],[MWA03], [Sha14]) of map-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ for $n \geq 3$ support the Mond conjecture that (5.2.1) should hold for all n for which $(n, n+1)$ are nice dimensions, but it remains unproven. Part of the difficulty in proving it lies in the fact that we do not have an effective method for computing image Milnor numbers. There is a conjectural formula (see below) but without a proof that a relative T^1 is Cohen-Macaulay, it only gives an upper bound for μ_I. Other information on the image Milnor number comes from the image-computing spectral sequence (see [GM93], [Gor95], [Hou99]), but so far this only yields an answer when f has corank 1 (see [GM93, Section 3]). In contrast, we do have a method for computing discriminant Milnor numbers, described in the next section.

5.3 Sections of stable images and discriminants

We begin by simplifying our initial description of $T^1(f)$, using an idea of Jim Damon. If $F : V \to W$ and $i : Y \to W$ are two maps, the *fibre product of V and Y over W*, denoted by $V \times_W Y$, is the space

$$V \times_W Y = \{(v, y) \in V \times Y : F(v) = i(y).\}$$

A *fibre square* is the commutative diagram

$$\begin{tikzcd} V \arrow[r, "F"] & W \\ V \times_W Y \arrow[u, "\pi_V"] \arrow[r, "f:=\pi_Y"'] & Y \arrow[u, "i"'] \end{tikzcd} \tag{5.3.1}$$

which results, where π_Y and π_V are the restrictions to $V \times_W Y$ of the projections $V \times_W Y \to Y$ and $V \times_W Y \to V$. If V, W and Y are smooth spaces and $i \pitchfork F$ then $V \times_W Y$ is smooth also. We say that the map $\pi_Y : V \times_W Y \to Y$, which here we will denote by f, is the *pull-back of F by i*, or *transverse pull-back* in the case where $i \pitchfork F$, and write $f = i^*(F)$. The transversality of i to F guarantees that $V \times_W Y$ is smooth, but there is no canonical choice of coordinate system on $V \times_W Y$, so the map $i^*(F)$ is really defined only up to right-equivalence.

A *standard fibre square* is a fibre square of the form

$$\begin{tikzcd} (X \times S, (x_0, s_0)) \arrow[r, "F"] & (Y \times S, (y_0, s_0)) \\ (X, x_0) \arrow[u, "j"] \arrow[r, "f"] & (Y, y_0) \arrow[u, "i"'] \end{tikzcd} \tag{5.3.2}$$

in which F is an unfolding of f, and i and j are standard immersions, with $i(y) = (y, 0)$ and $j(x) = (x, 0)$. Every map-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ of finite singularity type can be obtained by transverse pull-back from a stable map-germ: simply construct a stable unfolding $F : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$ and then recover f from F by means of a standard fibre square.

Example 5.3.1. Take $V = \mathbb{C}^2$, $W = \mathbb{C}^3$, $Y = \mathbb{C}^3$, and let $F(v_1, v_2) = (v_1, v_2^2, v_1v_2)$ and $i(y_1, y_2, y_3) = (p(y_1, y_2), y_2, y_3)$. Then $i\pitchfork F$ and

$$V \times_W Y = \{(v_1, v_2, y_1, y_2, y_3) : \ v_1 = p(y_1, y_2), \ v_2^2 = y_2, \ v_1v_2 = y_3.\}$$

The three equations defining $V \times_W Y$ allow us to dispense with the coordinates v_1, y_2 and y_3, retaining y_1, v_2 as coordinates on $V \times_W Y$. With respect to these coordinates, the maps π_V and π_Y are then given by

$$\begin{array}{rcl} \pi_Y(y_1, v_2) & = & (y_1, v_2^2, v_2 p(y_1, v_2^2) \\ \pi_V(y_1, v_2) & = & (p(y_1, v_2^2), v_2) \end{array}$$

Exercises 5.3.2.

1. Show that if V, W and Y are smooth and $i\pitchfork F$ then $V \times_Y W$ is smooth of dimension $\dim V + \dim Y - \dim W$.
2. Show that if f is obtained by transverse pull-back from F then
   (a) the set of critical points of f is the preimage by π_X of the set of critical points of F;
   (b) the set of critical values of f is the preimage by i of the set of critical values of F;
   (c) the local algebras $Q(f)$ and $Q(F)$ are isomorphic.
3. Let f be the germ of type H_2 given by $(x, y) \mapsto (x, y^3, xy + y^5)$ and let $F(a, b, c, y) = (a, b, c, y^3 + ay, by^2 + cy)$. Find $i : \mathbb{C}^3 \to \mathbb{C}^5$ such that $i^*(F) \simeq_{\mathcal{A}} f$.
4. Let $F(u, v, y) = (u, v, y^4 + uy^2 + vy)$. Find $i : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ such that $i^*(F) \simeq_{\mathcal{A}} f$ where $f(x, y) = (x, xy + y^4)$.
5. Let $f(x, y) = (x, y^3 + x^k y)$. Find a stable germ F and a germ i such that $f \simeq_{\mathcal{A}} i^*(F)$.
6. Find a stable bi-germ $F : (\mathbb{C}, \{0, 0'\}) \to (\mathbb{C}^2, 0)$ and $i : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that i^*F is $\mathcal{A}$-equivalent to
$$\left\{ \begin{array}{lcl} s & \mapsto & (s, s^2) \\ t & \mapsto & (t, -t^2) \end{array} \right.$$

Suppose that f is obtained from the stable map F by transverse pull-back by i. The main theorem of this section, 5.3.6, shows how to recover the module T^1_f in terms of the interaction of F and i.

Before stating it, we need a definition.

Definition 5.3.3. If $D \subset W$ is an analytic subvariety, $\mathrm{Der}(-\log D)$ is the $\mathcal{O}_W$-module (sheaf) of germs of vector fields on W tangent to D at its smooth points.

It is easy to show that if D is the variety of zeros of an ideal I then

$$\mathrm{Der}(-\log D) = \{\chi \in \theta_W : \chi \cdot g \in I \text{ for all } g \in I\},$$

and in particular if D is a hypersurface with equation h then

$$\mathrm{Der}(-\log D) = \{\chi \in \theta_W : \chi \cdot h = \alpha h \text{ for some } \alpha \in \mathcal{O}_W\}.$$

If D is any complex space and $x \in D$, the *isosingular locus of* D *at* w, $\mathscr{I}_{D,w}$, is the germ of the set of points

$$\{x \in (D, w) : \text{ the germs } (D, w) \text{ and } (D, x) \text{ are isomorphic}\}.$$

Theorem 5.3.4. (Ephraim, [Eph78]) *Let $D \subset W$. Then $T_w\mathscr{I}_{D,w} = \{\chi(w) : \chi \in Der(-\log D)_w\}$.*$\square$

The inclusion of right hand side in left in 5.3.4 is clear: if $\chi \in \mathrm{Der}(-\log D)_w$ and $\chi(w) \neq 0$ then the integral flow of χ preserves D, and thus induces a family of isomorphisms of D. Evidently the integral curve of χ through w is contained in $\mathscr{I}_{D,w}$, and so its tangent vector $\chi(w)$ is contained in $T_w\mathscr{I}_{D,w}$.

The vector space in 5.3.4 is known as the *logarithmic tangent space to D at w*; we denote it by $T_w^{\log}D$. If Y is a smooth space and $i : Y \to W$ a map, we say i is *logarithmically transverse to D at $y_0 \in Y$* if

$$d_{y_0}i(T_{y_0}Y) + T^{\log}_{i(y_0)}D = T_{i(y_0)}W. \tag{5.3.3}$$

Each of the three vector spaces in (5.3.3) is the evaluation at y_0 of the stalk of a sheaf of $\mathcal{O}_Y$-modules: the three sheaves are, respectively, $ti(\theta_Y)$, $i^*(\mathrm{Der}(-\log D))$ and $\theta(i)$.

Proposition 5.3.5. *The equality* (5.3.3) *holds if and only if*

$$\frac{\theta(i)}{ti(\theta_Y) + i^*(Der(-\log D))} = 0. \tag{5.3.4}$$

Proof. This is just Nakayama's Lemma: we have

$$\frac{\theta(i)}{ti(\theta_Y) + i^*(\mathrm{Der}(-logD)) + \mathfrak{m}_{Y,y_0}\,\theta(i)} \simeq \frac{T_{i(y_0)}W}{d_{y_0}i(T_{y_0}Y) + T^{\log}_{i(y_0)}D}.$$

$\square$

The quotient module on the left hand side of (5.3.4) thus measures the failure of logarithmic transversality of i to D. Its $\mathbb{C}$-vector-space dimension is finite if and only if i is logarithmically transverse to D outside y_0.

Let $F : (V, 0) \to (W, 0)$ be a stable map-germ, and let D be its discriminant.

Theorem 5.3.6. (J.N.Damon,[Dam91]) *If f is obtained from the stable map F by transverse pull back by i then*

$$T^1(f) \simeq \frac{\theta(i)}{ti(\theta_Y) + i^*(Der(-\log D))}.$$

To prove this we need

Lemma 5.3.7. *Let $F : (V, 0) \to (W, 0)$ be a map-germ of finite $\mathcal{A}_e$-codimension, and let D be its discriminant. Let $\chi \in \theta_{W,w_0}$. Then $\chi \in Der(-\log D)_{w_0}$ if and only if it can be lifted to a vector field $\tilde{\chi}$ on (V, v_0) – that is, there exists $\tilde{\chi} \in \theta_{V,v_0}$ such that*

$$tF(\tilde{\chi}) = \omega F(\chi).$$

Proof. Suppose that $\chi \in \theta_W$ has lift $\tilde{\chi} \in \theta_V$. By integrating χ and $\tilde{\chi}$ we obtain flows Ψ_t, Φ_t on W and V respectively, such that

$$F \circ \Phi_t = \Psi_t \circ F. \tag{5.3.5}$$

Suppose $y \in D$, and let $x \in \Sigma_F$ satisfy $y = F(x)$. For every t, (5.3.5) shows that the germs

$$F : (X, \Phi_t(x)) \to (Y, \Psi_t(y)) \quad \text{and} \quad F : (X, x) \to (Y, y)$$

are left-right equivalent. Since x is a critical point of F, so is $\Phi_t(x)$, and therefore $\Psi_t(y)$, which is equal to $F(\Phi_t(x))$, lies in D. That is, we have shown that the flows Φ_t and Ψ_t preserve Σ_F and D respectively. It follows that the vector fields $\tilde{\chi}$ and χ are tangent to Σ_F and D respectively. In particular, $\chi \in \mathrm{Der}(-\log D)$.

Reciprocally, if $\chi \in \mathrm{Der}(-\log D)$ then we can certainly lift $\chi|_D$ to a vector field $\tilde{\chi}_0$ on Σ_F. For Σ_F is the normalisation[1] of D, and vector fields lift to the normalisation by a theorem of Seidenberg[2]. Suppose $\tilde{\chi}_0$ is the restriction to Σ_F of a vector field $\tilde{\chi}_1 \in \mathrm{Der}_X$. We have no guarantee that $\tilde{\chi}_1$ is a lift of χ – i.e. that $tF(\tilde{\chi}_1) = \omega F(\chi)$ – only that this equality holds on Σ_F. But because J_F is radical, the fact that $\tilde{\chi}|_{\Sigma_F}$ is a lift of $\chi|_D$ means that $tF(\tilde{\chi}_1) - \omega F(\chi) \in J_F\theta(F)$. By Cramer's rule,

$$J_F\theta(F) \subset tF(\theta_V), \tag{5.3.6}$$

and thus there exists a vector field $\xi \in \theta_V$ such that

$$tF(\xi) = tF(\tilde{\chi}_1) - \omega F(\chi), \tag{5.3.7}$$

so that finally

$$tF(\tilde{\chi}_1 - \xi) = \omega F(\chi), \tag{5.3.8}$$

showing that χ is liftable. □

Proof of 5.3.6. We show first that we can assume that f, i, j and F form a standard fibre square as in(5.3.2). To see this, we may suppose that i is an immersion, for if we replace i by the immersion $i_1(y) = (i(y), y) \in X \times Y$, and F by $F_1 = F \times \mathrm{id}_Y : X \times Y \to Z \times Y$, then the discriminant of F_1 is equal to $D \times Y$, the pull-back map

$$i_1^*(F_1) : (X \times Y) \times_{Z\times Y} \ \to Y$$

is isomorphic (even the same as) the previous pull-back $i^*(F)$, and (by Exercise 5.3.8 below)

$$\frac{\theta(i_1)}{ti_1(\theta_Y) + i_1^*(\mathrm{Der}(-\log D \times Y)} \simeq \frac{\theta(i)}{ti(\theta_Y) + i^*(\mathrm{Der}(-\log D)}.$$

With this assumption, by 1.0.2 we can choose coordinates $y_1, \ldots, y_p$ on Y and $y_1, \ldots, y_p, u_1, \ldots, u_d$ on $Z \times Y$ so that i becomes the standard immersion $i(y) = (y, 0)$. Of course this changes the map f, but the new T^1_f is isomorphic to the old. As F is transverse to i, the map $(u_1 \circ F, \ldots, u_d \circ F)$ is a submersion, so its d component functions form part of a coordinate system on X. With respect to this new coordinate system, F is an unfolding of f.

Now that we are in the situation of the standard fibre square, we revert to the notation of 5.3.2 in which the parameter space is denoted by S. We denote by $\theta_{X\times S/S}$ the $\mathcal{O}_{X\times S}$-submodule of

[1]Since Σ_F is Cohen Macaulay, it is normal if and only if it is non-singular in codimension 1 (i.e. it set of singular points has codimension at least 2 in Σ_F). Because j^1F is transverse to the stratification $\{\Sigma^k : k \in \mathbb{N}\}$ of $L(n,p)$,

$$(\Sigma_F)_{\mathrm{Sing}} = j^1F^{-1}((\overline{\Sigma_1})_{\mathrm{Sing}}) = j^1F^{-1}(\overline{\Sigma_2});$$

it therefore has codimension in Σ_F equal to codim Σ^2 – codim Σ^1, which is greater than 1.

[2]Sketched argument: the normalisation is unique up to isomorphism, so any automorphism of D lifts to an automorphism of its normalisation Σ_F; given a vector field on D, integrate it to get a 1-parameter family of automorphisms Ψ_t, lift the Ψ_t to a 1-parameter family of automorphisms Φ_t of Σ_F, then differentiate Φ_t with respect to t and set $t = 0$ to get a vector field on Σ_F lifting χ.

$\theta_{X\times S}$ consisting of vector fields on $X \times S$ with zero component in the S direction, and, similarly, by $\theta(F/S)$ the $\mathcal{O}_{X\times S}$-submodule of $\theta(F)$ consisting of vector fields along F with zero component in the S direction. We define $\theta_{Y\times S/S}$ and $\mathrm{Der}(-\log D)/S$ analogously.

Let $\pi : Y \times S \to S$ be projection. Consider the following diagram.

$$\begin{tikzcd}
0 & & 0 & 0 & & \\
C_\bullet \arrow[u] & 0 \arrow[r] & t\pi(\mathrm{Der}(-\log D)) \arrow[r] \arrow[u] & \theta(\pi) \arrow[r] \arrow[u] & 0 & \\
B_\bullet \arrow[u] & 0 \arrow[r] & \mathrm{Der}(-\log D) \arrow[r] \arrow[u, "t\pi"] & \theta_{Y\times S} \arrow[r] \arrow[u, "t\pi"] & \frac{\theta(F)}{tF(\theta_{X\times S})} \arrow[r] \arrow[u] & 0 \\
A_\bullet \arrow[u] & 0 \arrow[r] & \mathrm{Der}(-\log D)/S \arrow[r] \arrow[u] & \theta_{Y\times S/S} \arrow[r] \arrow[u] & \frac{\theta(F/S)}{tF(\theta_{X\times S/S})} \arrow[r] \arrow[u] & 0 \\
0 \arrow[u] & & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{5.3.9}$$

Each column is exact. This is obvious for the first two columns; for the third, it is an easy calculation that the homomorphism

$$\frac{\theta(F/S)}{tF(\theta_{X\times S/S})} \to \frac{\theta(F)}{tF(\theta_{X\times S})},$$

induced by the inclusion $\theta(F/S) \hookrightarrow \theta(F)$, is an isomorphism. Each row in the diagram is a complex, and thus we have a short exact sequence of complexes. Let us give the columns indices $2, 1, 0$. The resulting long exact sequence of homology contains the portion

$$\cdots \longrightarrow H_1(B\bullet) \longrightarrow H_1(C\bullet) \longrightarrow H_0(A_\bullet) \longrightarrow H_0(B_\bullet) \longrightarrow \cdots$$

However, $B_\bullet$ is exact, by Lemma 5.3.7, and thus $H_1(C_\bullet) \simeq H_0(A_\bullet)$. Evaluating these homology modules, we obtain

$$\frac{\theta(\pi)}{t\pi(\mathrm{Der}(-\log D))} \simeq \frac{\theta(F/S)}{tF(\theta_{X\times S/S}) + \omega F(\theta_{Y\times S/S})}. \tag{5.3.10}$$

Dividing each side by $\mathfrak{m}_S$ times itself gives

$$\frac{\theta(\pi)}{t\pi(\mathrm{Der}(-\log D)) + \mathfrak{m}_S\,\theta(\pi)} \simeq \frac{\theta(F/S)}{tF(\theta_{X\times S/S}) + \omega F(\theta_{Y\times S/S}) + \mathfrak{m}_S\,\theta(F/S)}; \tag{5.3.11}$$

the right hand side in (5.3.11) is just T^1_f.

It remains to show that the left hand side in (5.3.11) is isomorphic to $\theta(i)/ti(\theta_Y)+i^*(\mathrm{Der}(-\log D))$. We have

$$\theta(\pi) = \sum_{j=1}^{d} \mathcal{O}_{Y\times S}\,\frac{\partial}{\partial s_j}$$

so

$$\frac{\theta(\pi)}{\mathfrak{m}_S\,\theta(\pi)} = \sum_{j=1}^{d} \mathcal{O}_Y\,\frac{\partial}{\partial s_i}.$$

Also

$$\theta(i) = \sum_{k=1}^{p} \mathcal{O}_Y \frac{\partial}{\partial y_k} \oplus \sum_{j=1}^{d} \mathcal{O}_Y \frac{\partial}{\partial s_j}$$

so

$$\frac{\theta(i)}{ti(\theta_Y)} = \sum_{j=1}^{d} \mathcal{O}_Y \frac{\partial}{\partial s_j}.$$

So $\theta(i)/ti(\theta_Y)$ can be identified with $\theta(\pi)/\mathfrak{m}_S\,\theta(\pi)$. Using this identification, we have

$$\frac{\theta(i)}{ti(\theta_Y) + i^*\mathrm{Der}(-\log D)} \simeq \frac{\theta(\pi)}{t\pi(\mathrm{Der}(-\log D)) + \mathfrak{m}_S\,\theta(\pi)} \simeq T^1_f.$$

□

Exercises 5.3.8. 1. Show that, as stated in the proof of 5.3.6, the natural map

$$\frac{\theta(F/S)}{tF(\theta_{X\times S/S})} \to \frac{\theta(F)}{tF(\theta_{X\times S})}$$

is an isomorphism.

2. Suppose that $D = D_0 \times S \subset Z \times S$, and $i : Y \to Z \times S$, let $\pi : Z \times S \to Z$ be projection and let $j_0 : Z \to Z \times S$ be the inclusion $z \mapsto (z, 0)$.

   (a) Show that

   $$\frac{\theta(i)}{ti(\theta_Y) + i^*\mathrm{Der}(-\log D)} \simeq \frac{\theta(\pi\circ i)}{t(\pi\circ i)(\theta_Y) + (\pi\circ i)^*(\mathrm{Der}(-\log D_0))}.$$

   Hint: $\mathrm{Der}(-\log D)$ contains all the vector fields $\partial/\partial s_i$. You can choose the remaining generators for $\mathrm{Der}(-\log D)$ in $\theta_{Z\times S/S}$.

   (b) Show that

   $$\frac{\theta(i)}{ti(\theta_Y) + i^*\mathrm{Der}(-\log D_0)} \simeq \frac{\theta(j_0\circ i)}{t(j_0\circ i)(\theta_Y) + (j_0\circ i)^*(\mathrm{Der}(-\log D))}.$$

   (c) Show that if $j : Z \to Z \times S$ is any inclusion of the form $z \mapsto (z, s(z))$ then

   $$\frac{\theta(i)}{ti(\theta_Y) + i^*\mathrm{Der}(-\log D_0)} \simeq \frac{\theta(j\circ i)}{t(j\circ i)(\theta_Y) + (j\circ i)^*(\mathrm{Der}(-\log D))}.$$

The module in the denominator of (5.3.4) is in fact the (extended) tangent space to the orbit of i under a variant of contact equivalence introduced by Damon in [Dam87] and called $\mathcal{K}_D$*-equivalence*, though we will not make use of this here. It was the key to his proof of 5.3.6 in [Dam91], where he showed that if i_t is a deformation of i then the family $i_t^*(F)$ is $\mathcal{A}$-trivial if and only if i_t is $\mathcal{K}_D$-trivial.

Definition 5.3.9. *Let $f, g : (Y, y_0) \to (W, w_0)$ and let $(D, w_0) \subset (W, w_0)$. We say that f is $\mathcal{K}_D$-equivalent to g if there exists diffeomorphisms $\Phi : (Y \times W, (y_0, w_0)) \to (Y \times W, (y_0, w_0))$ and $\varphi : (Y, y_0) \to (Y, y_0)$ such that*

1. *Φ lifts φ, i.e. $\pi_Y \circ \Phi = \varphi \circ \pi_Y$;*

2. *$\Phi(Y \times D) = Y \times D$,*

3. *$\Phi(graph(f)) = graph(g)$.*

In the usual version of contact equivalence ($\mathcal{K}$-quivalence), $D = \{y_0\}$.

The advantage of the quotient (5.3.4) over the expression (3.0.5) is that in (5.3.4) all the objects are finite modules over the same ring, $\mathcal{O}_Y$, whereas the first summand in the denominator in (3.0.5) is an $\mathcal{O}_{\mathbb{C}^n,S}$-module while the second is only an $\mathcal{O}_{\mathbb{C}^p,0}$ module. This makes (5.3.4) algebraically much simpler to work with.

Definition 5.3.10. If D is a divisor (hypersurface) in W, we say D is a *free divisor* if $\mathrm{Der}(-\log D)$ is a locally free $\mathcal{O}_W$-module.

Proposition 5.3.11. ([Loo84, 6.13]) *If $F : (\mathbb{C}^n, S) \to (\mathbb{C}^p, y_0)$ $(n \geq p)$ is stable then the discriminant D of F is a free divisor.*

Proof. Let us write $(\mathbb{C}^n, S) =: X$, $(\mathbb{C}^p, y_0) =: Y$. The proof uses the exact complex

$$0 \longrightarrow \mathrm{Der}(-\log D) \longrightarrow \theta_Y \xrightarrow{\bar{\omega}F} \frac{\theta(F)}{tF(\theta_X} \longrightarrow 0 \tag{5.3.12}$$

which already appeared (in the special case that $F$is a parameterised unfolding) as the complex $B_\bullet$ of (5.3.9). We may assume that F is not a trivial unfolding of a lower-dimensional germ; freeness of $\mathrm{Der}(-\log D)$ under this assumption implies freeness in general, since the discriminant of a trivial unfolding $F \times \mathrm{id}_S$ is equal to the product of S with the discriminant of F.

From the assumption, it follows that all of the members of $\mathrm{Der}(-\log D)$ vanish at y_0. For if $\chi \in \mathrm{Der}(-\log D)_{y_0}$ and $\chi(y_0) \neq 0$, then lifting χ to $\tilde{\chi}$ in θ_X (using 5.3.7), the Thom-Levine Lemma 2.1.3 implies that the integral flows of χ and $\tilde{\chi}$ give a 1-parameter trivialisation of F.

Let $\chi_1, \ldots, \chi_\ell$ be a minimal set of generators of $\mathrm{Der}(-\log D)$, with $\chi_i = \sum_j \chi_i^j \partial/\partial y_j$, and let χ be the $p \times \ell$ matrix of coefficients χ_i^j. Then

$$\mathcal{O}_Y^\ell \xrightarrow{\chi} \theta_Y \xrightarrow{\bar{\omega}F} \frac{\theta(F)}{tF(\theta_X)} \longrightarrow 0 \tag{5.3.13}$$

is exact. Since all of the entries in χ vanish at y_0, (5.3.13) is the right-hand end of a minimal free resolution of $\theta(F)/tF(\theta_Y)$. But such a free resolution must have length 1. We prove this in two steps:

Step 1: We show that $\theta(F)/tF(\theta_X)$ is Cohen-Macaulay of dimension $p-1$. The support of this module is the critical set Σ_F of F, the set of points where F is not a submersion. Theorem 1.6.1 deduces Cohen-Macaulayness from a classical theorem of Buchsbaum and Rim, [**?**], provided that the support has the same codimension in X as the set of $p \times n$ matrices of non-maximal rank in $L(n,p)$, namely $n - p + 1$. Its codimension can be no *greater*, by a standard argument [3]. That is,

[3] If $f : X \to Y$ is a map of complex manifolds, Z is a closed complex subspace of Y, and $f^{-1}(Z) \neq \emptyset$, then

$$\text{codimension in } X \text{ of } f^{-1}(Z) \quad \leq \quad \text{codimension in } Y \text{ of } Z.$$

This applies in our situation because Σ_F is the preimage, under the map $j^1F : X \to L(n,p)$, of the set of matrices of rank $< p$ in $L(n,p)$.

its dimension is *at least* $\dim \Sigma_F = p-1$. To prove equality, we first deduce from the $\mathcal{K}$-finiteness of F, that

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_X}{J_F + F^*(\mathfrak{m}_{Y,0})\,\mathcal{O}_X} < \infty;$$

this means that the restriction of F to its critical set $V(J_F)$ is finite. The argument is that

$$\frac{\mathcal{O}_X}{J_F + F^*(\mathfrak{m}_{Y,0})\,\mathcal{O}_X}$$

and

$$\frac{\theta(F)}{tF(\mathrm{Der}_X) + F^*\mathfrak{m}_{Y,0}\theta(F)} \tag{5.3.14}$$

have the same support, namely $F^{-1}(0) \cap V(J(F))$, and Theorem 3.4.1 implies that (5.3.14) has finite complex vector space dimension (in fact $\leq p$), so its support is just $\{0\}$. So the dimension of $V(J_F)$ is no greater than the dimension of its image in $D \subset Y$. This image is a closed variety, by finiteness, and cannot be all of Y, by Sard's theorem. Therefore it has dimension no *greater* than $p-1$. Hence $\dim V(J_F) = \dim D \leq p-1$, and $\theta(F)/tF(\mathrm{Der}_X)$ is Cohen Macaulay of dimension $p-1$ as required. For future use we note that J_F must in fact be radical: the condition on the codimension of the support of $\theta(F)/tF(\mathrm{Der}_X)$ guarantees that $\mathcal{O}_X\,/J_F$ also is Cohen-Macaulay; a Cohen-Macaulay space is reduced if and only if it is generically reduced (see e.g.[Loo84, page 50]), so one can check reducedness at a generic point, i.e. by a local calculation, and this is easily done for example at a fold point.
Step 2: Because $\theta(F)/tF(\mathrm{Der}_X)$ has depth $p-1$ over $\mathcal{O}_X$, and is finite over $\mathcal{O}_Y$, its $\mathcal{O}_Y$-depth is also $p-1$. Therefore by Auslander-Buchsbaum its projective dimension is 1. It follows that the kernel of $\overline{\omega F}$ is free.

□

Let h be an equation of $D(F)$, and define $\mathrm{Der}(-\log h)$ to be the $\mathcal{O}_Y$-module of germs of vector fields which annihilate h; that is, which are tangent not only to $D(F) = h^{-1}(0)$, but to all level sets of h. Clearly $\mathrm{Der}(-\log h)$ is a submodule of $\mathrm{Der}(-\log \Delta(F))$, but it depends on the choice of equation h, and is not determined by $D(F)$ alone. The following lemma gives a simple condition under which, if D is a free divisor, then not only $\mathrm{Der}(-\log D)$ but also $\mathrm{Der}(-\log h)$ is a locally free $\mathcal{O}_{\mathbb{C}^p}$-module.

Lemma 5.3.12. *If $h \in J_h$ (that is, if there exists a vector field χ such that $\chi \cdot h = h$) then*

$$\mathit{Der}(-\log D) = \mathit{Der}(-\log h) \oplus \mathcal{O}_Y \cdot\chi. \tag{5.3.15}$$

Proof. Any vector field $\vartheta \in \mathrm{Der}(-\log D)$ can be written in the form

$$\vartheta = (\vartheta - (\vartheta \cdot h)\chi) + ((\vartheta \cdot h)\chi)\,.$$

□

This applies in case h is weighted homogeneous, taking as χ the Euler vector field. But even if no such vector field exists for h, we can find an equation for the discriminant of a constant unfolding $F_1 := F \times \mathrm{id}_{\mathbb{C}}$, for which the splitting (5.3.15) holds. Take t as the coordinate on the extra copy of $\mathbb{C}$; since $D(F_1) = D(F) \times \mathbb{C}$, we can use $h_1(y,t) = e^t h(y)$ as equation for $D(F_1)$, and $\partial/\partial t$ as the vector field χ.

Exercise 5.3.13. *If $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is obtained from F by transverse pull back, then it can also be obtained from $F \times id_{\mathbb{C}}$ by transverse pull back.*

Proposition 5.3.14. ([DM91]) *Suppose that $F : (\mathbb{C}^N, 0) \to (\mathbb{C}^P, 0)$ is stable, and that* (5.3.15) *holds. If $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ has finite $\mathcal{A}_e$-codimension, with $n \geq p$ and (n, p) nice dimensions, and f is obtained from F by transverse pull back by i, then*

$$\mu_\Delta(f) = dim_{\mathbb{C}} \ \frac{\theta(i)}{ti\big(\theta_{\mathbb{C}^p,0}\big) + i^*\big(Der(-\log h)\big)}. \tag{5.3.16}$$

Proof. The proof involves four steps:

1. Let h be a reduced defining equation of the discriminant $D(f)$ of F, and let $I : (\mathbb{C}^p \times \mathbb{C}), (0,0)) \to (\mathbb{C}(P), 0$ be a deformation of i, inducing a stabilisation of f, that is, such that if $F(x, u) = (f_u(x), u)$ is the induced unfolding of f, then f_u is stable then for $u \neq 0$. Such a deformation exists since (n, p) are nice dimensions. Write $I(y, u) = i_u(y)$. Then for $u \neq 0$, i_u is logarithmically transverse to $D(F)$, and for each $u \in (\mathbb{C}, 0)$, $h \circ i_u$ is a reduced defining equation for $D(f_u)$. For each point $y \notin D(f_t)$, $d(h \circ i_t)$ gives rise to an isomorphism
$$\frac{\theta(i_t)_y}{ti_t(\theta_{\mathbb{C}^p,y}) + i_t^*(\mathrm{Der}(-\log h))_y} \simeq \frac{\mathcal{O}_{\mathbb{C}^p,y}}{J_{h \circ i_t}}$$
and thus
$$\sum_{y \notin D(f_t)} \dim_{\mathbb{C}} \frac{\theta(i_t)_y}{ti_t(\theta_{\mathbb{C}^p,y}) + i_t^*(\mathrm{Der}(-\log h))_y} = \sum_{y \notin D(f_t)} \mu(h \circ i_t, y) \tag{5.3.17}$$

2. The right hand side in (5.3.17) is equal to $\mu_\Delta(f)$, by Theorem 5.2.2.

3. At all points $y \in D(f_u)$,
$$\frac{\theta(i_u)_y}{ti_u(\theta_{\mathbb{C}^p,y}) + i_u^*(\mathrm{Der}(-\log D))_y} = 0$$
by the isomorphism of Theorem (5.3.6), for f_u is stable. In the nice dimensions, all stable germs are quasihomogeneous in suitable coordinates[4], and so
$$\frac{\theta(i_u)_y}{ti_u(\theta_{\mathbb{C}^p,y}) + i_u^*(\mathrm{Der}(-\log h))_y} = \frac{\theta(i_u)_y}{ti_u(\theta_{\mathbb{C}^p,y}) + i_u^*(\mathrm{Der}(-\log D))_y} = 0.$$
Thus,
$$\sum_y \dim_{\mathbb{C}} \frac{\theta(i_u)_y}{ti_u(\theta_{\mathbb{C}^p,y}) + i_u^*(\mathrm{Der}(-\log h))_y} = \sum_{y \notin D(f_u)} + \sum_{y \in D(f_u)} = \sum_{y \notin D(f_u)} = \mu_\Delta(f).$$

4. The final step is to show that
$$\sum_y \dim_{\mathbb{C}} \frac{\theta(i_t)_y}{ti_t(\theta_{\mathbb{C}^p,y}) + i_t^*(\mathrm{Der}(-\log h))_y} = \dim_{\mathbb{C}} \ \frac{\theta(i)}{ti\big(\theta_{\mathbb{C}^p,0}\big) + i^*\big(\mathrm{Der}(-\log h)\big)}$$

[4]This property of the nice dimensions can be checked by inspection of Mather's list of stable types in [Mat71]. In fact it *characterises* the nice dimensions, a fact which surely deserves explanation.

– in other words, that we have conservation of multiplicity. To show this, consider the relative module,

$$T^1_{\mathscr{K}_h/\mathbb{C}}I := \frac{\theta(I)}{tI(\theta_{\mathbb{C}^p\times\mathbb{C}/\mathbb{C}}) + I^*(\mathrm{Der}(-\log h)}.$$

This has presentation

$$\theta_{\mathbb{C}^p\times\mathbb{C}/\mathbb{C}} \oplus I^*(\mathrm{Der}(-\log h)) \to \theta(I) \to T^1_{\mathscr{K}_h/\mathbb{C}}I \to 0. \tag{5.3.18}$$

Now $\theta_{\mathbb{C}^p\times\mathbb{C}/\mathbb{C}}$ is free of rank p, and because $\mathrm{Der}(-\log h)$ is a free $\mathcal{O}_{\mathbb{C}^p}$-module of rank $P-1$, $I^*(\mathrm{Der}(-\log h))$ is free of rank $P-1$ over $\mathcal{O}_{\mathbb{C}^p\times\mathbb{C}}$; thus 5.3.18 can be written in the form

$$\mathcal{O}^p \oplus \mathcal{O}^{P-1} \;\to\; \mathcal{O}^P \;\to T^1_{\mathscr{K}_h/\mathbb{C}}I \to 0, \tag{5.3.19}$$

where $\mathcal{O} = \mathcal{O}_{\mathbb{C}^p\times\mathbb{C},0}$. A classical theorem of Buchsbaum and Rim states that the codimension of the support of the cokernel of an $\mathcal{O}$-linear map $\mathcal{O}^\ell \;\to\; \mathcal{O}^m$ (with $\ell \geq m$) is less than or equal to $\ell - m + 1$, and that if equality holds then the cokernel is Cohen Macaulay as $\mathcal{O}$-module. In our situation, this becomes

$$\mathrm{codim}(\mathrm{supp}(T^1_{\mathscr{K}_h/\mathbb{C}}I)) \leq (p+P-1) - P + 1 = p.$$

The inequality is an equality here because

$$\mathrm{supp}\left(T^1_{\mathscr{K}_h/\mathbb{C}}I\right) \cap (\mathbb{C}^p \times \{0\}) = \mathrm{supp}T^1_{\mathscr{K}_h}i = \{0\}; \tag{5.3.20}$$

if the codimension of $\mathrm{supp}(T^1_{\mathscr{K}_h/\mathbb{C}}I))$ were less than p, then its intersection with $\mathbb{C}^p \times \{0\}$ would have strictly positive dimension.

From Buchsbaum-Rim we therefore conclude that $T^1_{\mathscr{K}_h/\mathbb{C}}I$ is Cohen-Macaulay of dimension 1. Let $\pi : \mathbb{C}^p \times \mathbb{C} \to \mathbb{C}$ be projection; it is finite on $\mathrm{supp}\, T^1_{\mathscr{K}_h/\mathbb{C}}$ by (5.3.20), and it follows that the push-forward $\pi_*\left(T^1_{\mathscr{K}_h/\mathbb{C}}\right)$ to the base space $\mathbb{C}$ is a Cohen-Macaulay $\mathcal{O}_\mathbb{C}$-module of dimension 1, and therefore free. This is a standard argument: depth and dimension do not change under finite push-forward, so $\pi_*\left(T^1_{\mathscr{K}_h/\mathbb{C}}\right)$ is Cohen-Macaulay of dimension 1; since $\mathcal{O}_\mathbb{C}$ is regular, every finite module has a finite projective resolution; the Auslander-Buchsbaum formula

$$\mathrm{depth}_R M + \text{projective dimension}_R(M) = \mathrm{depth}_R R$$

for a finite module M over a Noetherian local ring R (which assumes the finiteness of both summands on the left) implies that the length of a minimal free resolution of $\pi_*\left(T^1_{\mathscr{K}_h/\mathbb{C}}\right)$ over $\mathcal{O}_\mathbb{C}$ is zero – i.e. that $\pi_*\left(T^1_{\mathscr{K}_h/\mathbb{C}}\right)$ is free. Finally

$$\sum_y \dim_\mathbb{C} \frac{\theta(i_t)_y}{ti_t(\theta_{\mathbb{C}^p,y}) + i_t^*(\mathrm{Der}(-\log h))_y} = \mathrm{rank}\, \pi_*\left(T^1_{\mathscr{K}_h/\mathbb{C}}\right)_u$$

$$= \mathrm{rank}\, \pi_*\left(T^1_{\mathscr{K}_h/\mathbb{C}}\right)_0 = \dim_\mathbb{C} \frac{\theta(i)}{ti\big(\theta_{\mathbb{C}^p,0}\big) + i^*\big(\mathrm{Der}(-\log h)\big)}.$$

□

Proof of Theorem 5.2.4: The inequality in Theorem 5.2.4 follows immediately from 5.3.14 and 5.3.6. Equality holds when f is weighted homogeneous, because then

$$\frac{\theta(i)}{ti(\theta_{\mathbb{C}^p,0}) + i^*(\mathrm{Der}(-\log h))} = \frac{\theta(i)}{ti(\theta_{\mathbb{C}^p,0}) + i^*(\mathrm{Der}(-\log D))}.$$

$\square$

Remark 5.3.15. *All except the last step in the proof just given work unchanged with the image Milnor number of a map-germ* $(\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ *in place of the discriminant Milnor number of a map* $(\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ *with* $n \geq p$*. However, without a proof of conservation of multiplicity, all that can be concluded is that*

$$\mu_I(f) \leq \dim_{\mathbb{C}} \frac{\theta(i)}{ti(\theta_{\mathbb{C}^p,0} + i^*(Der(-\log h))}.$$

Attempts to prove the statement of Theorem 5.2.5 for $n > 2$ *have generally focussed on proving this missing step. The crucial extra ingredient for discriminants is the fact that the discriminant of a stable map-germ is a free divisor.*

5.4 Open questions

1. The "Mond conjecture" asserts that if $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is a map-germ of finite codimension, and $(n, n+1)$ are in Mather's nice dimensions, then $\mu_I(f) \geq \mathcal{A}_e - \text{codim }(f)$, with equality if f is weighted homogeneous. It is proved for $n = 1$ and $n = 2$, and supported by many examples.

2. A famous theorem of Lê and Ramanujan states that a μ-constant family of isolated hypersurface singularities is topologically trivial, provided the ambient dimension is not 3. It is unknown whether this holds also in dimension 3. Do the image and discriminant Milnor numbers have an equally crucial role in determining the topology?

3. A stable perturbation of a finitely determined real map-germ $(\mathbb{R}^n, S) \to (\mathbb{R}^{n+1}, 0)$ is *maximal* if it exhibits all of the 0-dimensional stable singularities present in its complexification. It is a *good real perturbation* if the real image has n'th homology of rank $\mu_I(f)$ (so that inclusion of real image in complex image induces an isomorphism on H_n). Is it true that every good real perturbation is maximal? This is the case in all known examples. The same question is also open, concerning maps $(\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ with $n \geq p$, with "discriminant" replacing "image" and μ_Δ replacing μ_I.

Chapter 6

Multiple points in the source

The multiple point spaces of a map-germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with $n < p$ play an important rôle in the study of its geometry, as well as the topology of the image of a stable perturbation.

Introduction

We will study multiple-point spaces of maps $X \to Y$ in both the source and the target. Source k-tuple point spaces live in X, where they may be defined, roughly speaking, as the set of points x such that $|f^{-1}(f(x))| \geq k$, perhaps taking multiplicity into account, or in X^k, where they may be defined as the set of ordered k-tuples of points having the same image – though complications arise because of the presence of the diagonals (k-tuples of points of which some or even all are equal to one another), which are contained, willy nilly, in the spaces defined in the most naive way. The advantage of locating the space in X^k is the presence of the symmetric group action, by permutation of the copies of X, which carries the gluing data. In Section **??** we will see how the cohomology of the image of a map can be recovered from information on the symmetric group actions on the cohomology of the various multiple point spaces.

Target multiple point spaces no longer carry gluing data, but are integral to the understanding of the geometry of the image. They will be studied in Chapter 7.

6.0.1 Source Multiple Point Spaces

Let $f : X \to Y$ be a map of complex spaces. For each natural number $k \geq 2$ we define the idiot's multiple point space $ID^k(f)$ in the simplest possible way:

$$ID^k(f) = \{(x_1, \ldots, x_k) \in X^k : f(x_i) = f(x_j) \text{ for all } i, j\} \tag{6.0.1}$$

We note one obvious property, which will be shared by all of the versions of multiple point spaces that we consider, namely that the symmetric group S_k acts on $ID^k(f)$ by permuting the copies of X. The corresponding representation of the symmetric groups S_k on the cohomology of the multiple point spaces is crucial for relating the cohomology of the multiple point spaces with the cohomology of the image of f.

The space $ID^k(f)$ is easy to define and to write down equations for, but suffers from many disadvantages. For every f and every k, $ID^k(f)$ contains the small diagonal $\Delta_k = \{(x_1, \ldots, x_k) \in X^k : x_1 = \cdots = x_k\}$, which really does not deserve to be called a k-tuple point. Even when (x_1, x_2) is a genuine double point, with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$, the bogus triple points (x_1, x_1, x_2) and

(x_1, x_2, x_2) (and their permutations) lie in $ID^3(f)$. So $ID^k(f)$ has many irreducible components of differing dimensions and differing degrees of stupidity. It is only smooth when it is purely idiotic, in the case where f is a 1-1 immersion. Clearly if we want to derive information about f from the D^k we should find a way of removing, or disregarding, some of these components.

There are a number of different approaches to the definition of a more intelligent multiple point space. At one extreme, we define the "most genuine" multiple point space, $D^k_g(f)$ as follows. Let $X^{\langle k\rangle} = \{(x_1, \ldots, x_k) \in X^k : x_i \neq x_j \text{ if } i \neq j\}$. Then

$$D^k_g(f) = \text{closure } \{(x_1, \ldots, x_k) \in X^{\langle k\rangle} : f(x_i) = f(x_j) \text{ far all } i, j\} \tag{6.0.2}$$

Here the closure is taken in X^k, and we give it its reduced structure. This space excludes most points on the various diagonals, though it turns out that it does contain certain non-immersive points. For example, as we will see in ?? below, if f is of corank 1 at x_0, and $\dim_{\mathbb{C}} Q_f(x_0) = \ell < \infty$ then $(x_0, \ldots, x_0) \in D^k(f)$ if and only if $2 \leq k \leq \ell$.

Example 6.0.1. Consider the map-germ

$$f(x, y) = (x, y^2, xy)$$

parameterising the Whitney umbrella $Z^2 - X^2Y = 0$. We have

$$f(x_1, y_1) = f(x_2, y_2) \Leftrightarrow \begin{cases} x_1 = x_2 \\ y_1^2 = y_2^2 \\ x_1 y_1 = x_2 y_2 \end{cases}$$

and with the assumption that $(x_1, y_1) \neq (x_2, y_2)$ this means in particular that $y_1 - y_2 \neq 0$ and thus, from the second equation, $y_1 = -y_2$. The first and third equalities can be combined to give

$$x_1 y_1 - x_1 y_2 = 0$$

and this can be divided by $y_1 - y_2$, giving $x_1 = 0$. So $D^2(F)$ is a line in $\mathbb{C}^2 \times \mathbb{C}^2$, containing the point $\big((0, 0), (0, 0)\big)$ in its closure. We note that $(0, 0)$ is in fact the unique non-immersive point of F. From the calculation just made it is clear that $D^k(F) = \emptyset$ for $k \geq 3$.

One problem with this version of multiple point space is that it does not commute with restriction. That is, if $X \subset \mathcal{X}$, $Y \subset \mathcal{Y}$, and $F : \mathcal{X} \to \mathcal{Y}$ restricts to $f : X \to Y$, and if we denote by $i_k : X^k \to \mathcal{X}^k$ the obvious inclusion, then $D^k_g(f)$ may be different from $i_k^{-1}(D^k_g(F))$.

Example 6.0.2. Consider $F : \mathbb{C}^2 \to \mathbb{C}^3$ defined by $F(x, y) = (x, y^2, y^3 + xy)$. This is a twisted cross-cap; it is left-right equivalent to the map-germ $F_0(x, y) = (x, y^2, xy)$, but is preferable for the purposes of our example. Being equivalent to the stable germ F_0 means that F is stable. An easy modification of the calculation in Example 6.0.1 shows that

$$D^2(F) = \{\big((x_1, y_1), (x_2, y_2)\big) : x_1 = x_2, y_1 = -y_2, x_1 + y_1^2 = 0\}.$$

Now let $f : \mathbb{C} = \{0\} \times \mathbb{C} \to \{0\} \times \mathbb{C}^2 = \mathbb{C}^2$ be the restriction of F. Then $f(y) = (y^2, y^3)$, f is 1-1, and $D^k(f) = \emptyset$ for all $k \geq 2$. On the other hand, $i_2^{-1}(D^2(F)) = \{(0, 0)\}$.

For reasons which will become apparent later, this failure to commute with restriction is undesirable. Some of the most interesting applications of multiple point spaces rely precisely on the property that if $F : X \times P \to Y \times P$ is an unfolding of $f : X \to Y$ then the natural projection

$$D^k(F) \to P \tag{6.0.3}$$

is a deformation of $D^k(f)$, or, at the very least, $D^k(f)$ should be the fibre over 0 of this projection. To remedy this, we introduce a third variant of the definition, due originally to Terry Gaffney in [**?**], which assumes that all germs of f have finite singularity type - that is, possess a stable unfolding on a finite-dimensional parameter space. We work locally; that is, for each point $\mathbf{x} = (x_1, \ldots, x_k) \in ID^k(f)$ we define the germ of $D^k(f)$ at $\mathbf{x}$ by means of an ideal in $\mathcal{O}_{X^k,\mathbf{x}}$. We have then to show that all the ideals piece together to give a coherent sheaf of ideals. The definition is best given at two levels of generality. First we suppose that $\mathbf{x} \in ID^k(f)$ lies on the small diagonal: $x_i = x_j$ for all i, j; we denote the common value by x_0 and write $y_0 = f(x_0)$. We assume that the germ of $f : (X, x_0) \to (Y, y_0)$ is of finite singularity type. Choose a germ of stable unfolding $F : (X \times \Lambda, (x_0, 0)) \to (Y \times \Lambda, (y_0, 0))$ and set

$$D^k(F) = D^k_g(F) \tag{6.0.4}$$

with its reduced structure (i.e. $\mathcal{O}_{D^k(F)} = \mathcal{O}_{X^k} / I(D^k(F))$). Then *define* $D^k(f)$ to be the fibre over $0 \in P$ of the projection (6.0.3).That is, the defining ideal $\mathcal{I}^k(f)$ of $(D^k(f), \mathbf{x})$ is the restriction to $X^k = (X \times \{0\})^k$ of the defining ideal $\mathcal{I}^k(F)$ of $(D^k(F), (\mathbf{x}, 0))$. Now the structure may not be reduced, of course.

Any unfolding F takes the form

$$F(x, \lambda) = (f_\lambda(x), \lambda)$$

and the equations for $D^k_g(F)$ in $(X \times \Lambda)^k$ force $\lambda_1 = \cdots = \lambda_k$. Thus $D^k_g(F)$ naturally embeds in $X^k \times \Lambda$, where it is defined as the closure of the set of pairwise distinct k-tuples $(x_1, \lambda), (x_2, \lambda), \ldots, (x_k, \lambda)$ such that

$$f_\lambda(x_1) = f_\lambda(x_2) \cdots = f_\lambda(x_k). \tag{6.0.5}$$

From now on we make use of this embedding without further comment.

Example 6.0.3. We continue with Example 6.0.2. Here F is in fact a stable unfolding of f, so we can apply the definition without further calculation. We have

$$\mathcal{I}^2(F) = (y_1 + y_2, y_1^2 + x);$$

it follows that

$$\mathcal{I}^2(f) = (y_1 + y_2, y_1^2);$$

the germ of $D^2(f)$ at $(0, 0)$ is a fat point, unlike that of $(D^2_g(f)$, which is empty.

To extend the definition of $D^k(f)$ to the case where $\mathbf{x} = (x_1, \ldots, x_k)$ does not lie on the small diagonal, part of the difficulty we face is one of notation. To describe a multi-germ $f : (X, S) \to (Y, y_0)$, where $|S| = r$, we need labels for the r members of S and r systems of coordinates, one around each of the points of S. When we talk about multiple points we need multiple copies of each set of coordinates, which have to be distinguished from one another in some way. As we have seen, it is most natural to make each point of S the origin in the local coordinate system which describes its neighbourhood. We therefore adopt the following conventions:

Members of S are labelled $O^{(1)}, \ldots, O^{(r)}$
Points in the neighbourhood of $O^{(i)}$ are labelled $x^{(i)}$
The germ of f at $O^{(i)}$ is labelled $f^{(i)}$
Coordinates around $O^{(i)}$ are labelled $x_1^{(i)}, \ldots, x_n^{(i)}$
Where we consider a point in $ID^k(f)$ lying on one of the diagonals,
points in the neighbourhood of the j'th copy of $O^{(i)}$ are labelled $x^{(ij)}$
and coordinates around the j'th copy of $O^{(i)}$ are labelled $x_1^{(ij)}, \ldots, x_n^{(ij)}$.

In what follows, we make use of this perhaps cumbersome notation.

Consider a point $(x_1, \ldots, x_k) \in X^k$. After a permutation we may assume that

$$x_1 = \cdots = x_{k_1}, x_{k_1+1} = \cdots = x_{k_1+k_2}, \cdots, \cdots, x_{k_1+\cdots+k_{\ell-1}+1} = \cdots = x_k.$$

and there are no further equalities among the x_j. Following our labelling convention, $(x_1, \ldots, x_k)$ becomes

$$(\underbrace{O^{(1)}, \ldots, O^{(1)}}_{k_1}, \underbrace{O^{(2)}, \ldots, O^{(2)}}_{k_2}, \ldots, \ldots, \underbrace{O^{(\ell)}, \ldots, O^{(\ell)}}_{k_\ell})$$

Let $S = \{O^{(1)}, O^{(2)}, \ldots, O^{(\ell)}\}$. We consider the multi-germ

$$f : (X, S) \to (Y, y)$$

and choose a stable unfolding $F : (X \times \mathfrak{l}, S \times \{0\}) \to (Y \times \mathfrak{l}, (y_0, 0))$. Exactly as before, we define

$$D^k(F) = D^k_g(F) \tag{6.0.6}$$

and

$$D^k(f) = i_k^{-1}(D^k(F)) \tag{6.0.7}$$

Example 6.0.4. We consider the bi-germ $f : (\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$ defined by

$$f : \begin{cases} f^{(1)}(x_1^{(1)}, x_2^{(1)}) = (x_1^{(1)}, x_1^{(1)}, x_2^{(1)}) \\ f^{(2)}(x_1^{(2)}, x_2^{(2)}) = (x_1^{(2)}, (x_2^{(2)})^2, x_1^{(2)} x_2^{(2)}) \end{cases}$$

The two mono germs making up this bi-germ are both stable. As $f^{(1)}$ is an immersion, its isosingular locus is its image, the plane $\{(X, Y, Z) : X = Y\}$; the isosingular locus of $f^{(2)}$ is just the origin $\{(0, 0, 0)\}$, and the instability of f is due to the failure of transversality of the two, *cf.* In order to stabilise$f$ we take the unfolding

$$F : \begin{cases} F^{(1)}(x_1^{(1)}, x_2^{(1)}, \lambda) = (x_1^{(1)}, x_1^{(1)} + \lambda, x_2^{(1)}, \lambda) \\ F^{(2)}(x_1^{(2)}, x_2^{(2)}, \lambda) = (x_1^{(2)}, (x_2^{(2)})^2, x_1^{(2)} x_2^{(2)}, \lambda) \end{cases} \tag{6.0.8}$$

of f. The isosingular loci of $F^{(1)}$ and $F^{(2)}$ are now $\{(X_1, X_2, X_3, \lambda) \in \mathbb{C}^4 : X_2 - X_1 - \lambda = 0\}$ and $\{(0, 0, 0)\} \times \mathbb{C}$. These meet transversely, so that F is indeed stable. Then $D^2(F)$ has three components:

$$\begin{aligned} D^{(2)(2)}(F) :=\ & D^2(F^{(2)}) = D^2(f^{(2)}) \times \mathbb{C} \\ D^{(1)(2)}(F) :=\ & \{((x^{(1)}, \lambda), (x^{(2)}, \lambda)) : F_1(x^{(1)}, \lambda) = F_2(x^{(2)}\lambda)\} \\ D^{(2)(1)}(F) :=\ & \sigma(D^{(1)(2)}(F)) \end{aligned}$$

where σ is the involution interchanging (x_1, y_1, λ) and (x_2, y_2, λ). Note that $D^{(1)(1)}(F)$ is empty, since $F^{(1)}$ is an immersion. We see directly from (6.0.8) that $D^{(1)(2)}(F)$ is defined by equations

$$\begin{cases} x_1^{(1)} = x_1^{(2)} \\ x_1^{(1)} + \lambda = (x_2^{(2)})^2 \\ x_2^{(1)} = x_1^{(2)} x_2^{(2)}. \end{cases}$$

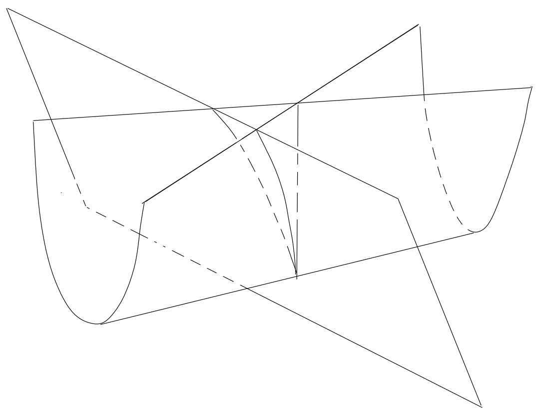


Figure 6.1: Image of the bi-germ F.

Further calculation shows that $D^3_g(F)$ is a union of three disjoint smooth curves:

$$D^3(F) = D^{(1)(2)(2)}(F) \cup D^{(2)(1)(2)}(F) \cup D^{(2)(2)(1)}(F); \tag{6.0.9}$$

$D^{(1)(2)(2)}(F)$ consists of points $\big((x^{(1)}, \lambda), (x^{(2,1)}, \lambda), (x^{(2,2)}, \lambda)\big)$ such that

$$\begin{cases} \big((x^{(2,1)}, \lambda), (x^{(2,2)}, \lambda)\big) \in D^2(F^{(2)}) \\ F^{(1)}(x^{(1)}, \lambda) = F^{(2)}(x^{(2,1)}, \lambda) \end{cases}$$

and thus has equations

$$\begin{cases} x_1^{(2,1)} = x_1^{(2,2)} = 0 \qquad x_2^{(2,1)} + x_2^{(2,2)} = 0 \\ x_1^{(1)} = 0 \qquad x_1^{(1)} + \lambda = (x_2^{(2,1)})^2 \qquad x_2^{(1)} = 0 \end{cases}$$

The other two terms on the right hand side of (6.0.9) are permutations of $D^{(1)(2)(2)}(F)$. It follows by (6.0.7) that the defining ideal of the germ of $D^3(f)$ at $(O^{(1)}, O^{(2)}, O^{(2)})$ is

$$\big(x_1^{(2,1)}, x_1^{(2,2)}, x_2^{(2,1)} + x_2^{(2,2)}, x_1^{(1)}, (x_2^{(2),1)})^2, x_2^{(1)}\big) \subset \mathcal{O}_{\left(\mathbb{C}^2\right)^3, (O^{(1)}, O^{(2)}, O^{(2)})};$$

this germ is thus a fat point. Altogether $D^3(f)$ consists of three fat points, forming a single orbit of S_3.

Although our definition of $D^k(f)$ is rather implicit, and gives no hint as to how defining equations may be found, the collection of spaces fit together to form an intricate structure. For each k there are projections $\pi^i_{k,k-1} : D^k(f) \to D^{k-1}(f)$, induced by the projection $X^k \to X^{k-1}$ which forgets the i'th copy of X.

Proposition 6.0.5. *1. If $(x^{(1)}, \ldots, x^{(k)}) \in X^{\langle k\rangle}$ and for some i, j $x^{(i)} \neq x^{(j)}$ then*

$$(x^{(1)}, \ldots, x^{(k)}) \in D^k(f) \Leftrightarrow f(x^{(1)}) = \cdots = f(x^{(k)}).$$

2. *The definition of $D^k(f)$ is independent of the choice of stable unfolding used in applying* (6.0.7)*.*

3. *If F is any unfolding of f on base-space $\mathfrak{l}$ then $D^k(f) = D^k(F)|_{\{\lambda=0\}}$.*

4. *Suppose that X and Y are complex manifolds and $f : X \to Y$ is an analytic map. Then the sheaf of ideals on X^k by which we have defined $D^k(f)$ is coherent.*

5. *If $f : (X, x_0) \to (Y, y_0)$ is a finitely determined map-germ and $D^k_g(f) \neq \emptyset$ then $D^k(f) = D^k_g(f)$.*

Proof. 1. This is easy to see: if $f(x_i) \neq f(x_j)$, let F be a stable unfolding of the germ of f at $(x^{(1)}, \ldots, x^{(k)})$ on parameter space P. Then there are neighbourhoods $U^{(i)}, U^{(j)}$ of $(x^{(i)}, 0)$ and $(x^{(j)}, 0)$ in $X \times P$ such that $F(U^{(1)}) \cap F(U^{(2)}) = \emptyset$. It follows that $(x^{(1)}, \ldots, x^{(k)}) \notin D^k_g(F)$.
2. Let G be a stable unfolding of F on parameters ξ. Then G is also a stable unfolding of f on parameters λ, ξ. By (6.0.7), $D^k(f) = D^k(G)|_{\{\lambda=\xi=0\}}$ and $D^k(F) = D^k(G)|_{\{\xi=0\}}$. Thus $D^k(f) = D^k(F)|_{\{\lambda=0\}}$.
3.
4. □

Lemma 6.0.6. *Let $M_k = \{y \in (\mathbb{C}^p, 0) : |f^{-1}(y)| \geq k\}$. If $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is finitely determined multi-germ then*

$$\dim M_k = n - (k-1)(p-n) \tag{6.0.10}$$

provided M_k is not empty.

Proof. Choose a good proper representative $\tilde{f} : U \to V$ for which every multi-germ $f_y : (U, f^{-1}(y)) \to (V, y)$ with $y \neq 0$ is stable (see 3.7.4) and let $U^* = U \setminus \{0\}$. Consider the map $\tilde{f}^k : (U^*)^{\langle k\rangle} \to V^k$. Because each multi-germ is stable, this map is transverse to the small diagonal $\Delta := \{(y_1, \ldots, y_k) \in V^k : y_1 = \ldots = y_k\}$. The preimage $(\tilde{f}^k)^{-1}(\Delta)$ is therefore a manifold of dimension $kn - (k-1)p$, if it is not empty. We have $M_k = f(\pi_i((\tilde{f}^k)^{-1}(\Delta))$, where $\pi_i : (U^*)^{\langle k\rangle} \to U$ is any one of the standard projections. Each projection $\pi_i| : (\tilde{f}^k)^{-1}(\Delta) \to U$ is finite, as is f, and so $f(\pi_i(\tilde{f}^k)^{-1}(\Delta))$ is a constructible set of dimension $kn - (k-1)p = n - (k-1)(p-n)$. □

Proposition 6.0.7. *Let $f : X \to Y$ be an analytic map, let $\Delta_k \subset X^k$ denote the small diagonal, and, for $x \in X$, denote by x^k the point $(x, \ldots, x) \in \Delta_k$.*
1. Then

$$D^k(f) \cap \Delta_k \subseteq \{x^k : \dim_{\mathbb{C}} Q(f)_x \geq k\} \tag{6.0.11}$$

2. If the local algebra $Q(f)_x$ is of type Σ^1 or Σ^2, or of discrete algebra type, then equality holds in (6.0.11)*. In particular, equality in* (6.0.11) *holds for all stable germs in the nice dimensions.*

Proof. 1. This is straightforward. Choose a proper representative $f : V \to W$ of the germ of f at x_0. Let $g_1, \ldots, g_\ell \in \mathcal{O}_{X,x_0}$ project to a basis of $Q(f)_x$. Then $\mathcal{O}_{X,x_0}$ is generated over $\mathcal{O}_{Y,y_0}$ by $g_1, \ldots, g_\ell$, and so there is a presentation

$$\mathcal{O}^m_{Y,y_0} \longrightarrow \mathcal{O}^\ell_{Y,y_0} \longrightarrow \mathcal{O}_{X,x_0} \longrightarrow 0\ .$$

By the finite coherence theorem, this extends to a presentation of $f_* \mathcal{O}_X$ on some neighbourhood U of y_0,

$$\mathcal{O}_U^m \longrightarrow \mathcal{O}_U^\ell \longrightarrow f_* \mathcal{O}_{V\cap f^{-1}(U)} \longrightarrow 0 \; .$$

It follows that for each $y \in U$,

$$\sum_{x\in V} \dim_{\mathbb{C}} \mathcal{O}_{X,x} / f^* \mathfrak{m}_{Y,y_0} \mathcal{O}_{X,x} = \dim_{\mathbb{C}} f_*(\mathcal{O}_{V\cap f^{-1}(U)})_y \leq \ell.$$

Thus, y can have no more than ℓ distinct preimages. Exactly the same argument applies to a stable unfolding F of f, showing that $(x,0)^k \notin D^k_g(F)$ if $k > \dim_{\mathbb{C}} Q(F)(x,0)$. Since $Q(F)_{(x,0} \simeq Q(f)_x$, this shows that $x^k \notin D^k(f)$ if $k > \ell$.

2. The equality in (6.0.11) for the stated algebra types is the principal result of [JG76] and we do not reproduce their arguments here, though see remark 6.0.11 for a discussion of the easiest case. Their statement is as follows: for a germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p,)$, choose a representative $\tilde{f} : U \to V$ and define its *topological multiplicity* $m(\tilde{f})$, as the greatest integer k such that in every neighbourhood V_1 of 0 in V and for every neighbourhood U_1 of 0 in U, there is a point $y \in V_1$ such that y has k distinct preimages in U_1. It is easy to see that $m(\tilde{f})$ is independent of the choice of representative, and thus that we have defined an invariant of the germ f itself, which we denote by $m(f)$. We will abbreviate this rather cumbersome definition of $m(f)$ by saying that it is the largest integer k such that there are points in $(\mathbb{C}^p, 0)$ with k distinct preimages in $(\mathbb{C}^n, 0)$. Let $\delta(f) = \dim_{\mathbb{R}} Q(f)$.

Theorem 6.0.8. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, with $n < p$, be a stable map germ of corank ≤ 2 or of discrete algebra type. Then $m(f) = \delta(f)$.*

Although this refers to $\mathbb{R}$ rather than $\mathbb{C}$, the proof in [JG76] works without change over $\mathbb{C}$. In fact we can circumvent the need to check this for a great number of algebra types. A large number of local algebra types are represented over $\mathbb{Z}$ - that is, by polynomial maps whose coefficients are integers. If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0$ is a stable complex germ whose local algebra is isomorphic to a local algebra represented over $\mathbb{R}$, then f is $\mathcal{A}$-equivalent to a polynomial germ $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ all of whose component functions are real polynomials, constructed by the standard procedure (*cf* Section **??**) from a real representative of the algebra. The results of Damon and Galligo for g evidently apply also to f.

The statement about the nice dimensions follows simply by inspection of Mather's list in [**?**] of stable germs in the nice dimensions: all have local algebras to which Damon and Galligo's theorem applies. □

Theorem 6.0.8 and its complex version easily extend to multi-germs. We first extend the definitions of $m(f)$ and $\delta(f)$ to multi-germs, in the obvious way: $\delta(f) = \sum_{O^{(i)}\in S} \delta(f^{(i)})$, and $m(f)$ is defined by replacing each occurrence of 0 (in $\mathbb{C}^n$), in the definition for mono-germs, by S.

Proposition 6.0.9. *Suppose that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ $(n < p)$ is a stable multi-germ and that for each point $O^{(i)} \in S$, the monogerm $f^{(i)} : (\mathbb{C}^n, O^{(i)}) \to (\mathbb{C}^n, 0)$ satisfies $m(f^{(i)}) = \delta(f^{(i)})$. Then $m(f) = \delta(f)$.*

Proof. Stability of f means that each monogerm $f^{(i)}$ is stable and that the analytic strata $T^{(i)}, i = 1, \ldots, k$ of the $f^{(i)}$ meet in general position at $0 \in \mathbb{C}^p$. After a coordinate change in $(\mathbb{C}^p, 0)$ and in each of the $(\mathbb{C}^n, O^{(i)})$ we may assume that each $Y^{(i)}$ is a linear subspace, and that $f^{(i)}$ is a product map of the form $\mathbb{C}^{n-d_i} \times T^{(i)} \xrightarrow{f_0^{(i)} \times \mathrm{id}_{T^{(i)}}} \mathbb{C}^{p-d_i} \times T^{(i)}$. By our hypothesis on the $f^{(i)}$, we can

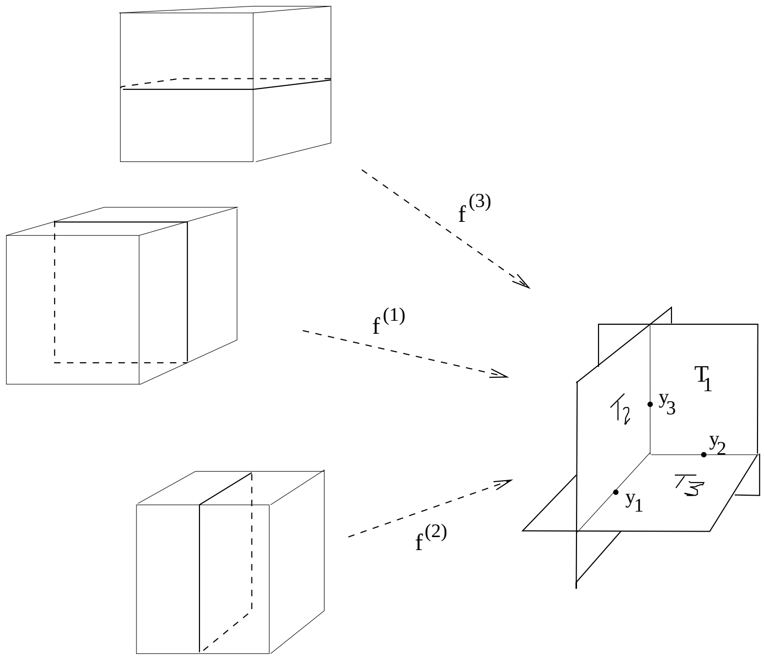


Figure 6.2: Schematic diagram of multi-germ.

choose $y^{(i)} \in (\mathbb{C}^p, 0)$ with $\delta(f^{(i)})$ preimages under $f^{(i)}$ (in $(\mathbb{C}^n, O^{(i)})$). Because $f^{(i)}$ is trivial in the $T^{(i)}$ direction, translation of $y(i)$ in a direction parallel to $T^{(i)}$ does not affect the number of its preimages. Thus for all points $y \in y^{(i)} + T^{(i)}$, we have $|\left(f^{(i)}\right)^{-1}(y)| = \delta(f^{(i)})$. The proposition now follows, because the general position of the $T^{(i)}$ implies that

$$\bigcap_{i=1}^{k} \left(y^{(i)} + T^{(i)}\right) \neq \emptyset;$$

any point in the intersection has $\delta(f^{(i)})$ preimages under $f^{(i)}$ and thus $\sum_i \delta(f^{(i)})$ preimages under f. To see that this intersection is non-empty, note that it is the preimage, under the epimorphism

$$\mathbb{C}^n \to \frac{\mathbb{C}^n}{T_1} \oplus \cdots \oplus \frac{\mathbb{C}^n}{T_k},$$

of the point $(y^{(1)} + T^{(1)}, \ldots, y^{(k)} + T^{(k)})$. $\square$

Remark 6.0.10. If $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a stable germ then all points y in $(\mathbb{C}^p, 0)$ satisfying $|f^{-1}(y)| = \delta(f)$ are normal crossing points. For by upper semicontinuity, $\sum_{x \in f^{-1}(y)} \delta(f)_x$ can be no greater than $\delta(f)$, and on the other hand $|f^{-1}(y)| = \delta(f)$, forcing $\delta(f)_x = 1$ for each $x \in f^{-1}(y)$. Thus $f : (\mathbb{C}^n, f^{-1}(y)) \to (\mathbb{C}^p, y)$ is a germ of immersions. As it is also stable, it must be the germ of immersions with normal crossings.

The hypothesis that the analytic strata of the $f^{(i)}$ be in general position is needed in 6.0.9. Example 6.0.4 shows an unstable bi-germ in which each of the component mono-germs is stable, and satisfies $m(f^{(i)}) = \delta(f^{(i)})$, but in which nevertheless $m(f) < \delta(f)$.

Remark 6.0.11. Theorem 6.0.8 is proved essentially by detailed calculation. One can get a tiny idea of the nature of the strategy from the following very easy case. Suppose $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is of corank 1. Then it is a stable unfolding of $f_0 : (\mathbb{C}\cdot 0) \to (\mathbb{C}^{p-n+1}, 0)$ given by $f_0(x) = (x^k, 0, \ldots, 0)$. By Mather's classification of stable germs by their local algebra, f is right-left equivalent to a constant unfolding of the germ

$$(x, \mathbf{u}, \mathbf{v_1}, \ldots, \mathbf{v_{p-1}}) \mapsto (x^k + u_1 x^{k-2} + \cdots + u_{k-1}x, \sum_{i=1}^{k-1} v_{1i}x^{k-i}, \ldots, \sum_{i=1}^{k-1} v_{p-1,i}x^i). \qquad (6.0.12)$$

We wish to show that in any neighbourhood of 0 in $\mathbb{C}^p$ there are points with k distinct preimages. But this is already true of the restriction of f to $\mathbb{C}\times\mathbb{C}^{k-1}$ (setting all the v_{ij} to 0), essentially by the fundamental theorem of algebra: for each fixed value of $\mathbf{u}$, the polynomial $x^k + u_1x^{k-2} + \cdots + u_{k-2}x$ has k roots, and for $\mathbf{u}$ not in the discriminant, these roots are all distinct. So there are points in $\mathbb{C}^p$ of the form $(0, \mathbf{u}, \mathbf{0}, \ldots, \mathbf{0})$ with k distinct preimages.

The simplest type of multiple point is a normal crossing, where k immersed branches of $\mathbb{C}^n$ meet in general position. The multi-germ parametrising a normal crossing is stable ([**?**]); in fact

Proposition 6.0.12. *Germs of immersions with normal crossings are the only stable multi-germ of immersions.*

Proof. We have to show

1. if $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a multi-germ of immersions parameterising a non-normal crossing, then an arbitrarily small perturbation of one or more of the $f^{(i)}$ places all crossings in general position.

2. two multi-germs of immersions parameterising crossings, with one normal and the other not normal, are not right-left equivalent.

□

Corollary 6.0.13. *Suppose that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ has corank ≤ 2, or is of discrete algebra type, and is finitely determined. Suppose also that $D^k(f) \supsetneq \{(0, \ldots, 0)\}$. Then the set of points $(x^{(1)}, \ldots, x^{(k)}) \in D^k(f)$ such that*

1. *the multigerm of f at $\{x^{(1)}, \ldots, x^{(k)}\}$ is an immersion with normal crossings*
2. *$f^{-1}(f(x^{(i)})) = \{x^{(1)}, \ldots, x^{(k)}\}$*

is open and dense in $D^k(f)$

Proof. Pick a good representative of f, $\tilde{f} : U \to V$ such that for every $y \in V \smallsetminus \{0\}$, $\tilde{f}_y : (U, f^{-1}(y)) \to (V, y)$ is stable. Let $(x^{(1}, \ldots, x^{(k)}) \in D^k(\tilde{f}) \smallsetminus \{(0, \ldots, 0)\}$ and let $y = \tilde{f}(x^{(i)})$. Note that the $x^{(i)}$ are not necessarily distinct from one another. The multi-germ $\tilde{f}_y : (\mathbb{C}^n, \tilde{f}^{-1}(y)) \to (\mathbb{C}^p, y)$ is stable, and all of its local algebras are of the type covered by Theorem 6.0.8. Evidently $\delta(\tilde{f}_y) \geq k$. By 6.0.9, for every neighbourhood collection of neighbourhoods $U^{(i)}$ of $x^{(i)}$ for $i = 1, \ldots, k$, and every neighbourhood V_y of y there are points $y_1 \in V_y$ with $\delta := \delta(\tilde{f}_y)$ distinct preimages $x_1^{(1)}, \ldots, x_1^{(\delta)}$ in $U_1 := \cup_i U^{(i)}$, and if y_1 is such a point then the germ $\tilde{f}_{y_1} : (U_1, f|_{U_1}^{-1}(y)) \to (V, y)$, is an immersion with normal crossings. Among these $\delta(\tilde{f}_y)$ preimages, choose k, say $x_1^{(i_1)}, \ldots, x_1^{(i_k)}$, with $x_1^{(i_j)} \in U^{(j)}$. The germ $\tilde{f} : (U, \{x_1^{(i_1)}, \ldots, x_1^{(i_k)}\}) \to (V, y_1)$ is an immersion with normal crossings, and $(x_1^{(i_1)}, \ldots, x_k^{(i_k)})$ lies in the neighbourhood $U^{(1)} \times \cdots \times U^{(k)} \cap D^k(\tilde{f})$ of $(x^{(1)}, \ldots, x^{(k)})$. □

Corollary 6.0.14. *Suppose that* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ *has corank* ≤ 2*, or is of discrete algebra type, and is finitely determined. Suppose also that* $M^k(f) \supsetneq \{0\}$*. Then the set of points* $y \in M^k(f)$ *such that at* y *the image of* f *is an immersed manifold with normal crossings is open and dense in* M_k*.*

Proof. □

Lemma 6.0.15. *Let* $M_k = \{y \in (\mathbb{C}^p, 0) : |f^{-1}(y)| \geq k\}$*. If* $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ *is finitely determined multi-germ then*

$$\dim M_k = n - (k-1)(p-n) \tag{6.0.13}$$

provided M_k *is not empty.*

Proof. Choose a good proper representative $\tilde{f} : U \to V$ for which every multi-germ $f_y : (U, f^{-1}(y)) \to (V, y)$ with $y \neq 0$ is stable (see **??**) and let $U^* = U \setminus \{0\}$. Consider the map $\tilde{f}^k : (U^*)^{\langle k \rangle} \to V^k$. Because each multi-germ is stable, this map is transverse to the small diagonal $\Delta := \{(y_1, \ldots, y_k) \in V^k : y_1 = \ldots = y_k\}$. The preimage $(\tilde{f}^k)^{-1}(\Delta)$ is therefore a manifold of dimension $kn - (k-1)p$, if it is not empty. We have $M_k = f(\pi_i((\tilde{f}^k)^{-1}(\Delta))$, where $\pi_i : (U^*)^{\langle k \rangle} \to U$ is any one of the standard projections. Each projection $\pi_i| : (\tilde{f}^k)^{-1}(\Delta) \to U$ is finite, as is f, and so $f(\pi_i(\tilde{f}^k)^{-1}(\Delta))$ is a constructible set of dimension $kn - (k-1)p = n - (k-1)(p-n)$. □

Proposition 6.0.16. *If* $(f : \mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ $(n < p)$ *is stable, and not an immersion at* 0*, then in every neighbourhood of* 0 *in* $\mathbb{C}^n$ *there are pairs of distinct points sharing the same image.*

Proof. If f is not an immersion then $I_2(f) \neq (1)$, so the codimension of $D^2(f)$ in $\mathbb{C}^n \times \mathbb{C}^n$ is no greater than the codimension of the corresponding generic variety, namely p, so its dimension is at least $2n - p$. On the other hand, if f is stable and not an immersion then the codimension of the non-immersive locus in $\mathbb{C}^n$ is $p - n + 1$, so that its dimension is $2n - p - 1$. Thus the complement of the non-immersive locus in $D^2(f)$ is non-empty. □

The k'th *source multiple point space* D^k of a finite proper map between topological spaces is the closure of the set of k-tuples of pairwise distinct points having the same image under the map. The k'th *target multiple point space* $M_k(f)$ is the closure in the image of the set of points having k or more distinct preimages. When $f\colon X \to Y$ is a finite analytic map of complex manifolds, the space $M_k(f)$ has a natural analytic structure as the subspace of Y defined by the $(k-1)$'st Fitting ideal $\mathrm{Fitt}_{k-1}(f_* \mathcal{O}_X)$ of the pushforward $f_* \mathcal{O}_X$ (see [Tei76], [MP89], [KLU96]). This structure is particularly good when X is Cohen-Macaulay, Y is smooth and $\dim Y = \dim X + 1$. One might hope for an analogous formula giving equations for $D^k(f)$ in X^k, in terms of f itself. No such formula is known in general, though for $k = 2$ the ideal defined, in terms of local coordinates $x_1, \ldots, x_n$ and $y_1, \ldots, y_p$ on X and Y, by

$$\mathcal{I}_2 := (f \times f)^* I_{\Delta_p} + \mathrm{Fitt}_0(I_{\Delta_n}/(f \times f)^* I_{\Delta_p}) \tag{6.0.14}$$

where I_{Δ_n} and I_{Δ_p} are the ideal sheaves defining the diagonals Δ_n in $\mathbb{C}^n \times \mathbb{C}^n$ and Δ_p in $\mathbb{C}^p \times \mathbb{C}^p$, gives $D^2(f)$ a scheme structure with many desirable qualities: if f is dimensionally correct – that is, if $D^2(f)$ has the expected dimension, $2n - p$, then $D^2(f)$ is Cohen Macaulay. If moreover f is finitely determined (for left-right equivalence), or, equivalently, has isolated instability, then provided its dimension is greater than 0, $\mathcal{I}_2$ is radical.

If the corank of f (the dimension of $\operatorname{Ker} df_0$), is equal to 1, much more is possible. An explicit list of generators for the ideal defining $D^k(f)$ in $(\mathbb{C}^n)^k$ is given in [MM89], where it is shown that a finite corank 1 map-germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is stable if and only if each $D^k(f)$ is smooth of dimension $p - k(p - n)$, or empty, for all $k \geq 2$. Moreover, it is finitely $\mathcal{A}$-determined if and only if D^k is an ICIS of dimension $p - k(p - n)$ or empty for those k with $p - k(p - n) \geq 0$, and D^k consists at most of only the origin if $p - k(p - n) < 0$ (see, e.g., [Mar93], [GM93] for other results).

We will say that f is *dimensionally correct* if for each k, $D^k(f)$ satisfies these dimensional requirements, including the requirement that when $p - k(p - n) < 0$, $D^k(f)$ consists at most of the origin.

6.1 Multiple point spaces

Given a map $f : X \to Y$, we set

$$^oD^k(f) = \{(x_1, \ldots, x_k) \in X^k | f(x_1) = \cdots = f(x_k), x_i \neq x_j \text{ if } i \neq j\} \tag{6.1.1}$$

and define the k'th *source multiple point space of* f, $D^k(f)$, by

$$D^k(f) = \text{closure } ^oD^k(f) \tag{6.1.2}$$

(where the closure in taken in X^k) *provided* $^oD^k(f)$ *is not empty.* We extend this definition to germs of maps by taking the limit over representatives; if $f \in \mathcal{E}^0_{n,p}$ is finite, the local conical structure guarantees that we obtain in this way a well defined germ at $\mathbf{0} \in (\mathbb{C}^n)^k$. We give $D^k(f)$ an analytic structure as follows. First, choose a stable unfolding $F : X \times \mathbb{C}^d \to Y \times \mathbb{C}^d$ and give $D^k(F)$ its reduced structure. Because F is an unfolding, $D^k(F)$ embeds naturally in $X^k \times \mathbb{C}^d$, with defining ideal $\mathcal{I}_k(F)$. Define

$$\mathcal{I}_k(f) = I_k(F)|_{\mathbf{u}=0}.$$

It is straightforward to check that this is independent of the choice of stable unfolding, and is compatible with unfolding in the sense that for *any* germ of unfolding $F : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$ of f, the diagram

$$\begin{tikzcd} D^k(f), 0 \arrow[r] \arrow[d] & D^k(F), 0 \arrow[d] \\ \{0\} \arrow[r] & \mathbb{C}^d, 0 \end{tikzcd} \tag{6.1.3}$$

in which the vertical arrows are projections to the base and the horizontal arrows are inclusions, is a fibre square.

This definition of $\mathcal{I}_k(f)$ is canonical, but gives no hint as to how to $\mathcal{I}_k(f)$ is to be calculated.

6.2 Equations for multiple point spaces

Our definition of multiple point spaces does not immediately suggest a means of finding equations. Here we describe a natural set of equations for $D^2(f)$. Denote the diagonals in $\mathbb{C}^p \times \mathbb{C}^p$ and $\mathbb{C}^n \times \mathbb{C}^n$ by Δ_p and Δ_n. Since $\Delta_n \subseteq ID^2(f)$ and $\mathcal{I}_{\Delta_n}$ is radical,

$$(f \times f)^*(\mathcal{I}_{\Delta_p}) \subseteq \mathcal{I}_{\Delta_n}$$

so there exist $\alpha_{ij}, 1 \le i \le p$, $1 \le j \le n$, such that

$$f_i(x^{(1)}) - f_i(x^{(2)}) = \sum_{j=1}^{n} \alpha_{ij}(x^{(1)}, x^{(2)})(x_j^{(1)} - x_j^{(2)}) \tag{6.2.1}$$

for $i = 1, \ldots, p$. Let A denote the matrix $(\alpha_{ij})_{1\le i\le p,\, 1\le j\le n}$, and let $\min_n(A)$ denote the ideal of $\mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^n,(0,0)}$ generated by its $n \times n$ minors. The equations (6.2.1) can be combined to give

$$f(x^{(1)}) - f(x^{(2)}) = A(x^{(1)}, x^{(2)})(x^{(1)} - x^{(2)}). \tag{6.2.2}$$

Define

$$\mathcal{I}^2(f) = (f \times f)^*(\mathcal{I}_{\Delta_p}) + \min_n(A) \tag{6.2.3}$$

Proposition 6.2.1. *1. $A(x^{(1)}, x^{(1)}) = [d_x^{(1)} f]$, and the restriction of $\mathcal{I}_2(f)$ to the diagonal Δ_n is equal to the ramification ideal $\mathcal{R}_f$ of f.*

2. $V(\mathcal{I}_2(f)) = \{(x^{(1)}, x^{(2)}) : x^{(1)} \neq x^{(2)}, f(x^{(1)}) = f(x^{(2)})\} \bigcup i_\Delta(V(\mathcal{R}_f))$, where i_Δ is the inclusion of the diagonal in $\mathbb{C}^n \times \mathbb{C}^n$.

3. $\mathcal{I}_2(f) = F_0\big((f \times f)^*(\mathcal{I}_{\Delta_p}/\mathcal{I}_{\Delta_n})\big) + (f \times f)^*(\mathcal{I}_{\Delta_p})$, and is thus independent of the choice of the α_{ij} satisfying (6.2.1).

4. The grade of $\mathcal{I}_2(f)$ is less than or equal to p; if it is equal to p then $\mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^n}/\mathcal{I}_2(f)$ is Cohen-Macaulay.

5. $\mathcal{I}_2(f) = ((f \times f)^*(\mathcal{I}_{\Delta_p}) : \mathcal{I}_{\Delta_n})$.

Proof. 1. From the Taylor series of $f_j(x^{(1)}) - f_j(x^{(2)})$ we see that

$$f_i(x^{(1)}) - f_i(x^{(2)}) = \sum_{j=1}^{n} (x_j^{(1)} - x_j^{(2)})\big(\frac{\partial f_i}{\partial x_j}(x^{(1)}) + \text{terms in } \mathcal{I}_{\Delta_n}\big).$$

Thus for each i the n-tuple

$$(\alpha_{i1}(x^{(1)}, x^{(2)}) - \frac{\partial f_i}{\partial x_1}(x^{(1)}, \ldots, \alpha_{in}(x^{(1)}, x^{(2)}) - \frac{\partial f_i}{\partial x_n}(x^{(1)})$$

is a relation between the n linear forms $x_1^{(1)} - x_1^{(2)}, \ldots, x_n^{(1)} - x_n^{(2)}$. Since these form a regular sequence, all relations among them have coefficients in $\mathcal{I}_{\Delta_n}$. It follows that

$$\frac{\partial f_i}{\partial x_j}(x^{(1)}) = \alpha_{ij}(x^{(1)}, x^{(1)}) \mod \mathcal{I}_{\Delta_n}.$$

Thus, on the diagonal $\mathcal{I}_{\Delta_n}$, α_{ij} coincides with $\partial f_i/\partial x_j$.

2. If $x^{(1)} \neq x^{(2)}$ and $f(x^{(1)}) = f(x^{(2)})$ then it is clear that in the equation (6.2.2), the matrix A has rank less than n, so $(x^{(1)}, x^{(2)}) \in V(\min_n(A))$, and $(x^{(1)}, x^{(2)}) \in V(\mathcal{I}_2(f))$. This shows that $V(\mathcal{I}_2(f)) \smallsetminus \Delta_n = D_g^2(f) \smallsetminus \Delta_n$. The result now follows from 1.

3. Let M be the set of n-tuples $(\bullet_1, \ldots, \bullet_n)$ of germs in $\mathcal{O}_{X\times X,(0,0)}$ such that

$$\sum_i \bullet_i \cdot (x_i^{(1)} - x_i^{(2)}) \in (f \times f)^*(\mathcal{I}_{\Delta_p}),$$

let M_0 be the set of n-tuples such that

$$\sum_i \bullet_i \cdot (x_i^{(1)} - x_i^{(2)}) = 0,$$

and consider the ideal $\bigwedge^n M$ of determinants of n-tuples of elements of M. We claim that

$$\mathcal{I}^2(f) = \bigwedge^n M + (f \times f)^*(\mathcal{I}_{\Delta_p}). \tag{6.2.4}$$

To see this, suppose that $\beta_1, \ldots, \beta_n$ is an n-tuple of elements of M, and let B be the $n \times n$ matrix wth rows $\bullet_1, \ldots, \beta_n$. By definition of the matrix A, there exists a matrix C over $\mathcal{O}_{X\times X,(0,0)}$ such that

$$B \cdot (x^{(1)} - x^{(2)}) = CA(x^{(1)} - x^{(2)})$$

so the rows of $B - CA$ are members of M_0. Thus

$$B - H = CA$$

where the rows of H lie in M_0. Now $x_1^{(1)} - x_1^{(2)}, \ldots, x_n^{(1)} - x_n^{(2)}$ is a regular sequence, so the members of M_0 are linear combinations of the trivial (Koszul) relations

$$(x_i^{(1)} - x_i^{(2)})\mathbf{e}_j - (x_j^{(1)} - x_j^{(2)})\mathbf{e}_i. \tag{6.2.5}$$

We wish to prove that $\det B \in \min_n A + S$. Evidently $\det(B - H) \in \min_n A$. By multilinearity, $\det(B - H)$ is a sum of determinants of the form

$$\pm \det(\bullet_{i_1}, \ldots, \bullet_{i_k}, \mathbf{h}_1, \ldots, \mathbf{h}_{n-k})^t \tag{6.2.6}$$

So it is enough to prove that any determinant of this with $k < n$ lies in $\min_n A + S$. Since $M_0 \subset M$, it is enough to show that it is true when $k = n - 1$. Suppose $k = n - 1$. Since det is linear in each row, we may suppose $\mathbf{h}_1$ is the Koszul relation (6.2.5). We have

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{n-1} \\ (x_i^{(1)} - x_i^{(2)})\mathbf{e}_j - (x_j^{(1)} - x_j^{(2)})\mathbf{e}_i \end{pmatrix} \begin{pmatrix} x_1^{(1)} - x_1^{(2)} \\ \vdots \\ \vdots \\ x_n^{(1)} - x_n^{(2)} \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_{p-1} \\ 0 \end{pmatrix} \tag{6.2.7}$$

By Cramer's rule, $\det B(x_i^{(1)} - x_i^{(2)})$ is equal to the determinant, D, of the matrix obtained from the matrix of coefficients in (6.2.7) by replacing its i'th column by the right hand side of (6.2.7). The last row of this matrix has unique non-zero entry $x_i^{(1)} - x_i^{(2)}$ in the j'th place, and all the elements in its i'th column lie in $(f \times f)^*(\mathcal{I}_{\Delta_p})$; so D lies in $(x_i^{(1)} - x_i^{(2)})(f \times f)^*(\mathcal{I}_{\Delta_p})$. We have shown that

$$(x_i^{(1)} - x_i^{(2)}) \det B \in (x_i^{(1)} - x_i^{(2)})(f \times f)^*(\mathcal{I}_{\Delta_p}) \tag{6.2.8}$$

and it follows that $\det B \in (f \times f)^*(\mathcal{I}_{\Delta_p})$, proving the inclusion of the right hand side in the left hand side in (6.2.4). The opposite inclusion is immediate from the definition of $\mathcal{I}_2(f)$. If K is the matrix of the first differential in the Koszul complex on $x_1^{(1)} - x_1^{(2)}, \ldots, x_n^{(1)} - x_n^{(2)}$ then (K, A) is the matrix of a presentation of $\mathcal{I}_{\Delta_n}/(f \times f)^*(\mathcal{I}_{\Delta_p})$. By what we have just proved, $\min_n(K, A) \subseteq \mathcal{I}_2(f)$, so

$$\mathcal{I}_2(f) \subseteq F_0\big(\mathcal{I}_{\Delta_n}/(f \times f)^*(\mathcal{I}_{\Delta_p})\big) + (f \times f)^*(\mathcal{I}_{\Delta_p}) \subseteq \mathcal{I}_2(f),$$

proving the required equality.

4. In [G.K75] and [Kem76], George Kempf studies the *variety of complexes*, W: the variety of m-uples of matrices $(M^{(1)}, \ldots, M^{(m)})$ (of some fixed size) satisfying $M_i M_{i+1} = 0$ for $i = 1, \ldots m-1$, and also its subvariety $W(k_1, \ldots, k_m)$ where in addition the ranks of the M_i are no greater than k_i for $i = 1, \ldots, m$. There is an obvious defining ideal, generated by the entries in the products $M_i M_{i-1}$ and the $(k_i + 1) \times (k_i + 1)$-minors of M_i for $i = 1, \ldots, m$. Kempf shows that with this structure, $W(k_1, \ldots, k_m)$ is reduced, and Cohen Macaulay. To derive the statement we want, we consider the case where the complex in question has just two maps:

$$\mathbb{C}^p \xrightarrow{M_2} \mathbb{C}^n \xrightarrow{M_1} \mathbb{C} \tag{6.2.9}$$

and apply Kempf's theorem to deduce that the quotient of $\mathcal{O}_{\mathrm{Mat}_{n\times p}\times \mathrm{Mat}_{1\times n}}$ by the ideal generated by the components of the product $M_2 M_1$ and the $n \times n$ minors of M_1 is Cohen-Macaulay. Our $V(\mathcal{I}_2(f))$ is the pull-back of this generic variety by the map

$$X \times X \to \mathrm{Mat}_{n\times p} \times \mathrm{Mat}_{1\times n}$$

sending $(x^{(1)}, x^{(2)})$ to $\big(A(x^{(1)}, x^{(2)}), x_1^{(1)} - x_1^{(2)}, \ldots, x_n^{(1)} - x_n^{(2)}\big)$. Our statement now follows from Kempf's theorem quoted above by the standard argument (our theorem 1.6.1).

5. Write A for the transporter ideal $\big((f \times f)^*(\mathcal{I}_{\Delta_p}) : \mathcal{I}_{\Delta_n}\big)$. By Cramer's rule, $\mathcal{I}_2(f) \subset A$. The two □

Corollary 6.2.2. *If f is finitely determined, then $D^2(f)$ is Cohen-Macaulay.*

6.2.1 Higher multiple point spaces

Various approaches are possible for finding equations for $D^3(f)$, but none have been described which have the good algebraic properties that we would like.

6.2.2 Multiple point towers

Let $f : X \to Y$. The usual projections

$$\mathbf{e}^{k,i} : X^{k+1} \to X^k, (x^{(1)}, \ldots \widehat{x^{(i)}}, \ldots, x^{(k+1)}) \tag{6.2.10}$$

specialise to maps $D^{k+1}(f) \to D^k(f)$. Because of the group actions, the image of $D^{k+1}(f)$ in $D^k(f)$ under $\mathbf{e}^{k,i}$ is independent of i. Indeed, if we denote by $(\to i)$ the permutation

$$(1, \ldots, k) \mapsto (1, \ldots, \hat{i}, \ldots, k, i),$$

we have

$$\mathbf{e}^{k+1,i} = \mathbf{e}^{k+1,k+1} \circ (\ \to i).$$

We denote the image of $D^{k+1}(f)$ in $D^{\ell}(f)$ under any composite of projections by $D^{k+1}_{\ell}(f)$. The collection of spaces and maps

$$\cdots D^{k+1}(f) \begin{array}{c}\longrightarrow\\ \longrightarrow\\ \cdots\\ \longrightarrow\\ \longrightarrow\end{array} D^k(f) \begin{array}{c}\longrightarrow\\ \cdots\\ \cdots\\ \longrightarrow\end{array} \quad \cdots \quad \begin{array}{c}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{array} D^2(f) \begin{array}{c}\longrightarrow\\ \longrightarrow\end{array} \mathbb{C}^n \tag{6.2.11}$$

is a *semi-simplicial object* in the category of analytic space-germs and maps. The formal definition will not be used and need not occupy us here, but what is important is to observe the wealth of structure that is present. This will be made use of with the *image computing spectral sequence*, see Section **??**.

In the absence of defining equations and a useable description of the $D^k(f)$ for $k > 2$, it would seem that little can be done with it. But there is one set of cases where we know a great deal more, and where the structure reveals unexpected beauty and complexity. This is where f is a map-germ (mono or multi-) of *corank 1*. If so, we can choose linearly adapted coordinates $x, y := x_1, \ldots, x_{n-1}, y$ so that f takes the form

$$f(x, y) = (x, f_n(x, y), \ldots, f_p(x, y)) \tag{6.2.12}$$

That is, we write f explicitly as an unfolding of a map-germ in the single variable y. Now any k points $x_1, y_1), \ldots, (x_k, y_k)$ sharing the same image must have equal x coordinates, and so $D^k(f)$ embeds naturally in $\mathbb{C}^{n-1} \times \mathbb{C}^k$. We take coordinates $x, y_1, \ldots, y_k$ on $\mathbb{C}^{n-1} \times \mathbb{C}^k$ and look for equations defining $D^k(f)$ in $\mathbb{C}^{n-1} \times \mathbb{C}^k$.

The following analysis will be applied to each of the component functions $f_j, j = n \ldots, p$ of f. To spare notation for the moment, let h be any function of $x, y_1, \ldots, y_k$. The map

$$(x, y_1, \ldots, y_k) \mapsto \big(h(x, y_1), \ldots, h(x, y_p)\big) \tag{6.2.13}$$

is equivariant with respect to the symmetric group actions on the source permuting the y_i and on the target permuting the $f_j(x, y_i)$. The $\mathscr{E}^{S_k}$ set of equivariant maps $\mathbb{C}^{n-1+k} \to \mathbb{C}^k$ is a module over the ring $\mathcal{O}^{S_k}$ of invariant functions on the source, and (although we will not *need* this fact) as such is generated by the gradient vectors of the generators of $\mathcal{O}^{S_k}$ [Poé76]. The ring $\mathcal{O}^{S_k}$ is generated over $\mathcal{O}_{\mathbb{C}^{n-1},0}$ by the sums of powers $\rho_1 = y_1 + \cdots + y_k, \ldots, \rho_k = y_1^k + \cdots + y_k^k$, and so every equivariant mapping can be written as linear combination, over $\mathcal{O}^{S_k}$, of the maps

$$\begin{array}{rcl} m_1(y_1, \ldots, y_k) & = & (1, \ldots, 1) \\ m_2(y_1, \ldots, y_k) & = & (y_1, \ldots, y_k) \\ \cdots & \cdots & \cdots \\ m_{k-1}(y_1, \ldots, y_k) & = & (y_1^{k-1}, \ldots, y_k^{k-1}) \end{array} \tag{6.2.14}$$

Thus there exist invariant functions $\alpha_0^k, \alpha_1^k, \ldots, \alpha_{k-1}^k$ such that

$$\begin{pmatrix} h(x, y_1) \\ \vdots \\ h(x, y_k) \end{pmatrix} = \alpha_0^k \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + \alpha_1^k \begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix} + \cdots + \alpha_{k-1}^k \begin{pmatrix} y_1^{k-1} \\ \vdots \\ y_k^{k-1} \end{pmatrix} \tag{6.2.15}$$

Solving for the α_i^k by by Cramer's rule gives

$$\alpha_\ell^k(x,y_1,\dots,y_k) = \frac{\begin{vmatrix} 1 & y_1 & \cdots & y_1^{\ell-1} & h(x,y_1) & y_1^{\ell+1} & \cdots & y_1^{k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & y_k & \cdots & y_k^{\ell-1} & h(x,y_k) & y_k^{\ell+1} & \cdots & y_k^{k-1} \end{vmatrix}}{\begin{vmatrix} 1 & y_1 & \cdots & y_1^{k-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & y_k & \cdots & y_k^{k-1} \end{vmatrix}}. \tag{6.2.16}$$

In fact we do not need Poenaru's statement, referred to above, to see that the α_ℓ^k are regular (analytic): the numerator in (6.2.16) vanishes whenever $y_i = y_\ell$ for any i, ℓ, and thus is divisible in $\mathcal{O}$ by $\prod_{i<\ell}(y_i - y_\ell)$, i.e. by the Vandermonde determinant, which is the denominator in (6.2.16). In other words the system of equations (6.2.15) has analytic solutions. As can be seen from (6.2.16), they are S_k-invariant . They are also unique, since the Vandermonde determinant vanishes only along a hypersurface.

Let $I_k(h)$ be the ideal generated by the α_ℓ^k for $\ell = 1, \dots, \ell - 1$.

Now in (6.2.15) subtract the first row from each of the others. Omitting the first row in the resulting equation gives

$$\begin{pmatrix} h(x,y_2) - h(x,y_1) \\ \vdots \\ h(x,y_k) - h(x,y_1) \end{pmatrix} = \begin{pmatrix} (y_2 - y_1) & \cdots & (y_2^{k-1} - y_1^{k-1}) \\ \vdots & \vdots & \vdots \\ (y_k - y_1) & \cdots & (y_k^{k-1} - y_1^{k-1}) \end{pmatrix} \begin{pmatrix} \alpha_1^k \\ \vdots \\ \alpha_k^{k-1} \end{pmatrix} \tag{6.2.17}$$

The determinant of the new matrix of coefficients is still $\mathrm{Vdm}(y_1, \dots, y_k)$. It follows that

$$I_k(h) \supseteq \big(h(x,y_2) - h(x,y_1), \dots, h(x,y_k) - h(x,y_1)\big) \tag{6.2.18}$$

and

$$y_1, \dots, y_k \text{ are pairwise distinct } \implies I_k(h) = \big(h(x,y_2) - h(x,y_1), \dots, h(x,y_k) - h(x,y_1)\big) \tag{6.2.19}$$

By contrast, when all the y_i are equal, we find that $I_k(h)$ reduces to an ideal of partial derivatives.

Lemma 6.2.3. $\alpha_0^k = h(x,y_i) \mod I_k(h)$ *and, for* $1 \le \ell \le k-1$, *we have*

$$y_1 = \cdots = y_k \implies \alpha_\ell^k + \binom{\ell+1}{\ell} y \alpha_{\ell+1}^k + \cdots + \binom{k-1}{\ell} y^{k-1-\ell} \alpha_{k-1}^k = \frac{1}{\ell!}\frac{\partial^\ell h}{\partial y^\ell} \tag{6.2.20}$$

and hence

$$(\alpha_\ell^k, \dots, \alpha_{k-1}^k) + (y_1 - y_2, \dots, y_1 - y_k) = \Big(\frac{\partial^\ell h}{\partial y^\ell}, \dots \frac{\partial^{k-1} h}{\partial y^{k-1}}\Big) + (y_1 - y_2, \dots, y_1 - y_k) \tag{6.2.21}$$

Proof. The first statement can be read off from from the i'th row in (6.2.15). For the rest, write $y_i = y + \delta_i$ for $i = 1, \dots, k$ and substitute this into (6.2.15). We get

$$\begin{aligned} \begin{pmatrix} h(x,y_1) \\ \vdots \\ h(x,y_k) \end{pmatrix} &= (\alpha_0^k + y\alpha_1^k + \cdots + y^{k-1}\alpha_{k-1}^k) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + \cdots + \Big(\alpha_\ell + \binom{\ell+1}{\ell} y \alpha_{\ell+1}^k + \cdots \\ &\quad + \binom{k-1}{\ell} y^{k-1-\ell}\alpha_{k-1}^k\Big) \begin{pmatrix} \delta_1^\ell \\ \vdots \\ \delta_k^\ell \end{pmatrix} + \cdots + \alpha_{k-1}^k \begin{pmatrix} \delta_1^{k-1} \\ \vdots \\ \delta_k^{k-1} \end{pmatrix} \end{aligned} \tag{6.2.22}$$

In the left hand side replace $h(x, y_i)$ by its Taylor expansion about (x, y). Writing $h^{(\ell)}$ for $\partial^\ell h/\partial y^\ell(x, y)$, and shifting powers of δ_i lower than k to the right hand side, we obtain

$$\begin{aligned}
\begin{pmatrix} \sum_{j\geq k}\big(h^{(j)}/j!\big)\delta_1^j \\ \vdots \\ \sum_{j\geq k}\big(h^{(j)}/j!\big)\delta_k^j \end{pmatrix} &= \left(\alpha_0^k + y\alpha_1^k + \cdots + y^{k-1}\alpha_{k-1}^k - h\right)\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} + \cdots \\
&\quad + \left(\alpha_\ell^k + \binom{\ell+1}{\ell}y\alpha_{\ell+1}^k + \cdots + \binom{k-1}{\ell}y^{k-1-\ell}\alpha_{k-1}^k - h^{(\ell)}/\ell!\right)\begin{pmatrix}\delta_1^\ell \\ \vdots \\ \delta_k^\ell\end{pmatrix} + \cdots \\
&\quad + \left(\alpha_{k-1}^k - h^{(k-1)}/(k-1)!\right)\begin{pmatrix}\delta_1^{k-1} \\ \vdots \\ \delta_k^{k-1}\end{pmatrix}
\end{aligned} \tag{6.2.23}$$

Divide both sides of the equation by the Vandermonde *matrix*

$$\begin{pmatrix} 1 & \delta_1 & \cdots & \delta_1^{k-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \delta_k & \cdots & \delta_k^{k-1}\end{pmatrix}; \tag{6.2.24}$$

it is obvious that we can do this on the right, while on the left it is possible by the statement of Poenaru already cited. All entries in the quotient on the left lie in the ideal $(\delta_1, \ldots, \delta_k)$, and so the same is true on the right. It follows that when the δ_i all vanish, we have

$$h^{(\ell)}/\ell! = \alpha_\ell^k + \binom{\ell+1}{\ell}y\alpha_{\ell+1}^k + \cdots + \binom{k-1}{\ell}y^{k-1-\ell}\alpha_{k-1}^k \tag{6.2.25}$$

This proves the second statement, and the third follows. □

What happens in situations intermediate between the situation of (6.2.19), where all the y_i are distinct from one another, and (6.2.20), where they are all equal? In other words, how does the ideal generated by the α_ℓ^k behave on the different diagonals (strata determined by S_k orbit type) in $\mathbb{C}^{n-1}\times\mathbb{C}^k$? Each stratum in $\mathbb{C}^{n-1}\times\mathbb{C}^k$ corresponds to a partition of $\{1, \ldots, k\}$. By re-ordering the y_i, any partition can be brought to the form

$$\Big(\{1, \ldots, r_1\}, \{r_1+1, \ldots, r_1+r_2\}, \ldots, \{r_1+\cdots+r_{m-1}+1, \ldots, r_1+\cdots+r_m = k\}\Big).$$

The corresponding stratum is

$$\Delta(\mathscr{P}) := \{(x, y_1, \ldots, y_k) \in \mathbb{C}^{n-1}\times\mathbb{C}^k \,:\, y_j = y_{j+1} \text{ iff } r_1+\ldots+r_{i-1} < j < r_1+\cdots+r_i \text{ and } 1 \leq i \leq m\} \tag{6.2.26}$$

whose closure has defining ideal

$$\mathcal{I}(\mathscr{P}) := \big(y_i - y_{i+1} : r_1+\ldots+r_{i-1} < j < r_1+\cdots+r_i \text{ for } 1 \leq i \leq m\}\big).$$

We call m the *length* of the partition. Observe that the dimension of $\Delta(\mathscr{P})$ is $n-1+m$.

We say the partition $\mathscr{P}'$ is *finer* than $\mathscr{P}$ if $\Delta(\mathscr{P}')$ is contained in the closure of $\Delta(\mathscr{P})$, and *strictly finer* if in addition they are not equal.

We need some notation. Define $s_i = r_1 + \cdots + r_i$ for $i = 1, \ldots, m$, and set $s_0 = 0$. Define $\pi_i^{\mathscr{P}} : \mathbb{C}^{n-1+k} \to \mathbb{C}^{n-1+r_i}$ and $\mathrm{pr}_i^{\mathscr{P}} : \mathbb{C}^{n-1+k} \to \mathbb{C}^n$ by

$$\pi_i^{\mathscr{P}}(x, y_1, \ldots, y_k) = (x, y_{s_{i-1}+1}, \ldots, y_{s_i}), \quad \mathrm{pr}_i^{\mathscr{P}}(x, y_1, \ldots, y_k) = (x, y_{s_i}) \tag{6.2.27}$$

For the purpose for which we will use $\mathrm{pr}_i^{\mathscr{P}}$, we could just as well have defined it using y_j, for any j between $s_{i-1}+1$ and s_i, instead of y_{s_i}. The point is to select one of the y_j's in the i'th block of the partition. The purpose will become clear in the next proposition.

When $(x, y_1, \ldots, y_k) \in \Delta(\mathscr{P})$, it will sometimes it will be useful to denote the common value of the y_j for $r_1 + \cdots + r_{i-1} < j < r_1 + \cdots + r_i$ by $y^{(i)}$. With this notation, $(y_1, \ldots, y_k)$ becomes

$$\big(\underbrace{y^{(1)}, \ldots, y^{(1)}}_{r_1 \text{ times}}, \ldots, \underbrace{y^{(m)}, \ldots, y^{(m)}}_{r_m \text{ times}}\big).$$

Let $I_{\mathscr{P}}(h) = I_k(h) + \mathcal{I}(\mathscr{P})$.

Proposition 6.2.4. *If $(x, \mathbf{y})$ does not lie in $\Delta(\mathscr{P}')$ for any partition $\mathscr{P}'$ strictly finer than $\mathscr{P}$, then the stalks in $\mathcal{O}_{\mathbb{C}^{n-1+k},(x,\mathbf{y})}$ of $I_k(h)$ and $I_{\mathscr{P}}(h)$ are equal, respectively, to*

$$\big(\pi_1^{\mathscr{P}}\big)^*\big(\mathcal{I}_{r_1}(h)\big) + \cdots + \big(\pi_m^{\mathscr{P}}\big)^*\big(\mathcal{I}_{r_m}(h)\big) + \Big(\Big\{h \circ pr_i^{\mathscr{P}} - h \circ pr_1^{\mathscr{P}} : i = 2, \ldots, m\Big\}\Big) \tag{6.2.28}$$

and

$$\mathcal{I}(\mathscr{P}) + \left(\left\{\frac{\partial^\ell h}{\partial y^\ell} \circ pr_i^{\mathscr{P}} : \; \ell = 1, \ldots, \; r_i - 1, i = 1 \ldots, m\right\}\right) + \Big(\Big\{h \circ pr_1^{\mathscr{P}} - h \circ pr_i^{\mathscr{P}} : i = 1, \ldots, m\Big\}\Big) \tag{6.2.29}$$

Proof. The subgroup $G = S_{r_1} \times \cdots \times S_{r_m}$ of S_k acts on $\mathbb{C}^{n-1} \times \mathbb{C}^k$ via the permutation representation of S_k. The ring of G-invariant functions $\mathcal{O}^G$ is generated by the power sums $g_{i,\ell} = (y_{r_1+\cdots+r_{i-1}+1})^\ell + \cdots + (y_{r_1+\cdots+r_i})^\ell$ for $0 \le j \le r_i$ and $i = 1, \ldots, m$. Let $\mathscr{E}^G$ be the $\mathcal{O}^G$-module of G-equivariant maps $\mathbb{C}^{n-1} \times \mathbb{C}^k \to \mathbb{C}^k$. By the theorem of Poenaru in *op cit*, $\mathscr{E}^G$ is generated over $\mathcal{O}^G$ by the gradients $e_{i,\ell}$ of the $g_{i,\ell+1}$; after dividing by constant coefficients, $e_{i,\ell}$ is the k-tuple with y_j^ℓ in the j'th place, for $r_1 + \cdots + r_{i-1} + 1 \le j \le r_1 + \cdots r_i$ and $0 \le \ell \le r_i - 1$, and with 0's elsewhere. Let $E(G)$ be the matrix with columns $e_{1,0}, \ldots, e_{1,r_1-1}, \ldots, e_{m,0}, \ldots, e_{m,r_m-1}$, and, as before, let $E(S_k)$ be the matrix whose columns are the generators of $\mathscr{E}(S_k)$ over $\mathcal{O}^{S_k}$. Because $\mathscr{E}(S_k) \subset \mathscr{E}(G)$, $E(S_k)$ is divisible by $E(G)$: there is a $k \times k$ matrix Q with entries in $\mathcal{O}^G$, such that $E(S_k) = E(G)Q$. The first column of Q has a 1 in the first, $r_1 + 1$'st, ..., $r_1 + \cdots + r_{m-1} + 1$'st places and 0's elsewhere.

Observe that

$$\det\, E(S_k) = \mathrm{Vdm}(y_1, \ldots, y_k)$$

$$\det\, E(G) = \mathrm{Vdm}(y_1, \ldots, y_{r_1}) \times \cdots \times \mathrm{Vdm}(y_{r_1+\cdots+r_{m-1}+1}, \ldots, y_k).$$

Since $(x, y) \notin \Delta(\mathscr{P}')$ for any $\mathscr{P}'$ strictly finer than $\mathscr{P}$, $\det Q = \det\, E(S_k)\big(\det\, E(G)\big)^{-1}$ does not vanish at (x, y). We have $\big(h(x, y_1), \ldots, h(x, y_k)\big)^t \in \mathscr{E}(S_k) \subset \mathscr{E}(G)$ and so

$$\big(h(x, y_1), \ldots, h(x, y_k)\big)^t = E(G)\beta = E(S_k)\alpha = E(G)Q\alpha. \tag{6.2.30}$$

for suitable column vectors $\beta = (\beta_{1,0}, \ldots, \beta_{1,r_1-1}, \beta_{2,0}, \ldots, \beta_{2,r_2-1}, \ldots, \beta_{m,0}, \ldots, \beta_{m,r_m-1})^t$ and $\alpha = (\alpha_0, \ldots, \alpha_{k-1})^t$. Notice that

$$(\beta_{i,1}, \ldots, \beta_{i,r_i-1}) = \left(\pi_i^{\mathscr{P}}\right)^*\left(I_{r_i}(h)\right).$$

Cancelling $E(G)$ in (6.2.30) we obtain $\beta = Q\alpha$. In order to obtain an equation relating the $\beta_{i\ell}$ to the generators $\alpha_1^k, \ldots, \alpha_{k-1}^k$ of $\mathcal{I}_k(h)$, we reduce the first column of Q to a 1 in the first place followed by zeros elsewhere, by subtracting the first row of Q from the r_1+1'st, r_1+r_2+1'st, $\ldots$, $r_1+\cdots+r_{m-1}+1$'st rows. We obtain

$$\left(\beta_{1,0}, \ldots, \beta_{1,r_1-1}, \beta_{2,0}-\beta_{1,0}, \beta_{2,1}, \ldots, \beta_{2,r_2-1}, \ldots, \beta_{m,0}-\beta_{1,0}, \beta_{m,1} \ldots, \beta_{m,r_m-1}\right)^t = \tag{6.2.31}$$
$$= \left(\begin{array}{c|c} 1 & \cdots \\ \hline 0 & Q' \end{array}\right) \begin{pmatrix} \alpha_0^k \\ \alpha_1^k \\ \vdots \\ \alpha_{k-1}^k \end{pmatrix}$$

Notice that by Lemma 6.2.3, $\beta_{i,0} = h \circ \mathrm{pr}_i^{\mathscr{P}} \mod (\beta_{i,1}, \ldots, \beta_{i,r_i-1})$, so $\beta_{i,0} - \beta_{1,0} = h \circ \mathrm{pr}_i^{\mathscr{P}} - h \circ \mathrm{pr}_1^{\mathscr{P}}$ mod $(\beta_{i,1}, \ldots, \beta_{i,r_i-1}, \beta_{1,1}, \ldots, \beta_{1,r_1-1})$. Since $\det Q' = \det Q \neq 0$ at (x, y), the ideal $\mathcal{I}_k(h)_{(x,y)}$ generated by the germs at (x, y) of $\alpha_1^k, \ldots, \alpha_{k-1}^k$ is equal to the ideal generated by the entries in rows $2, \ldots, k$ of the left hand side of (6.2.31); that is, to $\left(\{\beta_{i,\ell} : 1 \leq \ell \leq r_i - 1, 1 \leq i \leq m\}\right) + \left(h \circ \mathrm{pr}_2 - h \circ \mathrm{pr}_1, \ldots, h \circ \mathrm{pr}_m - h \circ \mathrm{pr}_1\right)$. Finally, by (6.2.20),

$$\left(\beta_{i,1}, \ldots, \beta_{i,r_i-1}\right) + \mathcal{I}(\mathscr{P}) = \left(\left\{\frac{\partial^\ell h}{\partial y^\ell} : 1 \leq \ell \leq r_i - 1\right\}\right),$$

and this completes the proof. $\square$

Corollary 6.2.5. *Let $(x, y_1, \ldots, y_k) \in \Delta(\mathscr{P})$ with $\mathscr{P}$ as in (6.2.26). Then $(x, y_1, \ldots, y_k) \in V(I_k(h))$ if and only if $h(x, y_1) = \cdots = h(x, y_k)$ and the ideal $(x, h)\,\mathcal{O}_{\mathbb{C}^n,(x,y^{(i)})}$ satisfies*

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^n,(x,y^{(i)})}}{(x, h)} \geq r_i$$

Now we use this construction to find defining equations for the multiple point spaces $D^k(f)$ for a corank 1 map-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ of the form (6.2.12). Apply (6.2.17) with f_j in place of h. Define

$$\mathcal{I}_k(f) = I_k(f_n) + \cdots + I_k(f_p). \tag{6.2.32}$$

and, for any partition $\mathscr{P}$ of $\{1, \ldots, k\}$,

$$\mathcal{I}_k(f, \mathscr{P}) = I_k(f_n, \mathscr{P}) + \cdots + I_k(f_p, \mathscr{P}) = \mathcal{I}_k(f) + \mathcal{I}(\mathscr{P}). \tag{6.2.33}$$

We wish to compare $D^k(f)$ and $V(\mathcal{I}_k(f))$. By (6.2.19), in $\mathbb{C}^{n-1} \times \mathbb{C}^{\langle k \rangle}$ (i.e. where the y_j are pairwise different) $V(\mathcal{I}_k(f))$ is just the set of points where $f(x, y_1) = \cdots = f(x, y_k)$. It follows that

$$V(\mathcal{I}_k(f)) \supseteq \mathrm{closure}\{(x, y_1 \ldots, y_k) \in \mathbb{C}^{n-1} \times \mathbb{C}^{\langle k \rangle} : f(x, y_i) = f(x, y_j) \text{ for all } i, j\} =: D_g^k(f) \tag{6.2.34}$$

(see (6.0.2)). Proposition 6.2.4 describes the intersection of $V(\mathcal{I}_k(f))$ with each stratum $\Delta(\mathscr{P})$ of the diagonal and enables us to turn this inclusion into an equality when f is stable. Suppose

$(x, y) \in \Delta(\mathscr{P})$, with $\mathscr{P}$ as in (6.2.26); then by 6.2.4, $(x, y) \in D^k(f)$ if and only if $f(x, y_1) = \cdots = f(x, y_k)$ and the partials $\partial^\ell f_j/\partial y^\ell$ vanish at $(x, y^{(i)})$ for $j = n, \ldots, p$ and for $i = 1, \ldots, r_i - 1$. This latter condition is equivalent to

$$\dim_{\mathbb{C}} Q(f)_{(x,y^{(i)})} \geq r_i;$$

if f is stable then by the Damon-Galligo theorem 6.0.8 (or, rather, its complex version) , this implies that $m(f)_{(x,y^{(i)})} \geq r_i$ also; now by Proposition 6.0.9, it follows that (x, y) is in the closure of the set of k-tuples of pairwise distinct points sharing the same image. That is, $(x, y) \in D^k_g(f)$. Taking into account (6.2.34), we have therefore shown that for stable map-germs,

$$V(\mathcal{I}_k(f)) = D^k_g(f). \tag{6.2.35}$$

It remains to show that this is an equality of schemes - i.e. that $\mathcal{I}_k(f)$ gives $D^k_g(f)$ its reduced structure. In the process of showing this we obtain a striking characterisation of stability and finite determinacy of corank 1 map-germs in terms of the singularities of $D^k(f)$.

The space $D^k(f)$ is the zero-locus of an ideal, $\mathcal{I}_k(f)$, generated by $(k-1)(p-n+1)$ elements. So its codimension in $\mathbb{C}^{n-1+k}$ is less than or equal to $(k-1)(p-n+1)$, and thus its dimension is at least $n - 1 + k - (k-1)(p-n+1)$, which simplifies to $n - k(p-n)$. Let us denote this number by $d_k(n, p)$. Note that this coincides with the estimate for the dimension of M_k, the space of points in $\mathbb{C}^p$ with at least k distinct preimages, given in Lemma 6.0.15. The composite $D^k(f) \xrightarrow{\mathbf{e}^{k,1}} \mathbb{C}^n \xrightarrow{f} \mathbb{C}^p$ is finite, and so $f\big(\mathbf{e}^{k,1}(D^k(f))\big)$ has the same dimension as $D^k(f)$. When f is stable, $D^k(f) = D^k_g(f)$ is defined as the closure of the set of k-tuples of pairwise distinct points, and so $f\big((\mathbf{e}^{k,1}(D^k(f))\big) =$ closure $M_k(f)$. It follows that if f is stable, $\dim D^k(f)$ *must* be equal to $d_k(n, p)$.

In what follows it will be useful to denote by $\alpha^k_\ell(f_j)$ the germ defined by (6.2.16) with f_j in place of h.

Proposition 6.2.6. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a corank 1 map-germ of the form* (6.2.12)*, and let $\dim_{\mathbb{C}} Q(f)0 = k$. Then f is stable if and only if $\mathcal{O}_{\mathbb{C}^{n-1+k},0} / \mathcal{I}_k(f)$ is regular of dimension $d_k(n, p)$.*

Proof. Define $A_k(f) : (\mathbb{C}^{n-1} \times \mathbb{C}^k, 0) \to (\mathbb{C}^{(k-1)(p-n)}, 0)$ by

$$\Big(\alpha^k_1(f_n), \ldots, \alpha^k_{k-1}(f_n), \ldots, \alpha^k_1(f_p), \ldots, \alpha^k_{k-1}(f_p)\Big). \tag{6.2.36}$$

That is, $A_k(f)$ is the map whose components are the generators of $\mathcal{I}_k(f)$; $\mathcal{O}_{\mathbb{C}^{n-1+k},0}$ is regular of dimension $d_k(n, p)$ if and only if $A_k(f)$ is a submersion. As $A_k(f)$ is invariant with respect to the S_k action on $\mathbb{C}^{n-1} \times \mathbb{C}^k$, It follows by Lemma 6.2.7 below, applied to the derivative $d_0 A$, that $A_k(f)$ is a submersion at 0 if and only if its restriction to the fixed point set of the group action is a submersion. The fixed point set is of course the small diagonal where all the y_i are equal. Here, the ideal $\mathcal{I}_k(f)$ is equal to the ideal $\Big(\{\partial^\ell f_j/\partial y^\ell : n \leq j \leq p, 1 \leq \ell \leq k-1\}\Big)$, by (6.2.20), so A_k is a submersion at $(0, 0) \in \mathbb{C}^{n-1} \times \mathbb{C}^k$ if and only if the map $B_k(f)$ with components

$$\Big(\frac{\partial f_n}{\partial y}, \ldots, \frac{\partial^{k-1} f_n}{\partial y^{k-1}}, \ldots, \frac{\partial f_p}{\partial y}, \ldots, \frac{\partial^{k-1} f_p}{\partial y^{k-1}}\Big) \tag{6.2.37}$$

is a submersion at $(0, 0)$. We can view $B_k(f)$ as the composite of the jet extension map, $j^{k-1}f : (\mathbb{C}^n, 0) \to J^{k-1}(n, p)$, with a projection $J^{k-1}(n, p) \to \mathbb{C}^{(k-1)(p-n)}$. John Mather characterised the stability of a map-germ f by the transversality of its jet extension map $j^\ell f$, for sufficiently high ℓ, to its contact class $\mathscr{K}j^\ell f(0)$ in $J^\ell(n,p)$ Given that f has the form (6.2.12), its contact class is determined by the smallest ℓ such that $\partial^\ell f_j/\partial y^\ell(0) \neq 0$ for some j. Let $LJ^{k-1}(n,p)$ be the subspace of jets of maps in the form (6.2.12). Since $\dim_{\mathbb{C}} Q(f) > k-1$, $\mathscr{K}j^{k-1}f(0) \cap LJ^{k-1}(n,p)$ consists of $(k-1)$-jets $j^k g(0)$ such that $\partial^\ell g_j/\partial y^\ell(0) = 0$ for $j = n,\ldots,p$ and $\ell = 1,\ldots,k-1$. So $B_k(f)$ is the composite of $j^{k-1}f$ with the projection $\pi : LJ^{k-1}(n,p) \to \mathbb{C}^{(k-1)(p-n)}$ for which $\pi^{-1}(0) = \big(\mathscr{K}j^{k-1}f(0)\big) \cap LJ^{k-1}(n,p)$. It follows that

$$B_k(f) \text{ is a submersion if and only if } j^{k-1}f \pitchfork \mathscr{K}j^{k-1}f(0).$$

Now because $\dim_{\mathbb{C}} Q(f) = k$, it follows that for any $\ell \geq k-1$, $\mathscr{K}j^\ell f(0) \cap LJ^\ell(n,p)$ is the preimage, under the projection $\pi^{\ell,k-1} : LJ^\ell(n,p) \to LJ^{k-1}(n,p)$, of $\mathscr{K}j^{k-1}f(0) \cap LJ^{k-1}(n,p)$. Since $j^{k-1}f = \pi^{\ell,k-1} \circ j^\ell f$, we deduce that

$$j^{k-1}f \pitchfork \mathscr{K}j^{k-1}f(0) \text{ if and only if } j^\ell f \pitchfork \mathscr{K}j^\ell f(0) \text{ for every } \ell \geq k-1.$$

This proves the proposition. □

Lemma 6.2.7. *Let the finite group G act linearly on the vector space V and let $H : V \to W$ be a linear map. Then H is an epimorphism if and only if $H| : Fix\ G \to W$ is an epimorphism.*

Proof. Here Fix G is the fixed point set of G. Because H is G-invariant,

$$H(v) = H\Big(\frac{1}{|G|}\sum_{g\in G} gv\Big).$$

The argument of H on the right hand side lies in Fix G. □

Because any regular ring is reduced, we deduce immediately

Corollary 6.2.8. *If f is stable then $\mathcal{I}_k(f)$ defines $D^k(f)$ with its reduced structure.*

Recall from Subsection 6.2.27 that when f is merely finite and not necessarily stable, we define $D^k(f)$ by taking a stable unfolding F of f on parameter space P; $D^k(f)$ is then the (scheme-theoretic) fibre of $D^k(F)$ over $0 \in P$. Since this is not always the reduced structure, we should give it a name; we call it the standard structure.

Theorem 6.2.9. *If f is a finite corank 1 map germ of the form* (6.2.12) *then $\mathcal{I}_k(f)$ defines $D^k(f)$ with the standard structure.*

Proof. If f has form (6.2.12), it is possible to choose a stable F unfolding of the same form. Then $\mathcal{I}_k(F)$ is generated by the $\alpha^k_\ell(F_j)$ for $j = n,\ldots,p$. It is clear that $\alpha^k_\ell(f_j) = \alpha^k_\ell(F_j)|_{\mathbb{C}^n\times\{0\}}$, so $\mathcal{I}_k(f)$ is the restriction to $\mathbb{C}^n \times \{0\}$ of $\mathcal{I}_k(F)$. The theorem now follows by Corollary6.2.8. □

The proof of Corollary 6.2.6 reveals the slightly surprising fact that for a finite corank 1 map-germ, if $D^k(f)$ is smooth at 0 then so is $D^\ell(f)$ for all $\ell < k$. For after a change of coordinates we may assume that f is of the form (6.2.12), and then

$$\begin{array}{ccccc} D^k(f) \text{ smooth at } 0 & \Longleftrightarrow & A_k(f) \text{ is a submersion} & \Longleftrightarrow & B_k(f) \text{ is a submersion} \\ & & & & \Downarrow \\ D^\ell(f) \text{ smooth at } 0 & \Longleftrightarrow & A_\ell(f) \text{ is a submersion} & \Longleftrightarrow & B_\ell(f) \text{ is a submersion} \end{array} \tag{6.2.38}$$

The reciprocal implication is easily seen to be false.

Example 6.2.10. Let $f(x_1, x_2, y) = (x_1, x_2, y^3 + x_1 y, y^5 + x_2 y)$. $\mathcal{I}_2(f)$ is generated by

$$\frac{\begin{vmatrix} 1 & y_1^3 + x_1 y_1 \\ 1 & y_2^3 + x_2 y_2 \end{vmatrix}}{\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix}}, \quad \frac{\begin{vmatrix} 1 & y_1^5 + x_1 y_1 \\ 1 & y_2^5 + x_2 y_2 \end{vmatrix}}{\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix}} \tag{6.2.39}$$

equal to

$$x_1 + y_1^2 + y_1 y_2 + y_2^2, \quad x_2 + y_1^4 + y_1^3 y_2 + y_1^2 y_2^2 + y_1 y_2^3 + y_2^4;$$

so $D^2(f)$ is smooth. $\mathcal{I}_3(f)$ is generated by

$$\frac{\begin{vmatrix} 1 & y_1^3 + x_1 y_1 & y_1^2 \\ 1 & y_2^3 + x_1 y_2 & y_2^2 \\ 1 & y_3^3 + x_1 y_3 & y_3^2 \end{vmatrix}}{\begin{vmatrix} 1 & y_1 & y_1^2 \\ 1 & y_2 & y_2^2 \\ 1 & y_3 & y_3^2 \end{vmatrix}}, \quad \frac{\begin{vmatrix} 1 & y_1 & y_1^3 + x_1 y_1 \\ 1 & y_2 & y_2^3 + x_1 y_2 \\ 1 & y_3 & y_3^3 + x_1 y_3 \end{vmatrix}}{\begin{vmatrix} 1 & y_1 & y_1^2 \\ 1 & y_2 & y_2^2 \\ 1 & y_3 & y_3^2 \end{vmatrix}}, \quad \frac{\begin{vmatrix} 1 & y_1^5 + x_2 y_1 & y_1^2 \\ 1 & y_2^5 + x_2 y_2 & y_2^2 \\ 1 & y_3^5 + x_2 y_3 & y_3^2 \end{vmatrix}}{\begin{vmatrix} 1 & y_1 & y_1^2 \\ 1 & y_2 & y_2^2 \\ 1 & y_3 & y_3^2 \end{vmatrix}}, \quad \frac{\begin{vmatrix} 1 & y_1 & y_1^5 + x_2 y_1 \\ 1 & y_2 & y_2^5 + x_2 y_2 \\ 1 & y_3 & y_3^5 + x_2 y_3 \end{vmatrix}}{\begin{vmatrix} 1 & y_1 & y_1^2 \\ 1 & y_2 & y_2^2 \\ 1 & y_3 & y_3^2 \end{vmatrix}}, \tag{6.2.40}$$

giving

$$x_1 + P_2(y_1, y_2, y_3), \quad y_1 + y_2 + y_3, \quad x_2 + P_4(y_1, y_2, y_3), \quad P_3(y_1, y_2, y_3),$$

where each of the P_i is a symmetric polynomial of the indicated degree. So $D^3(f)$ is a hypersurface singularity; it has Milnor number 4.

The next proposition is close in appearance to (6.2.38), and is proved by a similar strategy. Unfortunately to put this strategy into effect we need first to construct a different set of generators for $\mathcal{I}_k(f)$. Observe that for any function h,

$$\alpha_1^2(h) = \frac{\begin{vmatrix} 1 & h(x, y_1) \\ 1 & h(x, y_2) \end{vmatrix}}{\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix}} = \frac{h(x, y_2) - h(x, y_1)}{y_2 - y_1} \tag{6.2.41}$$

and

$$\alpha_2^3(h) = \frac{\begin{vmatrix} 1 & y_1 & h(x, y_1) \\ 1 & y_2 & h(x, y_2) \\ 1 & y_3 & h(x, y_3) \end{vmatrix}}{\begin{vmatrix} 1 & y_1 & y_1^2 \\ 1 & y_2 & y_2^2 \\ 1 & y_3 & y_3^2 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & y_1 & h(x, y_1) \\ 0 & y_2 - y_1 & h(x, y_2) - h(x, y_1) \\ 0 & y_3 - y_1 & h(x, y_3) - h(x, y_1) \end{vmatrix}}{(y_3 - y_2)(y_3 - y_1)(y_2 - y_1)} = \frac{\begin{vmatrix} 1 & \frac{h(x,y_2) - h(x,y_1)}{y_2 - y_1} \\ 1 & \frac{h(x,y_3) - h(x,y_1)}{y_3 - y_1} \end{vmatrix}}{y_3 - y_2}$$

$$= \frac{\frac{h(x,y_3) - h(x,y_1)}{y_3 - y_1} - \frac{h(x,y_2) - h(x,y_1)}{y_2 - y_1}}{y_3 - y_2}. \tag{6.2.42}$$

The right hand sides of (6.2.41) and (6.2.42) are the first and second *iterated divided differences* of h; we denote them by $d_{(1)}h$ and $d_{(2)}h$. Define the ℓ'th iterated divided difference of h inductively, by

$$d_{(\ell)}h(x, y_1, \ldots, y_{\ell+1}) = \frac{d_{(\ell-1)}h(x, y_1, \ldots, y_{\ell-1}, y_{\ell+1}) - d_{(\ell-1)}h(x, y_1, \ldots, y_{\ell-1}, y_\ell)}{y_{\ell+1} - y_\ell} \tag{6.2.43}$$

In fact $d_{(\ell)}h(x, y_1, \ldots, y_{\ell+1})$ is symmetric in $y_1, \ldots, y_{\ell+1}$, and moreover the slightly unnatural selection of indices in the inductive definition (6.2.43), with a distinguished role being given to y_ℓ and $y_{\ell+1}$, can be replaced by any other This will become clear in the following proof.

Lemma 6.2.11.

$$I_k(h) = \Big(d_{(1)}h(x, y_1, y_2),\, d_{(2)}h(x, y_1, y_2, y_3), \ldots, d_{(k-1)}h(x, y_1, \ldots, y_k)\Big)$$

Proof. By subtracting row i_1 of (6.2.15) from row i_2 and dividing by $y_{i_1} - y_{i_2}$ we obtain

$$d_{(1)}h(x, y_{i_1}, y_{i_2}) = \alpha_1^k + \alpha_2^k(y_{i_1} + y_{i_2}) \cdots + \alpha_{k-1}^k \frac{y_{i_1}^{k-1} - y_{i_2}^{k-1}}{y_{i_1} - y_{i_2}}; \tag{6.2.44}$$

so the divided difference $d_{(1)}(h)(x, y_{i_1}, y_{i_2})$ lies in $I_k(h)$. Now subtract (6.2.44), with i_3 in place of i_2, from (6.2.44); both sides can be divided by $y_{i_2} - y_{i_3}$, giving

$$d_{(2)}h(x, y_{i_1}, y_{i_2}, y_{i_3}) = \alpha_2^k + \alpha_3^k(y_{i_1} + y_{i_2} + y_{i_3}) + \cdots + \alpha_{k-1}^k P(y_{i_1}, y_{i_2}, y_{i_3}) \tag{6.2.45}$$

where P is a symmetric polynomial of degree $k-3$. Continuing in this way, we see the ideal $\big(\{d_{(\ell)}h(x, y_{i_1}, \ldots, y_{i_{\ell+1}}), 1 \le \ell \le k-1\}\big)$ of iterated divided differences is contained in $I_k(h)$. Moreover, after $k-1$ subtractions and divisions, we are left with the equation

$$d_{(k-1)}h(x, y_{i_1}, \ldots, y_{i_k}) = \alpha_{k-1}^k$$

(and this proves the symmetry of the $d_{(\ell)}h$). So α_{k-1}^k is in the ideal of iterated divided differences. By downward induction on ℓ, we see that the ideals are in fact equal: for any ordering $i_1, \ldots, i_k$ of $1, \ldots, k$, we have

$$\big(\{d_{(\ell)}h(x, y_{i_1}, \ldots, y_{i_{\ell+1}}), 1 \le \ell \le k-1\}\big) = I_k(h).$$

□

Proposition 6.2.12. *Suppose that $D^k(f)$ is smooth of dimension $d_k(n,p)$ at the point $(x, y_1, \ldots, y_k)$. Then for every $\ell < k$ and every projection $\mathbf{e} : D^k(f) \to D^\ell(f)$ (composite of the projections $\mathbf{e}^{i,j}$ described in Subsection 6.2.2), $D^\ell(f)$ is smooth of dimension $d_\ell(n,p)$ at $\mathbf{e}(x, y_1, \ldots, y_k)$.*

Proof. This follows easily from the preceding lemma. After permuting coordinates (which does not affect smoothness or dimension), we may assume that $\mathbf{e}(x, y_1, \ldots, y_k) = (x, y_1, \ldots, y_\ell)$. Let $C_k(f)$ be the map $\mathbb{C}^{n-1+k} \to \mathbb{C}^{(k-1)(p-n+1)}$ with components

$$\big(d_{(1)}f_j(x, y_1, y_2),\; d_{(2)}f_j(x, y_1, y_2, y_3),\; \ldots,\; d_{(k-1)}f_j(x, y, \ldots, y_k)\big)_{j=n,\ldots,p}.$$

Then

$$\begin{array}{ccc} D^k(f) \text{ smooth of dimension } d_k(n,p) \text{ at } (x, y_1, \ldots, y_k) & \Longleftrightarrow & C_k(f) \text{ is a submersion at } (x, y_1, \ldots, y_k) \\ & & \Big\Downarrow \\ D^\ell(f) \text{ smooth of dimension } d_\ell(n,p) \text{ at } (x, y_1, \ldots, y_\ell) & \Longleftrightarrow & C_\ell(f) \text{ is a submersion at } (x, y_1, \ldots, y_\ell) \end{array} \tag{6.2.46}$$

□

Suppose $f(x,y) = (x, f_n(x,y), \ldots, f_p(x,y))$ and $\mathscr{P}$ is a partition of $\{1,\ldots,k\}$ as in (6.2.26), of length m. We define $D^{\mathscr{P}}(f) = D^k(f) \cap \overline{\Delta(\mathscr{P})} = V\big(\mathcal{I}_k(f) + \mathcal{I}(\mathscr{P})\big)$. As $D^{\mathscr{P}}(f)$ is the zero locus, in $\overline{\Delta(\mathscr{P})}$, of an ideal generated by $(k-1)(n-p+1)$ elements, its dimension is at least

$$\dim \Delta(\mathscr{P}) - (k-1)(p-n+1) = n - 1 + m - (k-1)(p-n+1). \tag{6.2.47}$$

We denote this number by $d_{\mathscr{P}}(n,p)$.

Theorem 6.2.13. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a finite corank 1 map germ. Then the following are equivalent:*

1. *f is stable.*
2. *$D^k(f)$ is smooth of dimension $d_k(n,p)$, or empty, for all k such that $d_k(n,p) \geq 0$, and empty if $d_k(n,p) < 0$.*
3. *$D^{\mathscr{P}}(f)$ is smooth of dimension $d_{\mathscr{P}}(n,p)$, or empty, for every k and $\mathscr{P}$ such that $d_{\mathscr{P}}(n,p) \geq 0$, and empty if $d_{\mathscr{P}}(n,p) < 0$.*

Proof. (1⇔2): This is simply a way of stating 6.2.6 without mentioning the value of $\dim_{\mathbb{C}} Q(f)$. For suppose that $\dim Q(f) = k$. We know by 6.2.6 that f is stable if and only if $A_k(f)$ is a submersion, which holds, provided $d_k(n,p) \geq 0$, if and only if $D^k(f)$ is smooth of dimension $d_k(n,p)$. In this case, (6.2.38) shows that for all $\ell \leq k$, $D^\ell(f)$ is smooth of dimension $d_\ell(n,p)$. And for $\ell > k$, $D^\ell(f) = \emptyset$, since $A_\ell(f)(0) \neq 0$. If $d_k(n,p) < 0$ then $A_k(f)$ cannot be a submersion so f cannot be stable - unless $A_k(f)(0) \neq 0$, i.e. unless $D^k(f) = \emptyset$. But the fact that $\dim Q(f) = k$ implies, by the Damon-Galligo Theorem, that $D^k(f) \neq \emptyset$. So taken together, the facts that $d_k(n,p) < 0$ and that $\dim Q(f) = k$ imply that f is not stable.
(2⇔3): This is just the argument using Lemma 6.2.7 given in the proof of Proposition 6.2.4. Let $G = S_{r_1} \times \cdots \times S_{r_m}$. Then $V(\mathcal{I}(\mathscr{P}))$ is the fixed set of G. The map $A_k(f)$ is G-invariant, so $A_k(f)$ is a submersion if and only if $A_k(f)|_{V(\Delta(\mathscr{P}))}$ is a submersion. That is,

$$D^k(f) \text{ is smooth of dimension } d_k(n,p) \Leftrightarrow D^{\mathscr{P}}(f) \text{ is smooth of dimension } d_{\mathscr{P}}(n,p).$$

□

Lemma 6.2.14. *Let $f(x,y) = \big(x, f_n(x,y), \ldots, f_p(x,y)\big)$, and suppose that*

$$f(x, y^{(1)}) = \cdots = f(x, y^{(m)}) =: z$$

with $y^{(i)} \neq y^{(j)}$ if $i \neq j$. Suppose that $\delta(f)_{(x,y^{(i)}))} = r_i$. Let $k = r_1 + \cdots + r_m$ and let $\mathscr{P} = (r_1, \ldots, r_m)$. Then if $\mathbf{y} = (y^{(1)}, \ldots, y^{(1)}, \ldots, y^{(m)}, \ldots, y^{(m)})$ (where $y^{(m)}$ is iterated r_i times) and $S = \{(x,y^{(1)}), \ldots, (x,y^{(m)})\}$, the following are equivalent:

1. *The multi-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, z)$ is stable;*
2. *the map $A_k(f)$ defining $D^k(f)$ is a submersion at $(x, \mathbf{y})$;*
3. *the map $(A_k(f), I(\mathscr{P}))$ defining $D^{\mathscr{P}}(f)$ is a submersion at $(x, \mathbf{y})$.*

Proof. (2⇔3) This is just an application of Lemma 6.2.7. If $G = S_{r_1} \times \cdots \times S_{r_m}$ then $A_k(f)$ is G invariant, so $A_k(f)$ is a submersion at $(x, \mathbf{y})$ if and only if its restriction to the fixed point set of G, $\Delta(\mathscr{P})$, is a submersion. This is the case if and only if the map $(A_k(f), I(\mathscr{P}))$ is a submersion.

(1⇔3) By 6.2.7, (3) is equivalent to the restriction to $\Delta(\mathscr{P})$ of $A_k(f)$ being a submersion. As in the proof of 6.2.6, we let $B_r(f)$ be the map with components

$$\frac{\partial^\ell f_j}{\partial y^\ell} \text{ for } n \leq p, \quad 1 \leq \ell \leq r-1,$$

and we let $C^{\mathscr{P}}(f)$ be the map with components

$$f_j \circ \pi_i^{\mathscr{P}} - f_j \circ \pi_1^{\mathscr{P}} \text{ for } n \leq j \leq p \text{ and } 2 \leq i \leq m.$$

Then by 6.2.4, $A_k(f)|_{\Delta(\mathscr{P})}$ is a submersion if and only if the map $\Delta(\mathscr{P}) \to \mathbb{C}^{(l-1)(n-p+1)}$ defined by

$$(B_{r_1} \circ \pi_1^{\mathscr{P}}, \ldots, B_{r_m} \circ \pi_m^{\mathscr{P}}, C^{\mathscr{P}}) \tag{6.2.48}$$

is a submersion. As we saw in the proof of 6.2.6, submersiveness of $B_{r_i} \circ \pi_i^{\mathscr{P}}$ is equivalent to stability of the mono-germ $f : (\mathbb{C}^n, (x, y^{(i)}) \to (\mathbb{C}^p, z)$. The additional information contained in the submersiveness of (6.2.48) is that the images under f of the analytic strata of these germs meet in general position at z. Note that our embedding of $D^k(f)$ in $\mathbb{C}^{n-1} \times \mathbb{C}^k$ already takes into account the equality of the first $n-1$ components of these images. In any case, this general position, together with the stability of the mono-germs, is equivalent to the stability of the multi-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, z)$. □

Theorem 6.2.15. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a finite corank 1 map germ. Then the following are equivalent:*

1. *f is finitely determined.*
2. *$D^k(f)$ is an ICIS of dimension $d_k(n,p)$, or empty, for all k such that $d_k(n,p) \geq 0$, and consists at most of $\{0\}$ if $d_k(n,p) < 0$.*
3. *$D^{\mathscr{P}}(f)$ is an ICIS of dimension $d_{\mathscr{P}}(n,p)$, or empty, for every ℓ and $\mathscr{P}$ such that $d_{\mathscr{P}}(n,p) \geq 0$, and consists at most of $\{0\}$ if $d_{\mathscr{P}}(n,p) < 0$.*

Proof. (2⇔3) is just an application of Lemma 6.2.7.

(1⇔2). If f is finitely determined, there is a representative $\tilde{f} : U \to V$ with the property that for every $v \in V \smallsetminus \{0\}$, the multi-germ $\tilde{f} : (U, \tilde{f}^{-1}(v)) \to (V, v)$, which we denote by $\tilde{f}_v$, is stable. After shrinking U and V if necessary, we may choose linearly adapted coordinates so $\tilde{f}$ takes the form (6.2.12) and each multiple point space $D^\ell(\tilde{f})$ embeds in $\mathbb{C}^{n-1} \times \mathbb{C}^\ell$. Suppose $(x, y_1, \ldots, y_\ell) \in D^\ell(\tilde{f}) \smallsetminus \{0\}$, and let $v = f(x, y_j)$. Let $\tilde{f}^{-1}(v) = \{y^{(1)}, \ldots, y^{(m)}\}$, let $\delta(\tilde{f})_{(x, y^{(i)})} = r_i$, and let $k = \sum_i r_i$. Clearly $k \geq \ell$. We have

$$(x, \underbrace{y^{(1)}, \ldots, y^{(1)}}_{r_1 \text{ times}}, \ldots, \underbrace{y^{(m)}, \ldots, y^{(m)}}_{r_m \text{ times}}) \in D^k(\tilde{f}).$$

Denote this point by $(x, \mathbf{y})$. After permuting the y_j, $(x, y_1, \ldots, y_\ell) = \mathbf{e}(x, \mathbf{y})$ for some projection $\mathbf{e}$ as in 6.2.12. So by 6.2.12 it is enough to show that $D^k(f)$ is smooth at $(x, \mathbf{y})$. But this is what we proved in Lemma 6.2.14.

The converse holds by the same chain of reasoning. If all the $D^k(f)$ have isolated singularities at 0, and are of the right dimension $d_k(n,p)$, then we can choose a representative of f whose multi-germs outside 0 are all stable. $\square$

6.2.3 Disentangling a singularity: the geometry of a stable perturbation

Suppose that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a finitely determined germ, and let $f_t : U \to V$ be stable perturbation of f. Suppose that f_t has singularities of corank at most 1. Since the codimension of $\overline{\sum^2}$ in $J^1(n,p)$ is $2(p-n+2)$, and $j^1 f$ is transverse to any $\mathscr{A}$-invariant strata, this holds if $n < 2(p-n+2)$ and in particular, for maps from n-space to $n+1$-space, if $n < 6$. By Theorem 6.2.13, the multiple point spaces $D^k(f_t)$ are all smooth. It turns out that the projections $\mathbf{e}^{k,i} : D^k(f_t) \to D^{k-1}(f_t)$ are also stable maps. To see this, we invoke the *principle of iteration* – the observation, due to Salomonsen, that

$$D^\ell(\mathbf{e}^k) \simeq D^{k+\ell-1}(f). \tag{6.2.49}$$

We will prove this in a moment, but let us first apply it to prove that the $\mathbf{e}^{k,i}$ are stable. By 6.2.13, $D^k(f)$ is smooth of dimension $d_k(n,p)$, unless it is empty. Again by 6.2.13, it is enough to prove that $D^\ell(\mathbf{e})$ is smooth and of dimension $d_\ell(d_k(n,p), d_{k-1}(n,p))$, or empty. The smoothness, or emptiness, follow immediately from the principle of iteration, and as for the dimension, we have

$$\begin{aligned} d_\ell\big(d_k(n,p), d_{k-1}(n,p)\big) &= d_\ell\big(n-(k-1)(p-n), n-(k-2)(p-n)\big) \\ &= n-(k-1)(p-n)-(\ell-1)((p-n) \\ &= n-(k+\ell-2)(p-n) \\ &= d_{k+\ell-1}(n,p). \end{aligned} \tag{6.2.50}$$

Thus $\mathbf{e}^{k,i}$ is stable.

The collection of spaces and maps $(D^\bullet(f), \mathbf{e}^\bullet)$ already described in Subsection 6.2.2 is a "semi-simplicial stable map", although since the composite of two stable maps is not in general stable, stable maps do not form a category, and the term "semi-simplicial stable map" , which ought to mean "semi-simplicial object in the category of smooth spaces and stable maps", is not quite legitimate. We could recover legitimacy at the price of clarity by considering the category *generated by* smooth spaces and stable maps – that is, which allows *composites* of stable maps.

Whatever we call it, $(D^\bullet(f), \mathbf{e}^\bullet)$, together with the symmetric group actions, is a rich structure. Although our explicit description of multiple point spaces is only available when the maps in question have corank 1 singularities, this assumption need not be made for the map-germ f whose stable perturbation we are considering. All that is needed is to be sure that its stable perturbation f_t has only corank 1 singularities. This is guaranteed if, for example, the dimensions do not allow stable germs of corank > 1.

To prove (6.2.49), let us first define obvious morphisms in each direction. A point of $D^\ell(\mathbf{e}^{k,i})$ is an ℓ-tuple of points in $D^k(f)$ sharing the same image under $\mathbf{e}^{k,i}$. Since $\mathbf{e}^{k,i} = \mathbf{e}^{k,k} \circ (\ \to i)$, we may suppose, without loss of generality, that $i = k$. We will write $\mathbf{e}^k$ in place of $\mathbf{e}^{k,k}$. Since $\mathbf{e}^k$ simply forgets the last point in each k-tuple, a point in $D^\ell(\mathbf{e}^k)$ must take the form

$$\Big((x^{(1)}, \ldots, x^{(k,1)}),\ (x^{(1)}, \ldots, x^{(k-1)}, x^{(k,2)}), \ldots, (x^{(1)}, \ldots, x^{(k-1)}, x^{(k,\ell)})\Big). \tag{6.2.51}$$

Define $\varphi : D^\ell(\mathbf{e}^k) \to D^{k+\ell-1}(f)$ by sending the point (6.2.51) to the $k+\ell-1$-tuple

$$\left(x^{(1)}, \ldots, x^{(k-1)}, x^{(k,1)}, \ldots, x^{(k,\ell)}\right) \tag{6.2.52}$$

which certainly lies in $D^{k+\ell-1}(f)$; equally simply, we define $\varphi^{-1} : D^{k+\ell-1}(f) \to D^\ell(\mathbf{e})$ by

$$\varphi^{-1}\left(x^{(1)}, \ldots, x^{(k+\ell-1)}\right) =$$

$$\left(\left(x^{(1)}, \ldots, x^{(k-1)}, x^{(k)}\right), \left((x^{(1)}, \ldots, x^{(k-1)}, x^{(k+1)}\right), \ldots, \left(x^{(1)}, \ldots, x^{(k-1)}, x^{(k+\ell-1)}\right)\right). \tag{6.2.53}$$

If we knew that both $D^\ell(\mathbf{e})$ and $D^{k+\ell-1}(f)$ were reduced, this would enough to prove that φ is an isomorphism. However, although stability of f means that $D^{k+\ell-1}(f)$ is reduced, we do not know that $D^\ell(\mathbf{e})$ is reduced, since our only evidence that $\mathbf{e}$ is stable relies on the map (6.2.49) which we are trying to prove an isomorphism. So we have to supplement this construction with a more painstaking analysis. The argument comes from [Gor95], where it is a little more succinct than here. To begin, note that in case $\ell = 2$, the morphism φ of (6.2.52) fits into a commutative diagram

$$\begin{tikzcd} D^2(\mathbf{e}^k) \arrow[rr, "\varphi"] \arrow[dr, "\mathbf{e}^2"'] & & D^{k+1}(f) \arrow[dl, "\mathbf{e}^{k+1}"] \\ & D^k(f) & \end{tikzcd} \tag{6.2.54}$$

in which the map $\mathbf{e}^2(\mathbf{e}^k)$ is the projection associated to the double point space of the map $\mathbf{e}^k : D^k(f) \to D^{k-1}(f)$. We use this observation as the basis of an argument by induction on k. The argument makes use of one *ad hoc* calculation, for which we require an explicit formula for a stable map-germ of corank 1 and multiplicity $n+1$ from $(\mathbb{C}^N, 0)$ to $(\mathbb{C}^{N+r}, 0)$. By **??**, the smallest such N is equal to rn, and with respect to coordinates $y, u_1, \ldots, u_{n-1}$ and $v_{i,1}, \ldots, v_{i,n}$ for $i = 1, \ldots, r-1$ the map-germ we want takes the form

$$(u, v, y) \mapsto \left(u, v, y^{n+1} + \sum_{i=1}^{n-1} u_i y^{n-i}, \sum_{j=1}^{n} v_{1,j} y^{n-j+1}, \ldots, \sum_{j=1}^{n} v_{r-1,j} y^{n-j+1}\right). \tag{6.2.55}$$

Denote this map by $F_{n,r}$. To simplify notation, denote the source and target of $F_{n,r}$ as $D^1(F_{n,r})$ and $D^0(F_{n,r})$ respectively.

Lemma 6.2.16. *The projection $\mathbf{e}^2 : D^2(F_{n,r}) \to D^1(F_{n,r})$ is left-right-equivalent to $F_{n-1,r} \times I$, where I denotes the identity map $\mathbb{C} \to \mathbb{C}$.*

Proof. The defining equations for $D^2(F_{n,r})$,

$$\begin{aligned} \frac{y_1^{n+1} - y_2^{n+1} + \sum_{i=1}^{n-1} u_i (y_1^{n-i} - y_2^{n-i})}{y_1 - y_2} &= 0 \\ \frac{\sum_{i=1}^{n} v_{j,i} (y_1^{n-i+1} - y_2^{n-i+1})}{y_1 - y_2} &= 0 \text{ for } j = 1, \ldots, r-1 \end{aligned}$$

allow us to express u_{n-1} and the $v_{j,n}$, for $j=1,\ldots,r-1$, in terms of the other coordinates. Thus we can take y_1, y_2, $u' := u_1\ldots,u_{n-2}$ and $v' := v_{j,1}\ldots v_{j,n-1}$, for $j=1,\ldots,r-1$, as coordinates on $D^2(F_{n,r})$. Consider the projection $D^2(F_{n,r}) \to D^1(F_{n,r})$ when $y_1=0$. It takes the form

$$(y_2,u',v') \mapsto (u',v',y_2^n + \sum_{i=1}^{n-2} u_i y_2^{n-i-1}, \sum_{i=1}^{n-1} v_{1,i}y_2^{n-i}, \ldots, \sum_{i=1}^{n-1} v_{r-1,i}y_2^{n-i})$$

with respect to coordinates $(u',v',-u_{n-1},-v_{1,n},\ldots,-v_{r-1,n})$ on $\{y_1=0\}\subset D^1(F_{n,r})$. This is precisely (6.2.55) with $n-1$ in place of n. So $\mathbf{e}^2|_{\{y_1=0\}}$ is stable. It follows that $\mathbf{e}^2$, as an unfolding of $\mathbf{e}^2|_{\{y_1=0\}}$ on parameter y_1, is a trivial unfolding, and is therefore equivalent to $F_{n-1,r}\times I$. $\square$

Proposition 6.2.17. *The projection $\mathbf{e}^k : (D^k(F_{n,r}),0) \to (D^{k-1}(F_{n,r}),0)$ is left-right equivalent to the germ at 0 of the map $F_{n-k+1,r}\times I^{k-1}$, where I^{k-1} is the identity map on $\mathbb{C}^{k-1}$.*

Proof. We have just shown that this is true for $k=2$. Now suppose by induction that it is true for k: that there are isomorphisms making the diagram

$$\begin{tikzcd} D^k(F_{n,r}) \arrow[r, "\simeq"] \arrow[d, "\mathbf{e}^k"'] & D^1(F_{n-k+1,r}\times I^{k-1}) \arrow[d, "F_{n-k+1,r}\times I^{k-1}"] \\ D^{k-1}(F_{n,r}) \arrow[r, "\simeq"'] & D^0(F_{n-r+k,r}\times I^{k-1}) \end{tikzcd} \tag{6.2.56}$$

commute. Then by Lemma 6.2.16, there is a commutative diagram

$$\begin{tikzcd} D^2(\mathbf{e}^k) \arrow[r, "\simeq"] \arrow[d, "\mathbf{e}^2(\mathbf{e}^k)"'] & D^2(F_{n-k+1,r}\times I^{k-1}) \arrow[d, "\mathbf{e}^2"] \\ D^1(\mathbf{e}^k) \arrow[r, "\simeq"'] & D^1(F_{n-k+1,r}\times I^{k-1}) \end{tikzcd} \tag{6.2.57}$$

where the spaces in the top left and bottom left are the double-point space and domain of the map $\mathbf{e}^k : D^k(F_{n,r}) \to D^{k-1}(F_{n,r})$ on the left of (6.2.56). *It follows from 6.2.13 that $D^2(\mathbf{e}^k)$ is reduced.* This was all that was missing from our earlier argument that the map φ of (6.2.52) and (6.2.54) is an isomorphism: so we conclude that $\mathbf{e}^{k+1} : D^{k+1}(F_{n,r}) \to D^k(F_{n,r})$ is stable, and, juxtaposing (6.2.54) and (6.2.57), that there is a commutative diagram

$$\begin{tikzcd} D^{k+1}(F_{n,r})) \arrow[r, "\simeq"] \arrow[d, "\mathbf{e}^{k+1}"'] & D^2(F_{n-k+1,r}\times I^{k-1})\ . \arrow[d, "\mathbf{e}^2"] \\ D^k(F_{n,r}) \arrow[r, "\simeq"'] & D^1(F_{n-k+1,r}\times I^{k-1}) \end{tikzcd} \tag{6.2.58}$$

Juxtaposing *this* diagram with the diagram

$$\begin{tikzcd} D^2(F_{n-k+1}\times I^{k-1}) \arrow[r, "\simeq"] \arrow[d, "\mathbf{e}^2"'] & D^1(F_{n-k,r}\times I^k) \arrow[d, "F_{n-k,r}\times I^k"] \\ D^1(F_{n-k+1,r}\times I^{k-1}) \arrow[r, "\simeq"'] & D^0(F_{n-k,r}\times I^k) \end{tikzcd} \tag{6.2.59}$$

provided by Lemma 6.2.16 we obtain a commutative diagram

$$\begin{tikzcd} D^{k+1}(F_{n,r}) \arrow[r, "\simeq"] \arrow[d, "\mathbf{e}^k"'] & D^1(F_{n-k,r} \times I^k) \arrow[d, "F_{n-k,r}\times I^k"] \\ D^k(F_{n,r}) \arrow[r, "\simeq"'] & D^0(F_{n-k,r} \times I^k) \end{tikzcd} \quad . \tag{6.2.60}$$

This completes the induction. □

A stable singularity of corank 1 map $(\mathbb{C}^n, S) \to (\mathbb{C}^p, y)$, $n \le p$ with $|S| = k$ is characterised, up to isomorphism, by the k-tuple of its multiplicities at the k distinct points of S. This is simply because each of the mono-germs $f^{(i)} : (\mathbb{C}^n, x^{(i)}) \to (\mathbb{C}^p, y)$, for $x^{(i)} \in S$, is determined, up to isomorphism, by its local algebra, and thus by its mutiplicity, and because stability of f means that the analytic strata of the distinct mono-germs making up f are in general position in $\mathbb{C}^p$.

Lemma 6.2.18. *Up to isomorphism and trivial unfolding, all stable multi-germs of corank 1 maps $(M^n, S) \to (N^{n+r}, y)$ appear in the germs $F_{n,r}$, for n sufficently large.*

Proof. Suppose that $(M^m, S) \to (N^p, y)$, $n \le p$ is stable, $|S| = k < \infty$, and the germ of f at $x^{(i)} \in S$ is of corank ≤ 1, with multiplicity n_i, for $i = 1, \ldots, k$. Let $n = \sum_i n_i$. Then a (necessarily stable) multi-germ $(\mathbb{C}^n, S') \to (\mathbb{C}^p, y)$ with multiplicities $n_1, \ldots, n_k$ appears in every representative of $F_{n,r}$. To see this, identify the $n-1$-dimensional space of the parameters u_i with the space P_{n+1} of monic polynomials of degree $n+1$ with root sum and product both equal to 0. In P_n there is a stratum consisting of polynomials with k distinct roots with multiplicities $n_1, \ldots, n_k$. Let $u(t) = (u_1(t), \ldots, u_{n-1}(t))$, with $u(0) = 0$, parametrise an analytic curve which lies in in this stratum for $t \ne 0$. Let $x^{(1)}(t), \ldots, x^{(k)}(t)$ be the corresponding roots. They depend analytically on t, for $t \ne 0$. Then the multi-germ of $F_{n+1,r}$ at $S'_t = \{(u(t), 0, x^{(1)}t)), \ldots, (u(t), 0, x^{(k)}(t))\}$ has the required multiplicities. It must be stable, as the germ at 0 of $F_{n,r}$ is stable, stability is an open condition, and the weighted homogeneity of $F_{n+1,r}$ means that its stability in some neighbourhood of 0 is propagated to all of its domain. □

Corollary 6.2.19. (The principle of iteration) *If $f : U \to V$ is a stable map with only corank 1 singularities then all germs of projection $\mathbf{e}^k : D^k(f) \to D^{k-1}(f)$ are stable, and moreover the map $\varphi : D^\ell\big(\mathbf{e}^k : D^k(f) \to D^{k-1}(f)\big) \to D^{k+\ell-1}(f)$ of* (6.2.52) *is an isomorphism.*

Proof. This is now immediate from Proposition 6.2.17 and Lemma 6.2.18. □

Question Consider the full disentanglement $D^\bullet(f_t) := (D^\bullet(f_t), \mathbf{e}^{\bullet,\bullet})$ of a map-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, and suppose that f_t has only corank 1 singularities. To what extent do invariants of $D^\bullet(f_t)$ detect invariants of f? In particular, can the rank of $d_0 f$ be recovered from $D^\bullet(f_t)$?

Exercise 6.2.20. 1. Find equations for $D^2(f)$ and $D^3(f)$ when f is the map-germ given by

(a) $f(x_1, x_2, x_3, y) = (x_1, x_2, x_3, y^3 + x_1 y, x_2 y^2 + x_3 y)$ (stable map-germ of type $\sum^{1,1,0}$).

(b) $f(x, y) = (x, y^2, y^3 + x^{k+1} y)$ (type S_k in [Mon85] – here $D^3(f) = \emptyset$)

(c) $f(x, y) = (x, y^3, xy + y^5)$ (type H_2 in [Mon85])

(d) $f(x, y) = (x, y^3, xy + y^{3k-1})$ (type H_k in [Mon85]).

2. In 1(a), check that $D^k(f)$ is smooth whenever non-empty.

3. For 1(b),1(c) and 1(d), check that $D^2(f)$ has isolated singularity.

4. For 1(c) and 1(d), check that $D^3(f)$ is zero-dimensional. What is the complex vector space dimension of $\mathcal{O}_{\mathbb{C}\times\mathbb{C}^3,0}/\mathcal{I}_3(f)$ in these two cases (your answer should be divisible by 6)?

5. Suppose that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ has corank 1 and is finitely determined. Show that

   (a) $\dim_{\mathbb{C}}\ \mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^{n+1},0}/\mathcal{I}_{n+1}(f)$ is divisible by $(n+1)!$.

   (b) and use Theorem 1.3.3 to show that if f_t is a stable perturbation of f, then the image of f_t contains
   $$\frac{1}{(n+1)!}\left(\dim_{\mathbb{C}}\ \mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^{n+1},0}/\mathcal{I}_{n+1}(f)\right)$$
   ordinary $(n+1)$-tuple points (at which it is locally isomorphic to the union of the $n+1$ coordinate hyperplanes in $(\mathbb{C}^{n+1}, 0)$).

As a result of the two parts of Theorem **??**, it follows that when f_t is a stable perturbation of a finitely determined corank 1 germ f, then $D^k(f_t)$ is a smoothing, and therefore a Milnor fibre, of the ICIS $D^k(f)$.

Exercise 6.2.21. Find the Milnor numbers of $D^2(f)$ and $D^3(f)$ for the map germs of type S_k, H_2 and H_k in Exercise 6.2.20.

6.3 Computing the homology of the image

Let $f : X \to Y$ be a finite map. For each $k \geq 2$ there are projections $D^k(f) \to D^{k-1}(f)$ defined by forgetting one of the copies of X. These give rise to maps on the vanishing homology of the Milnor fibres $D^k(f_t)$ when f_t is a stable perturbation of f; there is thus a rather rich structure of homology groups and homomorphisms associated to a stable perturbation. It turns out that from this one can obtain information about the homology of the image of the stable perturbation. The action of the symmetric group S_k on $D^k(f)$ determines the gluing which takes place when the domain of f_t is mapped to the image, and it is therefore no surprise that in the computation of the homology of the image, this action should play a rôle. In fact it is the *alternating* part of the homology which enters into the calculation of $H_*(\text{image}(f_t))$. This was first observed in [GM93] at the level of rational homology. For any map $f : X \to Y$, we define

$$\text{Alt}_k H_q(D^k(f);\mathbb{Q}) = \{[c] \in H_q(D^k(f);\mathbb{Q}) : \sigma_*([c]) = \text{sign}(\sigma)[c] \text{ for all } \sigma \in S_k\},$$

and refer to it as the *alternating part* of $H_q(D^k(f);\mathbb{Q})$. Later the construction was greatly clarified by Goryunov in [Gor95], by the introduction of the *alternating chain complex*, which we now describe. The description here differs from Goryunov's only in that it uses singular homology in place of cellular homology.

6.4 The alternating chain complex

Let D^k be any space on which the symmetric group S_k acts, and let $C_\ell(D^k)$ be the usual free abelian group of singular ℓ-chains in D^k. A chain $c \in C_\ell(D^k)$ is *alternating* if for each $\sigma \in S_k$, $\sigma_\#(c) = \text{sign}(\sigma)c$. We denote the set of all alternating chains (with integer coefficients) on D^k by $C_\ell^{\text{Alt}}(D^k)$. It is, evidently, a subgroup of $C_\ell(D^k)$, and therefore free abelian. The $C_\ell^{\text{Alt}}(D^k)$ form a complex under the usual boundary map; we call its homology the *alternating homology of* D^k), and denote it by $H_*^{\text{Alt}}(D^k)$.

Proposition 6.4.1.

$$H_*^{Alt}(D^k) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq Alt_k H_q(D^k; \mathbb{Q}).$$

Proof. **Exercise** □

We will use this as a heuristic guide to later constructions. In particular, if $D^k = D^k(f_t)$, where f_t is a stable perturbation of a corank 1 map-germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, then $D^k(f_t)$ is the Milnor fibre of an ICIS of dimension $p - k(p-n)$ provided $p - k(p-n) \geq 0$, and empty if $p - k(p-n) < 0$; thus $H_q(D^k(f); \mathbb{Q}) = 0$ unless $q = 0$ or $q = p - k(p-n)$. Now if $p - k(p-n) > 0$, $D^k(f)$ is connected and so S_k acts trivially on $H_0(D^k(f); \mathbb{Q})$, and it follows that $\text{Alt}_k H_0(D^k(f); \mathbb{Q}) = 0$. Thus

Proposition 6.4.2. *If f_t is a stable perturbation of a a corank 1 map-germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, then $Alt_k(H_q(D^k(f_t); \mathbb{Q}) = 0$ if $q \neq p - k(p-n)$.*

In other words, *for all k $Alt_k H_*(D^k(f_t); \mathbb{Q})$ is concentrated in middle dimension.*

Let us return to the situation of a map $f : X \to Y$, and let $D^k(f)$ be the usual multiple point spaces. Denote by π^k the projection $D^k(f) \to D^{k-1}(f)$ defined by

$$\pi^k(x_1, \ldots, x_k) = (x_1, \ldots, x_{k-1}).$$

Proposition 6.4.3. $\pi^k_\#(C_\ell^{Alt}(D^k(f)) \subset C_\ell^{Alt}(D^{k-1}(f))$.

Proof. There is an obvious embedding $i : S_{k-1} \hookrightarrow S_k$ such that for $\sigma \in S_{k-1}$ then, as maps on $D^k(f)$,

$$\sigma \circ \pi^k = \pi^k \circ i(\sigma);$$

as a map $\{1, \ldots, k\} \to \{1, \ldots, k\}$,

$$i(\sigma)(j) = \begin{cases} \sigma(j) & \text{if } j < k \\ k & \text{if } j = k \end{cases}$$

The sign of $i(\sigma)$ is the same as the sign of σ; it follows that if $c \in C_\ell^{\text{Alt}}(D^k(f))$ then for any $\sigma \in S_{k-1}$,

$$\sigma_\#(\pi^k_\#(c)) = \pi^k_\# i(\sigma)_\#(c) = \pi^k_\#(\text{sign}(i(\sigma))c) = \text{sign}(\sigma)\pi^k_\#(c).$$

Thus $\pi^k_\#(c) \in C_\ell^{\text{Alt}}(D^{k-1}(f))$. □

Proposition 6.4.4. $\pi^{k-1}_\# \circ \pi^k_\# = 0$ *on* $C_\bullet^{Alt}(D^k(f))$, *and* $f_\# \pi^2_\# = 0$ *on* $C_\bullet^{Alt}(D^2(f))$.

Proof. Let $\sigma \in S_k$ be the transposition $(k-1\ k)$. Clearly $\pi^{k-1} \circ \pi^k = \pi^{k-1} \circ \pi_k \circ \sigma$, and it follows that for $c \in C_\ell^{\mathrm{Alt}}(D^k(f))$,

$$(\pi^{k-1} \circ \pi^k)_\#(c) = (\pi^{k-1} \circ \pi_k)_\#(\sigma_\#(c)) = (\pi^{k-1} \circ \pi^k)_\#(-c) = -(\pi^{k-1} \circ \pi^k)_\#(c).$$

Since $C_\ell^{\mathrm{Alt}}(D^{k-2}(f)$ is free abelian, this proves that $(\pi^{k-1} \circ \pi^k)_\#(c) = 0$.

The second statement is proved by essentially the same argument. □

Suppose that $c_2 \in C_\ell^{\mathrm{Alt}}(D^2(f))$ represents a homology class in $H_\ell^{\mathrm{Alt}}(D^2(f))$. Then $\pi^2_\#(c_2)$ is also closed in $C_\bullet(X)$. *Now let us make the assumption that $H_\ell(X) = 0$.* This is certainly justified if X is the (contractible) domain of a stable perturbation of a corank 1 map-germ. The assumption also tallies with the evidence provided by Propositions 6.4.1 and 6.4.2 in the case of a stable perturbation of a corank 1 map-germ, for these suggest (though they do not prove) that if $H_\ell^{\mathrm{Alt}}(D^k(f_t)) \neq 0$ then $H_\ell^{\mathrm{Alt}}(D^{k-1}(f_t)) = 0$. We make it now in order to motivate a later more formal construction.

We will refer to this assumption as the *Vanishing Assumption*.

Under this assumption, since $\pi^2_\#(c_2)$ is a cycle, it must also be a boundary: there exists $c_1 \in C_{\ell+1}(X)$ such that $\partial c_1 = \pi^2_\#(c_2)$. Then $f_\#(c_1)$ is a cycle in the image of f, for $\partial f_\#(c_1) = f_\#(\partial c_1) = f_\# \pi^2_\#(c_2)$, and this is equal to 0 by 6.5.13.

Conclusion: From the alternating homology class $[c_2] \in H_\ell^{\mathrm{Alt}}(D^2(f)$, under the assumption that $H_\ell(X) = 0$, we have constructed a homology class $[f_\#(c_1)] \in H_{\ell+1}(Y)$.

Warning: We have not constructed a map $H_\ell^{\mathrm{Alt}}(D^2(f) \to H_{\ell+1}(Y)$; there was an element of arbitrariness in the choice of c_1. In fact if c_1' is any other choice of $\ell+1$-chain on X such that $\partial c_1' = \pi^2_\#(c_2)$ then $c_1 - c_1'$ represents a homology class in $H_{\ell+1}(X)$, and thus the homology classes of $f_\#(c_1)$ and $f_\#(c_1')$ in $H_{\ell+1}$ differ by an element of $f_* H_{\ell+1}(X)$. Our construction in fact yields a map $H_\ell^{\mathrm{Alt}}(D^2) \to H_{\ell+1}(Y)/f_* H_{\ell+1}(X)$.

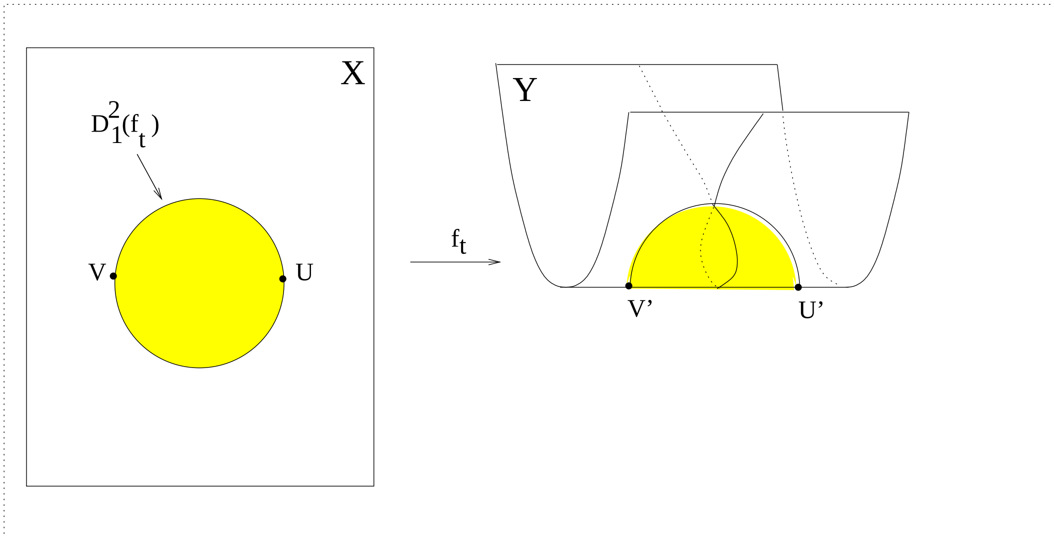
Example 6.4.5. In this example X is contractible, so the imprecision in the choice of the cycle $f_\#(c_2)$ does not arise. Consider the stable perturbation $f_t : \mathbb{R}^2 \to \mathbb{R}^3$, defined by $f_t(x, y) = (x, y^2, y^3 + x^2 + ty)$, of the singularity $f = f_0$ of type S_1. We have

$$D^2(f_t) = \{(x, y_1, y_2) : y_1 + y_2 = 0 = x^2 + y_1^2 + y_1 y_2 + y_2^2 + t\};$$

The projection $\pi^2(x, y_1, y_2) = (x, y_1)$ (with inverse $(x, y) \mapsto (x, y, -y)$) maps this isomorphically to the conic

$$D_1^2(f_t) := \{(x, y) \in \mathbb{C}^2 : x^2 + y^2 + t = 0\},$$

with the involution $\sigma(x, y_1, y_2) = (x, y_2, y_1)$ now induced by $(x, y) \mapsto (x, -y)$.



Let a and b be 1-simplices running from U to V, a on the upper arc and b on the lower arc of D^2, such that $\sigma \circ a = b$. Then the alternating homology $H_1^{\mathrm{Alt}}(D^2(f_t)$ is generated by $a - b$. Since here $D^2(f_t)$ is embedded in the domain X of f_t, we identify $a - b \in C_1^{\mathrm{Alt}}(D^2(f_t))$ with its image in $C_1(X)$.Taking as c_2 a suitable triangulation of the interior of the shaded disc, we have $\partial c_2 = a - b$. As can be seen in the picture, $f_\#(c_2)$ forms a bubble whose homology class generates $H_2(Y)$.

In fact this picture shows all the action of the complexified map $\mathbb{C}^2 \to \mathbb{C}^3$. Here $D^2(f_t) \simeq D^2_1(f_t)$ is the complex Milnor fibre of an A_1 singularity, and is diffeomorphic to a cylinder. However its alternating homology is generated by the cycle shown in the real picture, and from there on the construction is the same.

Exercise 6.4.6. 1. Check that our map $H_\ell^{\mathrm{Alt}}(D^2(f)) \to H_{\ell+1}(Y)/f_*H_{\ell+1}(X)$ is well-defined in the sense that if c_2 and c_2' represent the same alternating homology class in $H_\ell^{\mathrm{Alt}}(D^2(f))$ then the resulting homology classes are the same in $H_{\ell+1}(Y)/f_*H_{\ell+1}(X)$.

2. Show that if we dispense with the Vanishing Assumption (that $H_\ell(X) = 0$), our construction yields a map

$$\ker\big(\pi^2_* : H_2^{\mathrm{Alt}}(D^2(f)) \to H_2(X)\big) \to H_{\ell+1}(Y).$$

3. Under the Vanishing Assumption (to simplify notation) let F_ℓ be the image of $H_\ell^{\mathrm{Alt}}(D^2)$ in $H_{\ell+1}(Y)/f_*H_{\ell=1}(X)$, and let $\bar{F}_\ell$ be the preimage of F_ℓ in $H_{\ell+1}(Y)$. Show that if we assume also that $H_{\ell-1}^{\mathrm{Alt}}(D^2(f)) = 0$, the construction of the last two pages can be extended to give a map $H_{\ell-1}^{\mathrm{Alt}}(D^3(f)) \to H_{\ell+1}(Y)/\bar{F}_\ell$. The scheme of the argument is shown in the following diagram, in which we begin with an alternating $(\ell-1)$-cycle a_3 on $D^3(f)$ and successively

choose $a_2 \in C_\ell^{\mathrm{Alt}} D^2(f)$ and $a_1 \in C_{\ell+1}(X)$.

(6.4.1)

$$\begin{tikzcd}
 & \ell-2 & \ell-1 & \ell & \ell+1 \\
D^3(f) \arrow[d, "\pi^3"'] & 0 & a_3 \arrow[l, mapsto, "\partial"'] \arrow[d, mapsto] & & \\
D^2(f) \arrow[d, "\pi^2"'] & 0 & \pi^3_\# a_3 \arrow[l, mapsto, "\partial"'] \arrow[d, mapsto] & a_2 \arrow[l, mapsto, "\partial"'] \arrow[d, mapsto] & \\
X \arrow[d, "f"'] & & \pi^2_\# \pi^3_\# a_3 = 0 & \pi^2_\# a_2 \arrow[l, mapsto, "\partial"'] \arrow[d, mapsto] & a_1 \arrow[l, mapsto, "\partial"'] \arrow[d, mapsto] \\
Y & & & f_\# \pi^2_\# a_2 = 0 & f_\# a_1 \arrow[l, mapsto, "\partial"']
\end{tikzcd}$$

4. Show how to modify the construction if the Vanishing Assumptions are dropped.

6.5 The image computing spectral sequence

The rather complicated combinatorics of the previous constructions are all bundled up together in a spectral sequence which was first described in [GM93] and later developed and extended in [Gor95], [Hou97] and [Hou99]. The main theorems of [Gor95] on this topic are the following. We give the first in approximate form in order not to hide its statement in a technical fog.

Theorem 6.5.1. *Let $f : X \to Y$ be a finite surjective map of topological spaces. Then there is a spectral sequence with $E^1_{pq} = H^{Alt}_p(D^q(f))$, converging to $H_{p+q-1}(Y)$.*

Spectral sequences are sometimes rather complicated to use, and to understand, and have a rather bad reputation. They arise wherever we have a double complex, as here, and in many other situations. In the case of a double complex, say $C_{\bullet,\bullet}$, with differentials $d_h : C_{i,j} \to C_{i-1,j}$ and $d_v : C_{i,j} \to C_{i,j-1}$ satisfying the anti-commutation relation

$$d_v \circ d_h = -d_h \circ d_v,$$

it is easy to see that by first applying the horizontal differential d_h, from the array

(6.5.1)

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \arrow[d] & \\
\arrow[r] & C_{i+1,j+1} \arrow[r] \arrow[d] & C_{i,j+1} \arrow[r] \arrow[d] & C_{i-1,j+1} \arrow[r] \arrow[d] & {} \\
\arrow[r] & C_{i+1,j} \arrow[r] \arrow[d] & C_{i,j} \arrow[r] \arrow[d] & C_{i-1,j} \arrow[r] \arrow[d] & {} \\
\arrow[r] & C_{i+1,j-1} \arrow[r] \arrow[d] & C_{i,j-1} \arrow[r] \arrow[d] & C_{i-1,j-1} \arrow[r] \arrow[d] & {} \\
 & {} & {} & {} &
\end{tikzcd}$$

we obtain an array of "horizontal" homology groups

$$\begin{array}{ccc}
\downarrow & \downarrow & \downarrow \\
H^{(1)}_{i+1,j+1} & H^{(1)}_{i,j+1} & H^{(1)}_{i-1,j+1} \\
\downarrow & \downarrow & \downarrow \\
H^{(1)}_{i+1,j} & H^{(1)}_{i,j} & H^{(1)}_{i-1,j} \\
\downarrow & \downarrow & \downarrow \\
H^{(1)}_{i+1,j-1} & H^{(1)}_{i,j-1} & H^{(1)}_{i-1,j-1} \\
\downarrow & \downarrow & \downarrow
\end{array} \tag{6.5.2}$$

in which the vertical arrows d_v pass to the quotient to give the vertical arrows shown, since by the anti-commutation relation the vertical differential is, up to sign, a morphism of the horizontal chain complexes. We could equally well apply d_v first and obtain an array of vertical homology groups and horizontal arrows. Applying the vertical arrows in (6.5.2) we obtain a third array. Further differentials can be defined, leading to a sequence of arrays, each with its differential. Rather than attempting a complete description at this point, we give an operational version as it applies in the case of a stable perturbation $f_t : U_t \twoheadrightarrow Y_t$ of a finitely determined map germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with $n < p$, beginning with $C_{i,j} = C_i^{\text{Alt}}(D^{j+1}(f_t))$. In this case, the horizontal homology is $H^{(1)}_{i,j} = H_i^{\text{Alt}}(D^{j+1}(f_t))$. The general theory of spectral sequences says that if the homology of the sequence of arrays eventually stabilises – i.e if for some N all of the arrows in the N'th array , and in all succeeding arrays, are 0 – then the homology of the image Y_t can be read off from homology groups $H^{(N)}_{i,j}$ in the N'th array. More precisely, for each $\ell \geq 0$ there is a filtration

$$F_0^\ell = 0 \subseteq F_1^\ell \subseteq \cdots \subseteq F_m^\ell = H_\ell(Y_t)$$

on the ℓ'th homology of the image, such that for $k = 1, \cdots, m$

$$F_k^\ell / F_{k-1}^\ell \simeq H^{(N)}_{i,\ell-i}.$$

In particular, the ℓ'th betti number of Y_t is the sum, over i, of the ranks of the $H_{i,\ell-i}$. What makes this rather easily useable in our case is the fact that the alternating homology of the multiple point spaces $D^k(f_t)$ vanishes except in middle dimension:

Theorem 6.5.2. ([[Hou97]) $H_j^{Alt}(D^k(f_t)) = 0$ *except when* $k = \dim D^k(f_t)$. □

This vanishing is easy to see for alternating homology with rational coefficients, in in the case that f has corank 1. Here all of the multiple point spaces $D^k(f_t)$ are Milnor fibres of isolated complete intersection singularities, and hence have reduced homology only in middle dimension. The rational alternating homology $H_i^{\text{Alt}}(D^k(f_t); \mathbb{Q})$ is a direct summand in $H_i(D^k(f_t); \mathbb{Q})$, so it too vanishes outside middle dimension. The proof without these hypotheses is long and rather difficult, and is omitted here. If $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+r}, 0)$ is no longer assumed to have corank 1, then we know very little about its multiple point spaces $D^k(f)$ and those of a stable perturbation f_t. In particular, $D^k(f)$ is not in general an ICIS, and, if the dimensions $(n, n+r)$ are such that there may be corank 2 stable singularities of maps $\mathbb{C}^n \to \mathbb{C}^{n+r}$, then $D^k(f_t)$ is not in general a smoothing of $D^k(f)$.

An immediate consequence of Theorem 6.5.2 is that in each of the successive arrays derived from (6.5.1) in the case that $C_{i,j} = C_i^{\mathrm{Alt}}(D^{j+1}(f_t))$, all of the arrows vanish, for the simple reason that for each one, either the domain or the range is 0. One says, under these circumstances, that "the spectral sequence collapses at E^1".

Corollary 6.5.3. *Suppose that* $f_t : X_t \twoheadrightarrow Y_t$ *is a stable perturbation of a map germ* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+r}, 0)$, *with* $r > 0$.

1. *If* $r \geq 2$, *then*

$$H_q(Y_t) = \begin{cases} H^{Alt}_{n-(k-1)r}(D^k(f_t)) & \text{if } q = n-(k-1)(r-1) \text{ for some } k \\ \mathbb{Z} & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

2. *If* $r = 1$, *then* $H_q(Y) = 0$ *if* $q \neq 0, n$, *and there is a filtration on* $H_n(Y_t)$ *such that the associated graded module is isomorphic to the direct sum*

$$\bigoplus_{k=2}^{n+1} H^{Alt}_{n-k+1}(D^k(f_t)).$$

If D^k is an S_k-invariant ICIS of dimension r with S_k-invariant Milnor fibre D^k_t, let us refer to the rank of $H_r^{\mathrm{Alt}}(D^k_t)$ as the *alternating Milnor number* of D^k. Then we have

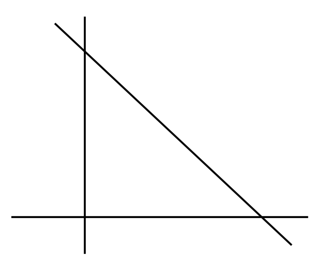
Corollary 6.5.4. *In the situation of 6.5.3(2), the image Milnor number of* f *is the sum of the alternating Milnor numbers of the ICISs* $D^k(f)$ *for* $k = 2, \ldots, n+1$.

Exercise 6.5.5.

1. Viewing $\mathbb{RP}^2$ as the image of the upper unit disc under the map which identifies opposite points on the boundary, find an alternating homology class in $H_0^{\mathrm{Alt}}(D^2(f))$ which gives rise to a generator of $H_1(\mathbb{RP}^2) \simeq \mathbb{Z}/2\mathbb{Z}$. Generalise this to $\mathbb{RP}^n$, taking care to distinguish between the case n even and n odd.

2. Let X be the disjoint union of 3 real lines and $f : X \to \mathbb{R}^2$ be the map

$$\begin{cases} u & \mapsto & (u, 0) \\ v & \mapsto & (0, v) \\ w & \mapsto & (w, 1-w) \end{cases}$$



(a) Where does the 1-cycle in the image of f come from?

(b) Does complexifying f into a map from the disjoint union of three complex lines into $\mathbb{C}^2$ make any difference?

3. Generalising the previous exercise, consider the map from the disjoint union of $n+2$ copies of $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$, mapping the j'th copy of $\mathbb{R}^n$ to the coordinate plane $\{x_j = 0\}$ for $j = 1, \ldots, n+1$ and mapping the last copy of $\mathbb{R}^n$ by

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, (1 - \sum_i x_i)).$$

The image, Y, is the boundary of an $n+1$ simplex, and topologically a sphere. Where does the n- cycle generating $H_n(Y)$ come from?

6.5.1 Towards the ICSS

Our first derivation of the image computing spectral sequence uses sheaf cohomology. However, there is a
Preliminary step: We compare the sheaf of germs of locally constant real valued functions on Y, $\mathbb{R}_Y$, with the corresponding sheaves on the $D^k(f)$, $\mathbb{R}_{D^k(f)}$. Denote by f_k the natural evaluation map $D^k(f) \to Y$, $f_k(x_1, \ldots, x_k) = f(x_1)$. The symmetric group S_k acts on $D^k(f)$, permuting the copies of X, and this action extends to an action on $f_{k*}(\mathbb{R}_{D^k(f)})$. Recall that for an open set $U \subset Y$, $f_*^{(k)}(\mathbb{R}_{D^k})(U)$ is by definition equal to $\Gamma(f_k^{-1}(U), \mathbb{R}_{D^k(f)})$. Because f_k is S_k-invariant, $f_k^{-1}(U)$ is mapped to itself by the permutation action, and so S_k acts on $\Gamma(f_k^{-1}(U), \mathbb{R}_{D^k(f)})$, by $\sigma \cdot s = \sigma^{-1*}(s) := s \circ \sigma^{-1}$. For any representation V of S_k, we define

$$\mathrm{Alt}_k(V) = \{v \in V : \sigma \cdot v = \mathrm{sign}(\sigma)v \text{ for all } \sigma \in S_k\} \tag{6.5.3}$$

Then $\mathrm{Alt}_k f_{k*}(\mathbb{R}_{D^k(f)})$ is the sheaf on Y obtained by the standard limiting procedure from the presheaf of sections just defined:

$$\mathrm{Alt}_k f_{k*}(\mathbb{R}_{D^k(f)})_y = \lim_{U \searrow y} \mathrm{Alt}_k \Gamma(f_k^{-1}(U), \mathbb{R}_{D^k(f)}). \tag{6.5.4}$$

Observe that composition with $\mathbf{e}^{k+1,i} : D^{k+1}(f) \to D^k(f)$ gives rise to a map

$$\mathbf{e}^{k+1,i*} : \Gamma(f_k^{-1}(U), \mathbb{R}_{D^k(f)}) \to \Gamma(f_{k+1}^{-1}(U), \mathbb{R}_{D^{k+1}(f)});$$

we define $\delta_k : \Gamma(f_k^{-1}(U), \mathbb{R}_{D^k(f)}) \to \Gamma(f_{k+1}^{-1}(U), \mathbb{R}_{D^{k+1}(f)})$ by

$$\delta_k = \sum_{i=1}^{k+1} (-1)^{i-1} \big(\mathbf{e}^{k+1,i}\big)^*. \tag{6.5.5}$$

Lemma 6.5.6. *If $s \in Alt_k \Gamma\big(f_k^{-1}(U), \mathbb{R}_{D_k(f)}\big)$ then $\delta_k(s) \in Alt_{k+1} \Gamma\big(f_{k+1}^{-1}(U), \mathbb{R}_{D^{k+1}(f)}\big)$.*

Proof. Let (i,j) be the permutation interchanging i and j and assume $i < j$. Then

$$(i,j)^*\big(\delta_k(s)\big)(x_1, \ldots, x_{k+1}) =$$

$$(-1)^{j-1}s(x_1,\ldots,x_j,\ldots,\widehat{x_i},\ldots,x_{k+1})+(-1)^{i-1}s(x_1,\ldots,\widehat{x_j},\ldots,x_i,\ldots,x_{k+1})+$$
$$+\sum_{\ell\neq i,j}(-1)^{\ell-1}s(x_1,\ldots,x_j,\ldots,\widehat{x_\ell},\ldots,x_i,\ldots,x_{k+1}). \tag{6.5.6}$$

In the first summand, x_j is in the i'th place of the argument and x_i is missing from the j'th. The argument in standard order, now with x_i mising fom the i'th place, is obtained from this one by $|i-j-1|$ transpositions. In the second, the roles of i and j are reversed and again $|i-j-1|$ transpositions put the argument in the standard order. In each of the summands in the third term on the right hand side of (6.5.6), just one transposition is required to place the argument in standard order. Because s is itself alternating, the effect of all of these transpositions is to multiply the first two summands by $(-1)^{i-j-1}$ and the remainder by -1. Combining these powers of -1 with those in (6.5.6), we see that $(i,j)^*(\delta_k(s))=-\delta_k(s)$. Since transpositions generate S_{k+1}, the result follows. $\square$

It is easy to check that $\delta_{k+1}\circ\delta_k=0$. Thus the sequence

$$0\to\mathbb{R}_Y\to f_*\mathbb{R}_X\to \mathrm{Alt}_2 f_{2*}(\mathbb{R}_{D^2(f)})\to \mathrm{Alt}_3 f_{3*}(\mathbb{R}_{D^3(f)})\to\cdots \tag{6.5.7}$$

is a complex. For brevity we denote it by $\mathbf{Alt}f_*\mathbb{R}_{D^\bullet}$.

Example 6.5.7. Consider the bi-germ f of Example 6.0.4, consisting of two mono-germs, one an immersion and the other a cross-cap, with images meeting along a curve which passes through the centre of the cross-cap, 0 (the image of the unique non-immersive point). Denote by a and b the origins of coordinates in the domains of the immersion and of the cross-cap, respectively. Then

$$\begin{aligned} f^{-1}(0) &= \{a,b\}\\ f_2^{-1}(0) &= \{(a,b),(b,a),(b,b)\}\subset D^2(f)\\ f_3^{-1}(0) &= \{(a,b,b),(b,a,b),(b,b,a)\}\subset D^3(f)\end{aligned}$$

For $(x_1,x_2)\in f_2^{-1}(0)$ let $\chi_{(x_1,x_2)}$ be the germ of locally constant function taking the value 1 at (x_1,x_2) and 0 at all other points of $f_2^{-1}(0)$, and for $(x_1,x_2,x_3)\in f_3^{-1}(0)$ define $\chi_{(x_1,x_2,x_3)}$ analogously. We have

$$\begin{aligned} f_{2,*}\mathbb{R}_{D^2,0} &= \mathbb{R}\langle\chi_{(a,b)},\chi_{(b,a)},\chi_{(b,b)}\rangle\\ \mathrm{Alt}\,f_{2,*}\mathbb{R}_{D^2,0} &= \mathbb{R}\langle\chi_{(a,b)}-\chi_{(b,a)}\rangle\\ f_{3*}\mathbb{R}_{D^3,0} &= \mathbb{R}\langle\chi_{(a,b,b)},\chi_{(b,a,b)},\chi_{(b,b,a)}\rangle\\ \mathrm{Alt}\,f_{3*}\mathbb{R}_{D^3,0} &= 0\end{aligned} \tag{6.5.8}$$

The complexes $f_{\bullet *}\mathbb{R}_{D^\bullet,0}$ and $\mathbf{Alt}f_*\mathbb{R}_{D^\bullet,0}$ are thus isomorphic to

$$0\longrightarrow\mathbb{R}\xrightarrow{\delta_0}\mathbb{R}^2\xrightarrow{\delta_1}\mathbb{R}^3\xrightarrow{\delta_2}\mathbb{R}^3\xrightarrow{\delta_3}0$$

and

$$0\longrightarrow\mathbb{R}\xrightarrow{\delta_0}\mathbb{R}^2\xrightarrow{\delta_1}\mathbb{R}\xrightarrow{\delta_2}0$$

respectively. Clearly for dimensional reasons the first cannot be exact while the second may be – though we have to look closely at the maps δ_k in order to see whether this is really so.

At this point it is interesting to compare these with the complexes obtained from the *idiot's* multiple point spaces $ID^k(f)$ by the analogous procedures, in which we denote by $f_k : ID^k(f) \to Y$ the evaluation map and use the same differential $\delta_k : f_{k*}\mathbb{R}_{ID^k(f)} \to f_{k+1*}\mathbb{R}_{ID^{k+1}(f)}$ as on $f_{k*}\mathbb{R}_{D^k(f)}$. Observe that in $ID^k(f)$, we have

$$\begin{array}{rcl} f^{-1}(0) & = & \{a,b\} \\ f_2^{-1}(0) & = & \{(a,b),(b,a),(b,b),(a,a)\} \subset D^2(f) \\ f_3^{-1}(0) & = & \{(a,b,b),(b,a,b),(b,b,a),(a,a,b),(a,b,a),(b,a,a),(a,a,a),(b,b,b)\} \subset D^3(f) \\ \cdots & = & \cdots \end{array}$$

Thus $f_{\bullet *}\mathbb{R}_{ID^\bullet,0}$ is an infinite complex. On the other hand, $\mathbf{Alt} f_{\bullet *}\mathbb{R}_{ID^\bullet,0}$ coincides with Alt $f_{\bullet *}\mathbb{R}_{D^\bullet,0}$, since an alternating section of $f_{k*}\mathbb{R}_{ID^k,y}$ necessarily vanishes at any point $(x_1,\dots,x_k) \in f_k^{-1}(y)$ where there is an equality $x_i = x_j$ for some $i \neq j$, and $ID^k(f) \smallsetminus D^k(f)$ consists entirely of such points.

Proposition 6.5.8. *If f is finite-to-one and proper,* $\mathbf{Alt} f_*\mathbb{R}_{D^\bullet}$ *is exact.*

Proof. Suppose $y \in Y$ has exactly $m+1$ preimages, $x_0,\dots,x_m$. We prove exactness of the complex of stalks $\mathbf{Alt}\mathbb{R}_{\mathbf{D}^\bullet,y}$ by showing that it is isomorphic to the simplicial cochain complex $C^\bullet(\Delta^m;\mathbb{R})$ of the standard m-simplex Δ^m. Since Δ^m is contractible, this complex is acyclic, with H^0 equal to $\mathbb{R}$.

To lighten the notation we write D^k in place of $D^k(f)$, and in keeping with our notation in the previous section, we write $X = D^1$ and $Y = D^0$.

Since f is finite and proper,

$$f_{k*}(\mathbb{R}_{D^k})_y = \bigoplus_{x \in (f_k)^{-1}(y)} \mathbb{R}_{D^k,x}.$$

Let $x = (x_{i_1},\dots,x_{i_k}) \in f_k^{-1}(y)$ (we do not suppose all the i_j are distinct here) and let $\chi_{(x_{i_1},\dots,x_{i_k})}$ denote the germ of locally constant function on D^k which takes the value 1 at $(x_{i_1},\dots,x_{i_k})$ and the value 0 at all other points $x' \in f_k^{-1}(y)$. Define

$$\mathrm{Alt}\chi_{(x_{i_1},\dots,x_{i_k})} = \sum_{\sigma \in S_k} \mathrm{sign}(\sigma)\chi_{(x_{i_{\sigma(1)}},\dots,x_{i_{\sigma(k)}})};$$

it lies in $\mathrm{Alt}_k f_{k*}(\mathbb{R}_{D^k})$. If $i_j = i_\ell$ for any $j \neq \ell$, then Alt $\chi_{(x_{i_1},\dots,x_{i_k})} = 0$. As basis for $\mathrm{Alt}_k f_{k*}(\mathbb{R}_{D^k})_y$ we can take the collection $\mathrm{Alt}\chi_{(x_{i_1},\dots,x_{i_k})}$ with $0 \le i_1 < \cdots < i_k \le m$. Notice that this construction effectively deletes multiple points $(x_{i_1},\dots,x_{i_k})$ in which there is a repetition.

Let $v_0,\dots,v_m$ be the vertices of the standard m-simplex Δ^m, and for $0 \le i_1 < \cdots < i_k \le m$, let $(v_{i_1},\dots,v_{i_m})$ be the $k-1$ face with vertices $v_{i_1},\dots,v_{i_k}$. These faces together give the standard triangulation of Δ^m. Let $\xi_{(v_{i_1},\dots,v_{i_k})}$ be the simplicial $(k-1)$-cochain in $C^m(\Delta^m;\mathbb{R})$ (with respect to this triangulation), which takes the value 1 on $(v_{i_1},\dots,v_{i_k})$, and 0 on all other $(k-1)$-faces. These cochains form a basis for $C^{k-1}(\Delta^m;\mathbb{R})$. Denote by ∂^{k-1} the coboundary operator $C^{k-1}(\Delta^m;\mathbb{R}) \to C^k(D^m;\mathbb{R})$ Define a map of complexes $\varphi^\bullet : \mathbf{Alt} f_*\mathbb{R}_{D^\bullet,y} \to C^\bullet(\Delta^m;\mathbb{R})$ by taking, as $\varphi^{(k-1)} : (\mathrm{Alt} f_{k*}\mathbb{R}_{D^k})_y \to C^{k-1}(\Delta^m;\mathbb{R})$ the map sending $\mathrm{Alt}\chi_{(x_{i_1},\dots,x_{i_k})}$ to $\xi_{(v_{i_1},\dots,v_{i_k})}$. Clearly $\varphi^{(k-1)}$ is an isomorphism for all k, and we have to check only that $\partial^{k-1} \circ \varphi^{(k-1)} = \varphi^k \circ \delta_k$. It's enough to evaluate both sides on a basis element $\chi_{(x_{i_1},\ldots,x_{i_k})}$. Denote the $k+1$-tuple $(j_1,\ldots,j_{k+1})$ by J and the $k+1$-tuple $(x_{j_1},\ldots,x_{j_{k+1}})$ by x_J. Since the *Alt* χ_{x_J}, for strictly increasing sequences J, form a basis for Alt $f_{k+1*}\mathbb{R}_{D^{k+1}(f),y}$, we have

$$\varphi^{(k)}\big((\delta_k(\text{Alt}\,\chi_{(x_{i_1},\ldots,x_{i_k})})\big) = \sum_{\substack{x_J\in f_{k+1}^{-1}(y)\\ J\text{ increasing}}} \lambda_J \text{Alt}\,\chi_{x_J} \tag{6.5.9}$$

for some values λ_J which we have to determine. Clearly $\lambda_J = 0$ unless there is a (necessarily unique) ℓ such that $(i_1,\ldots,i_k) = (j_1,\ldots,\widehat{j_\ell}\ldots,j_{k+1})$, and in this case $(x_{i_1},\ldots,x_{i_k}) = \mathbf{e}^{k+1,\ell}(x_{j_1},\ldots,x_{j_{k+1}})$ and

$$\delta_k \text{Alt}\,\chi_{(x_{i_1},\ldots,x_{i_k})}(x_{j_1},\ldots,x_{j_{k+1}}) = (-1)^{\ell-1}.$$

The value of the right hand side of (6.5.9) at x_J is λ_J, so we conclude

$$\varphi^{(k)}\big((\delta_k(\text{Alt}\,\chi_{(x_{i_1},\ldots,x_{i_k})})\big) = \sum_{\substack{x_I=\mathbf{e}^{k+1,\ell}(x_J)\\ J\text{ increasing}}} (-1)^{\ell-1}\text{Alt}\,\chi_{x_J}. \tag{6.5.10}$$

On the other hand, $\partial^{k-1}\varphi^{k-1}\text{Alt}\,\chi_{(x_{i_1},\ldots,x_{i_k})} = \partial^{k-1}(\xi_{(v_{i_1},\ldots,v_{i_k})})$ satisfies an identical formula, since

$$\partial^{k-1}\xi_{(v_{i_1},\ldots,v_{i_k})} = \sum_{\substack{I=\mathbf{e}^{k+1,\ell}(J)\\ J\text{ increasing}}} (-1)^{\ell-1}\xi_{v_J}. \tag{6.5.11}$$

This proves the isomorphism of complexes Alt$f_*\mathbb{R}_{D^\bullet,y} \simeq C^\bullet(\Delta^m;\mathbb{R})$, and we conclude that Alt$f_*\mathbb{R}_{D^\bullet,y}$ is exact, as required. □

6.5.2 A short rehearsal

Another theorem much more widely known than the ICSS can be obtained by an argument which begins just as ours does. Suppose that X is a space on which a finite group G acts, and let X/G denote the quotient topological space. For any vector space on which G acts, let V^G denote the invariant subspace.

Theorem 6.5.9. *In these circumstances, $H^k(X/G;\mathbb{R}) = H^k(X;\mathbb{R})^G$.*

The proof begins exactly as does the one we are embarked for the construction of the ICSS. We compare the sheaf $\mathbb{R}_Y$ of germs of locally constant functions on X/G with the push-forward of the sheaf $\mathbb{R}_X$ by the quotient map $q : X \to X/G$. Once the relation between the two is understood, the rest follows by standard means. It is worth seeing how this goes in this case, since in one important respect it is simpler than the remainder of the construction of the ICSS: it does not need a spectral sequence.

Lemma 6.5.10. *There is an isomorphism $\mathbb{R}_{X/G} \to (q_*\mathbb{R}_X)^G$.*

Proof. Just as in the previous case, for U open in X/G, G acts naturally on $\Gamma(q^{-1}(U), \mathbb{R}_X)$: for $g \in G$ and $s \in \Gamma(q^{-1}(U), \mathbb{R}_X)$, $g \cdot s(x) = s(g^{-1}.x)$. If $\bar{s} \in \Gamma(U, \mathbb{R}_{X/G})$, then composition with q gives rise to a G-invariant section $q^*(\bar{s}) \in \Gamma(q^{-1}(U), \mathbb{R}_X)$, and conversely if $s \in \Gamma(q^{-1}(U), \mathbb{R}_X)$ is G-invariant then s passes to the quotient to define a locally constant function on U. Thus $q^* : \Gamma(U, \mathbb{R}_{X/G}) \to \Gamma(q^{-1}(U), \mathbb{R}_X)^G$ is an isomorphism. Taking limits over open sets, we arrive at an isomorphism $\mathbb{R}_{X/G} \to \big(q_* \mathbb{R}_X\big)^G$. □

Thus, in this case the long exact sequence (6.5.7) is replaced by the very short exact sequence

$$0 \to \mathbb{R}_{X/G} \to \big(q_* \mathbb{R}_X\big)^G \to 0. \tag{6.5.12}$$

The next step is to apply a similar procedure (push forward and take the invariant part) to a suitable resolution of the sheaf $\mathbb{R}_X$. If X is smooth, we can use the Poincaré Lemma:

$$0 \to \mathbb{R}_X \to \Omega^0_X \to \Omega^1_X \to \cdots$$

is exact. Even if X is not smooth, a suitable resolution exists, though getting one's hands on one may be hard. The crucial point is that there should be a sequence of sheaves on X,

$$0 \to \mathscr{S}^0 \to \mathscr{S}^1 \to \cdots \tag{6.5.13}$$

such that

1. (6.5.13) is exact except that the kernel of $\mathscr{S}^0 \to \mathscr{S}^1$ is isomorphic to $\mathbb{R}_X$, and
2. the q'th cohomology of X with coefficients in each of the sheaves $\mathscr{S}^i$ is equal to 0.

Property (2) is guaranteed if all the $\mathscr{S}^i$ are *injective*. Godement in [God73] describes a universal procedure for constructing an injective resolution. The objects he constructs are essentially impossible to see, but all that one really needs to know is that they exist, and have some rather general properties.

The argument continues as follows: by general nonsense, as in our discussion of de Rham cohomology, $H^q(X; \mathbb{R})$ is equal to the q'th cohomology of the complex of global sections

$$0 \to \Gamma(X, \mathscr{S}^0) \to \Gamma(X, \mathscr{S}^1) \to \cdots \tag{6.5.14}$$

Because q is a finite and proper map, the sheaves $q_* \mathscr{S}^j$ are still injective (finite push-forward preserves injectivity), and the complex

$$0 \to q_* \mathbb{R}_X \to q_* \mathscr{S}^0 \to q_* \mathscr{S}^1 \to \cdots \tag{6.5.15}$$

is still exact (finite push forward preserves exactness). Now comes the next significant step in the argument: we split off the invariant part of each of the sheaves $q_* \mathscr{S}^j$ to get a subcomplex

$$0 \to \big(q_* \mathbb{R}_X\big)^G \to \big(q_* \mathscr{S}^0\big)^G \to \big(q_* \mathscr{S}^1\big)^G \to \cdots \tag{6.5.16}$$

That this makes sense at all requires

1. that G act on each of the pushed-forward sheaves $q_* \mathscr{S}^j$

2. that each differential map invariant sections to invariant sections. In fact we need slightly more:

3. Each of the differentials must commute with the group action.

We will refer to these properties together as "naturality" of the resolution (6.5.13). Both are clear if $\mathscr{S}^j = \Omega^j_X$ and if G acts smoothly: there is a natural operation of pull back so that a diffeomorphism $\varphi : U \to V$ gives rise to an isomorphism $\varphi^* : \Omega^j(V) \to \Omega^j(U)$, and in particular an action of G by diffeomorphisms on $q^{-1}(U)$ induces a representation of G on $\Omega^j(q^{-1}(U))$. Moreover the exterior derivative commutes with pull-back, so d maps invariant sections to invariant sections. This issue is discussed at a more abstract level in Godement's book [God73]; the injective resolution Godement constructs has both of the required properties.

Lemma 6.5.11. (6.5.16) *is exact.*

Proof. The crucial ingredient in the proof is the existence of a projection operator

$$A : q_*\mathscr{S}^j \to \big(q_*\mathscr{S}^j\big)^G;$$

that is, a G-linear operator whose restriction to $\big(q_*\mathscr{S}^j\big)^G$ is the identity. Here such an operator A can be defined by averaging over the group:

$$A(s) = \frac{1}{|G|}\sum_{g\in G} g^*(s).$$

Clearly $A(s)$ is invariant, for

$$g_1^*(A(s)) = \frac{1}{|G|}\sum_{g\in G} g_1^*(g^*(s)) = \frac{1}{|G|}\sum_{g\in G}(gg_1)^*(s);$$

since $\{gg_1 : g \in G\} = G$, $g_1^*(A(s)) = A(s)$. The same argument also shows that if s is G-invariant then $A(s) = s$.

Because each of the differentials in (6.5.16) is simply the restriction of the corresponding differential in (6.5.15), (6.5.16) is a complex. All that is required is to show that the kernel of each differential is contained in the image of the preceding differential. Suppose that $s_j \in \big(q_*\mathscr{S}^j\big)^G_y$ is in the kernel of d_j. By exactness of (6.5.15), there exists $s_{j-1} \in q_*\mathscr{S}^{j-1}_y$ such that $d_{j-1}s_{j-1} = s_j$. Now apply A. Because the group action commutes with the differentials, and the differentials are linear, we have

$$d_{j-1}A(s_{j-1}) = A\big(d_{j-1}(s_{j-1})\big) = A(s_j),$$

and because s_j is G-invariant, $A(s_j) = s_j$. This completes the proof. □

Now because of the isomorphism (6.5.12), (6.5.16) gives us an injective resolution of $\mathbb{R}_{X/G}$. General theory tells us that $H^q(X/G;\mathbb{R})$ can be computed as the q'th cohomology of the complex of global sections

$$0 \to \Gamma(X/G, \big(q_*\mathscr{S}^0\big)^G \to \Gamma(X/G, \big(q_*\mathscr{S}^1\big)^G \to \cdots. \tag{6.5.17}$$

But this complex is just the same as

$$0 \to \Gamma(X, \mathscr{S}^0)^G \to \Gamma(X, \mathscr{S}^1)^G \to \cdots \tag{6.5.18}$$

and, finally, the cohomology of this complex is the invariant part of the cohomology of the complex

$$0 \to \Gamma(X, \mathscr{S}^0) \to \Gamma(X, \mathscr{S}^1) \to \cdots. \tag{6.5.19}$$

The q'th cohomology here is just $H^q(X;\mathbb{R})$. So we have proved Theorem 6.5.9.

6.5.3 The Image Computing Spectral Sequence from Sheaf Cohomology

The long exact sequence (6.5.7) relates the constant sheaf on Y to the alternating parts of the push-forward to Y of the constant sheaves on the $D^k(f)$. In the light of Subsection 6.5.2, we should expect that the cohomology of Y with coefficients in $\mathbb{R}_Y$, i.e. $H^*(Y;\mathbb{R})$, is related to the alternating part of the cohomology of the $D^k(f)$. This is exactly what happens. Suppose $\mathscr{S}_k^\bullet$ is an injective resolution of $\mathbb{R}_{D^k(f)}$ with the property of naturality. Then as in Subsection 6.5.2,

$$0 \to \mathrm{Alt}_k\, f_{k*}\mathscr{S}_k^0 \to \mathrm{Alt}_k\, f_{k*}\mathscr{S}_k^1 \to \cdots \tag{6.5.20}$$

is an injective resolution of $\mathrm{Alt}_k\, f_{k*}\mathbb{R}_{D^k(f)}$. Therefore the q'th cohomology of the complex of its global sections is equal to $H^q(Y, \mathrm{Alt}_k\, f_{k*}\mathbb{R}_{D^k(f)})$. As before,

$$H^q(\Gamma(Y, \mathrm{Alt}_k\, f_{k*}\mathscr{S}_k^\bullet)) = H^q\big(\mathrm{Alt}_k\Gamma(D^k(f), \mathscr{S}_k^\bullet)\big) = \mathrm{Alt}_k\, H^q\big(\Gamma(D^k(f), \mathscr{S}_k^\bullet)\big) = \mathrm{Alt}_k\, H^q(D^k(f);\mathbb{R}). \tag{6.5.21}$$

Now we seem to have a large number of cohomology groups, indexed by q and by k, all of which, we expect, are related in some way to the cohomology of Y. What is the exact nature of this relation? The answer is, of course, given by the ICSS. Let us go a stage back and put all the injective resolutions together into a single diagram.

$$\begin{tikzcd}
 & & & {} & {} & {} & {} & \\
2 & & & f_*\mathscr{S}_1^2 \arrow[u, "d_1^2"] \arrow[r, "\delta_1^2"'] & \mathrm{Alt}_2\, f_{2*}\mathscr{S}_2^2 \arrow[u, "d_2^2"] \arrow[r, "\delta_2^2"'] & \mathrm{Alt}_3\, f_{3*}\mathscr{S}_3^2 \arrow[u, "d_3^2"] \arrow[r, "\delta_3^2"'] & \mathrm{Alt}_4\, f_{4*}\mathscr{S}_4^2 \arrow[u, "d_4^2"] \arrow[r] & {} \\
1 & & & f_*\mathscr{S}_1^1 \arrow[u, "d_1^1"] \arrow[r, "\delta_1^1"'] & \mathrm{Alt}_2\, f_{2*}\mathscr{S}_2^1 \arrow[u, "d_2^1"] \arrow[r, "\delta_2^1"'] & \mathrm{Alt}_3\, f_{3*}\mathscr{S}_3^1 \arrow[u, "d_3^1"] \arrow[r, "\delta_3^1"'] & \mathrm{Alt}_4\, f_{4*}\mathscr{S}_4^1 \arrow[u, "d_4^1"] \arrow[r] & {} \\
0 & & & f_*\mathscr{S}_1^0 \arrow[u, "d_1^0"] \arrow[r, "\delta_1^0"'] & \mathrm{Alt}_2\, f_{2*}\mathscr{S}_2^0 \arrow[u, "d_2^0"] \arrow[r, "\delta_2^0"'] & \mathrm{Alt}_3\, f_{3*}\mathscr{S}_3^0 \arrow[u, "d_3^0"] \arrow[r, "\delta_3^0"'] & \mathrm{Alt}_4\, f_{4*}\mathscr{S}_4^0 \arrow[u, "d_4^0"] \arrow[r] & {} \\
-1 & 0 \arrow[r] & \mathbb{R}_Y \arrow[r] & f_*\mathbb{R}_X \arrow[u, "i_1"] \arrow[r, "\delta_1"'] & \mathrm{Alt}_2\, f_{2*}\mathbb{R}_{D^2} \arrow[u, "i_2"] \arrow[r, "\delta_2"'] & \mathrm{Alt}_3\, f_{3*}\mathbb{R}_{D^3} \arrow[u, "i_3"] \arrow[r, "\delta_3"'] & \mathrm{Alt}_4\, f_{4*}\mathbb{R}_{D^4} \arrow[u, "i_4"] \arrow[r, "\delta_4"'] & {} \\
 & & & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \\
 & & & 0 & 1 & 2 & 3 &
\end{tikzcd} \tag{6.5.22}$$

Here the columns are the resolutions $\mathrm{Alt}_k\, f_{k*}\mathscr{S}_k^\bullet$ shown in (6.5.20), with i_k the inclusion of $\mathrm{Alt}_k\, f_{k*}\mathbb{R}_{D^k}$ as kernel of the arrow from $\mathrm{Alt}_k\, f_{k*}\mathscr{S}_k^0$. Injectivity of the sheaves $\mathrm{Alt}_k\, f_{k*}\mathscr{S}_k^j$ means that δ_k :

$\mathrm{Alt}_k\, f_{k*}\mathbb{R}_{D^k} \to \mathrm{Alt}_{k+1}\, f_{k+1*}\mathbb{R}_{D^{k+1}}$ lifts to a morphism of complexes $\mathrm{Alt}_k\, f_{k*}\mathscr{S}_k^\bullet \to \mathrm{Alt}_{k+1}\, f_{k+1*}\mathscr{S}_{k+1}^\bullet$, that is, to horizontal arrows making the diagram commutative.

Lemma 6.5.12. [?, Théorème 5.1.3]. *It is possible to choose the injective resolutions $\mathcal{S}_i^\bullet$ and the lifts $Alt_k\, f_{k*}\mathscr{S}_k^\bullet \to Alt_{k+1}\, f_{k+1*}\mathscr{S}_{k+1}^\bullet$ so that the rows of the above diagram are complexes – that is, so that $\delta_{i+1}^j \circ \delta_i^j = 0$ for all $i \geq 1$ and $j \geq 0$.*

Now we modify the diagram: we multiply every one of the horizontal arrows in the even rows (the rows containing $\mathrm{Alt}_k\, f_{k*}\mathscr{S}_k^j$ for j even) by -1. Denote by δ_k^j the horizontal arrow from $\mathrm{Alt}_k\, f_{k*}\mathscr{S}_k^j$ obtained in this way, and denote by d_k^j the vertical arrow. The rows (and columns) are still complexes, but now, every small square in the diagram is anticommutative:

$$d_{k+1}^j \delta_k^j + \delta_{k+1}^j d_k^j = 0 \tag{6.5.23}$$

An array of objects and maps with these three properties is called a *double complex*. From this double complex we obtain a single complex, the so-called *total complex* $K^\bullet$ of the double complex, which is an injective resolution of $\mathbb{R}_Y$. The procedure is as follows. As ℓ'th object in the total complex, we take the direct sum of all the injective sheaves along the line joining locations $(0, \ell)$ and $(\ell, 0)$. Because the numbering of the D^k is shifted by 1 from the numbering of the columns – so that D^{k+1} appears in column k – the formula for this sum has a slightly uncomfortable asymmetry:

$$K^\ell = \bigoplus_{p+q=\ell} \mathrm{Alt}_{p+1}\, f_{p+1*}\mathscr{S}_{p+1}^q. \tag{6.5.24}$$

The arrows in the diagram lead to arrows $d_\ell : K^\ell \to K^{\ell+1}$: the summand $\mathrm{Alt}_{p+1}\, f_{p+1*}\mathscr{S}_{p+1}^q$ in K^{p+q} is mapped to the two summands $\mathrm{Alt}_{p+2}\, f_{p+2*}\mathscr{S}_{p+2}^q$ and $\mathrm{Alt}_{p+1}\, f_{p+1*}\mathscr{S}_{p+1}^{q+1}$ in K^{p+q+1} by the two maps δ_{p+1}^q and d_{p+1}^q, and this extends by linearity to define d_ℓ. The fact that the rows and columns of the double complex are complexes, together with the anticommutation relation (6.5.23), mean that $d_{\ell+1} \circ d\ell = 0$, so $K^\bullet$ is a complex. In fact more is true:

Lemma 6.5.13. *$K^\bullet$ is an injective resolution of $\mathbb{R}_Y$.*

Proof. As a direct sum of injective sheaves on Y, each sheaf K^ℓ is injective. It remains to show that

1. the kernel of $d_0 : K^0 \to K^1$ is isomorphic to $\mathbb{R}_Y$, and
2. $K^\bullet$ is exact at K^ℓ for $\ell > 0$.

The first of these is easy: the horizontal arrow from $\mathbb{R}_Y$ followed by a vertical arrow defines an injective map into $f_*\mathcal{S}_1^0$, and it is clear that both δ_1^0 and d_1^0 are zero on the image, which is therefore contained in $\ker d_0$. Conversely, $s \in \ker d_0$ if and only if $d_1^0 s = 0$ and $\delta_1^0 s = 0$. The first of these means that $s = i_0(s')$ for some $s' \in f_*\mathbb{R}_X$, by exactness of column 0. Then anticommutativity of the bottom left square means that $i_1(\delta_1(s')) = 0$ in $\mathrm{Alt}_2 f_{2*}\mathbb{R}_{D^2}$, and this forces $\delta_1(s') = 0$. Now exactness of row -1 means that s' comes from $s'' \in \mathbb{R}_Y$.

The argument for the second is scarcely more difficult, especially if carried out with an eye on the diagram on the previous page. Suppose that $s := (s_0^k, s_1^{k-1}, \ldots, s_k^0) \in \ker d_k$. Then the vertical differential of the first component must vanish. That is, $d_0^k(s_0^k) = 0$ in $f_*\mathscr{S}_0^{k+1}$. Exactness of column 0 means that $s_0^k = d_0^{k-1}(s_0^{k-1})$ for some s_0^{k-1}. So modulo $d_{k-1}(f_*\mathscr{S}_0^{k-1})$, s is equal to a

$(k+1)$-tuple s' whose first component is 0. In making this replacement, the second component, s_1^{k-1} is replaced by $s_1^{k-1} \pm \delta_0^{k-1}(s_0^{k-1})$, but there is no need to keep track of this. Since $K^\bullet$ is a complex, $s' = s - d_{k-1}s_0^{k-1}$ is still in $\ker d_k$. Because the first component in s' is 0, the vertical differential of its *second* component must vanish. Exactness of column 1 means there is a s_1^{k-2} such that $d_1^{k-1}(s_1^{k-1})$ is the second component of s'. So again modulo the image of d_{k-1}, s', and therefore s, are equal to an element whose first two components are zero. Iterating this argument k times, we conclude that modulo the image of d_{k-1}, s is equal to a $k+1$-tuple of the form $(0, \ldots, 0, s_k^0)$, which as before is still in $\ker d_k$. Now both the vertical and horizontal differentials of s_k^0 must vanish. Because $d_k^0(s_k^0) = 0$, $s_k^0 = i_k(s'')$ for some $s'' \in \mathrm{Alt}_k\, f_{k*}\mathbb{R}_{D^k}$. Then $i_{k+1}\delta_k(s'') = \delta_k^0(s_k^0) = 0$. Hence $\delta_k(s'') = 0$. Now by exactness of the (-1)'st row of the diagram, $s'' = \delta_{k-1}s'''$ for some $s''' \in \mathrm{Alt}_{k-1}\, f_{k-1*}\mathbb{R}_{D^{k-1}}$, and $s_k^0 = \delta_{k-1}^0(i_{k-1}(s''')) = d_{k-1}(i_{k-1}(s'''))$. Thus our original section s lies in the image of d_{k-1}. □

By general nonsense, it follows from 6.5.13 that $H^q(Y;\mathbb{R})$ is the q'th cohomology of the complex of global sections of $K^\bullet$.

Key question: how do we compute this cohomology, and how do we relate it to the alternating cohomology of the $D^k(f)$?

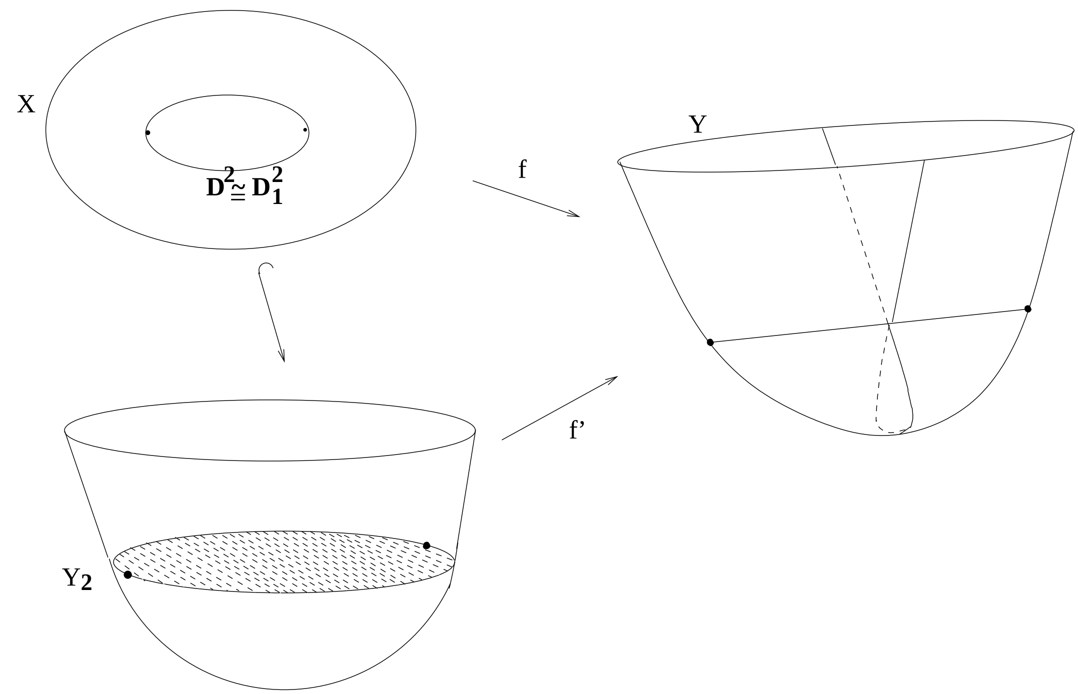
6.6 The ICSS from Vassiliev's geometric realisation

One of the mysteries remaining from the last two sections is the appearance of *alternating* homology. The image Y is filtered by the spaces $M_k := \{y \in Y : |f^{-1}(y)| \geq k\}$, where we count each preimage point with suitable multiplicity, so that M_k is the image of $D^k(f)$ under the natural map induced by f. At first sight M_k looks rather like the quotient of $D^k(f)$ by the S_k action, so one might expect, in line with e.g. Theorem 6.5.9, that $H_*(M_k)$ is perhaps the S_k-invariant part of $H_*(D^k(f))$, and that in the course of using the spectral sequence for the homology of a filtered space see [McC01], though we give a brief introduction below) to compute $H_*(Y)$, one would find oneself using these S_k invariant parts of the homology of $D^k(f)$. Instead, it is the alternating (co)homology that plays the key role. In [**?**] Victor Goryunov gave a completely different construction of the ICSS, which explains the appearance of the alternating (co)homology, and leads to a stronger result, valid over $\mathbb{Z}$ and not merely over $\mathbb{Q}$.

Goryunov's approach is more obviously topological than the approach in Section 6.5.3, and begins with a geometric construction due to V. Vassiliev. Let $f : X \twoheadrightarrow Y$ be a map, with Y Hausdorff. Vassiliev constructs a "geometric realisation" of the semi-simplicial object $D^\bullet(f)$ as follows. Let m be the greatest number of distinct preimages of any point $y \in Y$. Choose an embedding of X in some Euclidean space $\mathbb{R}^N$, such that no m distinct points of the image lie in any $m-2$-dimensional affine subspace, and identify X with its image under this embedding. If X is a smooth manifold, such an embedding can be shown to exist by an application of Mather's multi-jet version of the Thom Transversality Theorem (see [GG73]): the set of multi-jets ${}_mj^0(z)$ in $J^0(X,\mathbb{R}^N)$ whose m target points lie in an $(m-2)$-dimensional affine subspace is an algebraic bundle over X whose codimension in ${}_rJ^0(X,\mathbb{R}^N)$, $c_{m,N}$, can easily be calculated. When N is sufficiently great that $c_{m,N} > m\,\mathrm{dim}X$, the multi-jet transversality theorem ensures that the set of embeddings of X in $\mathbb{R}^N$ with the requisite property is non-empty.

If $f(x_0) = \cdots = f(x_k) = y \in Y$, let $\Delta_{(x_0,\ldots,x_k)}$ be the standard k-simplex in $\{y\} \times \mathbb{R}^N$ with vertices $(y, x_0), \ldots, (y, x_k)$. We define $Y_k \subset Y \times \mathbb{R}^N$ to be the union of all such simplices of dimension $\leq k-1$. Then Y_1 is the graph of f, and so homeomorphic to X, Y_2 is homeomorphic to the union of X together with all the segments joining pairs of points with the same image under f, and so on. We let $Y' = \cup_k Y_k$. The assumption about the embedding guarantees that each of the k-simplices we have glued in is non-degenerate (is not contained in any $k-2$-plane).

This definition has its subtleties. The construction is not simply that for each k-tuple of points $x_1, \ldots, x_k$ with the same image, we glue an abstract $(k-1)$-simplex to X with vertices at $x_1, \ldots, x_k$, in the sense of an abstract identification space. This would leave the interiors of the different simplices separated from one another by X. By using the affine structure of $\mathbb{R}^N$ we ensure, for example, that where $D^2(f)$ has dimension 1, the 1-simplices we glue in together form a surface, as shown in the following drawing.



Here X is the disc shown, and $f : X \to Y$ is a stable perturbation of the S_1^+ singularity, with equation

$$f(x, y) = (x, y^2, y^3 + x^2 y - ty)$$

for small positive t. There are two Whitney umbrellas on the image, joined by a line of double-points. They, and their preimages in X and in Y_1, are indicated by black dots. We show $D^2_1(f)$ as a small circle contained in X; because $D^3(f) = \emptyset$, $D^2(f)$ is isomorphic to its projection $D^2_1(f)$.

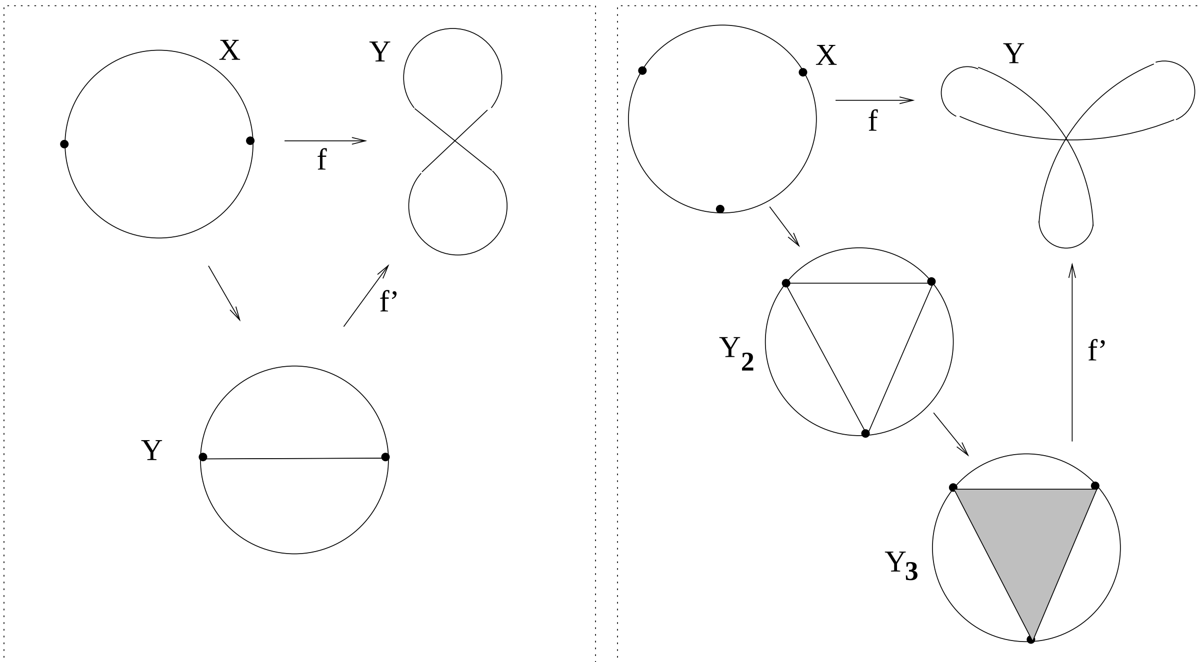
The involution on $D^2(f)$ is reflection in the line joining the two Whitney umbrella points. Goryunov constructs Y' in $Y \times \mathbb{R}^N$, which we cannot draw, of course. Fortunately in this example it is possible to embed $Y' = Y_2$ in $\mathbb{R}^3$, as shown. We have drawn the 1-simplices making up $Y_2 \smallsetminus Y_1$ as dashed lines. Together they form a disc.

The projection $Y \times \mathbb{R}^N \to Y$ induces a natural map $f' : Y' \to Y$. Observe that if $y \in Y$ with $f^{-1}(y) = \{x_1, \ldots, x_k\}$ then $(f')^{-1}(y)$ is a $(k-1)$ simplex in $\{y\} \times \mathbb{R}^N$. That is, every point in y

has, as preimage in Y', a simplex of some dimension. Contracting each of the simplices to a point reduces Y' to Y; since contracting a (reasonably embedded) contractible subspace does not alter the homotopy type, we have

Lemma 6.6.1. [Vas01, Lemma 1 page 86] *$f' : Y' \to Y$ is a homotopy-equivalence.* □

We do not prove this lemma here. [1].



In the second figure here, gluing in segments joining the three points in X with the same image creates some unwanted homotopy, that of the triangle in Y_2. This is killed by gluing in the 2-simplex to create Y_3, which is homotopy-equivalent to Y.

Vassiliev's geometric realisation gives us a natural increasing filtration on the space Y'. From any filtration on a topological space, one obtains a spectral sequence which, under favourable circumstances converges, to the homology of the space. We will shortly give a sketch of this. Goryunov explained the appearance of the alternating cohomology of the multiple point spaces in the cohomology of the image, by showing that the E^1 term of this spectral sequence is isomorphic to the alternating homology of $D^k(f)$. We will spend the rest of this chapter showing this.

6.6.1 Background on the spectral sequence for the homology of a filtered space

Any filtration $Z_1 \subset Z_2 \subset \ldots \subset Z_m = Z$ on a space Z gives rise to a spectral sequence in homology, with $E^1_{p,q} = H_{p+q}(Z_p, Z_{p-1})$, which converges to $H_{p+q}(Z)$. This is in fact part of a more general statement about filtered complexes. The filtration on Z gives rise in an obvious way to a filtration

[1]Vassiliev's proof is criticised in [?], where instead it is shown that f' induces an isomorphism on homology

on $C_\bullet(Z;\mathbb{Z})$, whose p'th term is the image in $C_\bullet(Z)$ of $C_\bullet(Z_p)$, and the spectral sequence of the filtration on Z is a special case of the more general *spectral sequence arising from a filtered complex.* Let us briefly describe this. To tally with our later exposition of Goryunov's work, we describe only the case of a filtered complex of homological type, where the degree of the differential is -1, and we assume the filtration is increasing, i.e. $F_p \subset F_{p+1}$, exhaustive and finite: there exists a finite p such that for all q, $F_pR_q = R_q$.

Consider a complex $R_\bullet$ with differential d of degree -1. Suppose that $R_\bullet$ is equipped with a filtration $F^\bullet$ as described, and that the differential d respects the filtration – that is, for all p, q, d maps F^pR_q to F^pR_{q-1}. Then $F_pR_\bullet$ is a subcomplex, whose homology we denote by $H_*(F_pR_\bullet)$. The map of complexes

$$F_pR_\bullet \to R_\bullet$$

induces a morphism of homology

$$H_*(F_pR_\bullet) \to H_*(R_\bullet)$$

and we obtain a filtration on $H_*(R_\bullet)$ by taking

$$F_pH_*(R_\bullet) = \text{image } H_*(F_pR_\bullet) \to H_*(R_\bullet).$$

The fact that d respects the filtration means that it passes to the quotient to define a differential on the p'th graded part of $R_\bullet$:

$$d : F^pR_q/F^{p-1}R_q \to F^pR_{q-1}/F^{p-1}R_{q-1}.$$

What does the homology of this complex tell us about the homology of $R_\bullet$? The answer is that there is a spectral sequence, with $E^1_{p,q} = H_{p+q}(F^pR_\bullet/F^{p-1}R_\bullet)$, which converges to the graded module associated to the filtration on $H^{p+q}(R_\bullet)$. That is,

$$E^\infty_{p,q} \simeq F^pH_{p+q}(R_\bullet)/F^{p-1}H_{p+q}(R_\bullet).$$

We will not describe the construction of the r'th page of the spectral sequence, but limit ourselves to pointing out that there is a natural differential $d_1 : E^1_{p,q} \to E^1_{p-1,q}$ satisfying $d_1 \circ d_1 = 0$, which the reader will easily construct.

Lemma 6.6.2. *Suppose the filtered complex $(R_\bullet, F^\bullet R_\bullet)$ is the chain complex of a filtered space X, with filtration $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X$. Then the E^1 term of the spectral sequence is*

$$E^1_{p,q} = H_{p+q}(X_p, X_{p-1}),$$

and the differential $d_1 : E^1_{p,q} \to E^1_{p-1,q}$ is .

6.6.2 $D^k(f) \times \Delta_{k-1}$ as a covering of (Y_k, Y_{k-1})

Now recall the definition of $D^k_g(f)$ as the closure of the set of ordered k-tuples of pairwise distinct points of X sharing the same image under f. We consider the standard $k-1$ simplex $\Delta_{k-1} = \{(t_1, \ldots, t_k) \in \mathbb{R}^k : t_i \geq 0 \text{ for all i}, \sum_i t_i = 1\}$. Define a map $h_k : D^k(f) \times \Delta_{k-1} \to Y_k$ by

$$h_k\big((x_1, \ldots, x_k), (t_1, \ldots, t_k)\big) = \big(f(x_1), t_1x_1 + \cdots + t_kx_k\big).$$

Let $\operatorname{diag}(D^k(f))$ be the intersection of $D^k(f)$ with the big diagonal (where not all the x_i are distinct). Then h_k maps both $\operatorname{diag}(D^k(f))$ and $D^k(f) \times \partial\,\Delta_{k-1}$ to Y_{k-1}, so we have a map of pairs

$$(D^k(f) \times \Delta_{k-1},) \xrightarrow{h_k} (Y_k, Y_{k-1})\ .$$

Indeed, $h_k^{-1}(Y_{k-1}) = \operatorname{diag}(D^k(f)) \cup D^k(f) \times \partial\,\Delta_{k-1}$. There is a natural action of S_k on $D^k \times \Delta_{k-1}$, the product of the action of S_k on $D^k(f)$ already discussed and the action on Δ_{k-1} permuting the coordinates $t_1, \ldots, t_k$. The map h_k is invariant with respect to this action.

Lemma 6.6.3. *On the complement of $\big(diag(D^k(f)) \times \Delta_{k-1}\big) \cup D^k(f) \times \partial\,\Delta_{k-1}$, h_k is a $k!$-fold covering map.*

Proof. Each point in $(\mathbb{R}^N)^k \smallsetminus \operatorname{diag}(\mathbb{R}^N)^k$ has a covering by open sets U whose image under any non-trivial permutation in S_k is disjoint from U. For such a U, h_k is $1-1$ on $\big(U \cap D^k(f)\big) \times \big(\Delta_{k-1} \smallsetminus \partial\,\Delta_{k-1}\big)$: any equality

$$\big(f(x_1), \sum_j t_j x_j)\big) = \big(f(x_1'), \sum_j t_j' x_j'\big)$$

where $\sum_j t_j = \sum_j t_j' = 1$ and $(x_1, \ldots, x_k) \neq (x_1', \ldots, x_k') \in U \cap D^k(f)$ leads to the conclusion that some ℓ distinct preimages of $y = f(x_1)$ (with $k < \ell \leq 2k$) lie in an $\ell - 2$-dimensional affine subspace of $\mathbb{R}^N$, contradicting our hypothesis about the embedding.

The map $U \times \operatorname{int}\,\Delta_{k-1} \to \mathbb{R}^N$ defined by the same formula as h_k is open. It follows that on $\big(U \cap D^k(f)\big) \times \operatorname{int}\,\Delta_{k-1}$, h_k is a homeomorphism onto its image. The same is true if we replace U by the disjoint set $\sigma(U)$ for $\sigma \in S_k \smallsetminus \{\text{identity}\}$. This replacement does not affect the image. $\square$

We want to calculate $H_*(Y_k, Y_{k-1})$ with integer coefficients. For this we use cellular homology . Take a CW-structure on $D^k(f)$ on which S_k acts by permutation of cells, and with the property that if $\sigma(e) = e$, for some simplex e and $\sigma \in S_k$, then σ leaves e pointwise fixed. Such a decomposition can be obtained from a decomposition with the weaker requirement that cells are mapped pointwise to one another, by means of a subdivision. A proof of the existence of such a decomposition can be found in e.g. [**?**].

6.6.3 Background on cellular homology

A good reference here is [Hat02, Section 2.2]. Let X be a CW complex and for each k let X^k be its k-skeleton. The relative homology group $H_k(X^k, X^{k-1})$ is freely generated by the k-cells of X; for $H_k(X^k, X^{k-1}) \simeq \tilde{H}_k(X^k/X^{k-1})$ and X^k/X^{k-1} is the wedge sum of the k-cells of X with their boundaries contracted to a (common) point. The *cellular chain complex* of X with respect to the decomposition is the complex

$$\cdots \longrightarrow H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \xrightarrow{d_{n-1}} \cdots$$

where each differential d_n is the composite of arrows in the long exact sequences of pairs (X^n, X^{n-1}) and (X^{n-1}, X^{n-2}), as shown in the following diagram.

$$\begin{tikzcd}
& H_{n-1}(X^{n-2}) = 0 \arrow[dr] & & H_{n-1}(X^n) \simeq H_{n-1}(X) & \\
& & H_{n-1}(X^{n-1}) \arrow[ur, "p_n"'] \arrow[dr, "j_{n-1}"] & & \\
\cdots \arrow[r, "d_{n+1}"] & H_n(X^n, X^{n-1}) \arrow[ur, "\partial_n"] \arrow[rr, "d_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "d_{n-1}"] \arrow[dr] & {} \\
H_n(X^n) \arrow[ur] & & & & H_{n-2}(X^{n-2})
\end{tikzcd}$$

Note that $H_k(X^n, X^{n-1}) = 0$ for $k \neq n$ and $H_n(X^{n+1}) = H_n(X)$ (both by [Hat02, Lemma 2.34]). It follows easily ([Hat02, Theorem 2.35]) that the homology of the cellular chain complex is isomorphic to the singular homology of X.

If X is a CW-complex and Y is a subcomplex, the *relative cellular homology* is the homology of the complex

$$\cdots \longrightarrow H_{n+1}(X^{n+1}, X^n \cup Y^{n+1}) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1} \cup Y^n) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2} \cup Y^{n-1}) \xrightarrow{d_{n-1}} \cdots \tag{6.6.1}$$

Note that $X^n / X^{n-1} \cup Y^n$ is homeomorphic to the wedge sum of the n-cells in X and not in Y modulo their boundaries, and so $H^n(X^n, X^{n-1} \cup Y^n)$ is the free abelian group on these n-cells, and $H_k(X^n, X^{n-1} \cup Y^n) = 0$ for $k \neq n$. The arrows in (6.6.1) are constructed as follows. We have

$$H_k(X^{n-1} \cup Y^n, X^{n-1}) = H_k(Y^n, Y^{n-1}) \tag{6.6.2}$$

for all k, for since Y is a subcomplex of X, it follows that $Y^{n-1} = Y^n \cap X^{n-1}$ so inclusion $Y^n \to X^{n-1} \cup Y^n$ induces a homeomorphism

$$Y^n / Y^{n-1} \simeq X^{n-1} \cup Y^n / X^{n-1}.$$

Therefore $H_k(X^{n-1} \cup Y^n, X^{n-1}) = 0$ for $k \neq n$, by [Hat02, Lemma 2.34]. The long exact sequence of the triple $(X^n, X^{n-1} \cup Y^n, X^{n-1})$ therefore collapses to the short exact sequence

$$0 \longrightarrow H_n(X^{n-1} \cup Y^n, X^{n-1}) \longrightarrow H_n(X^n, X^{n-1}) \longrightarrow H_n(X^n, X^{n-1} \cup Y^n) \longrightarrow 0\ , \tag{6.6.3}$$

in which, by (6.6.2), we can replace $H_n(X^{n-1} \cup Y^n, X^{n-1})$ by $H_n(Y^n, Y^{n-1})$. Making this replacement, we obtain a short exact sequence of (vertically displayed) complexes

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \arrow[d] & \\
0 \arrow[r] & H_n(Y^n, Y^{n-1}) \arrow[r] \arrow[d] & H_n(X^n, X^{n-1}) \arrow[r] \arrow[d] & H_n(X^n, Y^n \cup X^{n-1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_{n-1}(Y^{n-1}, Y^{n-2}) \arrow[r] \arrow[d] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] \arrow[d] & H_{n-1}(X^{n-1}, Y^{n-1} \cup X^{n-2}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_{n-2}(Y^{n-2}, Y^{n-3}) \arrow[r] \arrow[d] & H_{n-2}(X^{n-2}, X^{n-3}) \arrow[r] \arrow[d] & H_{n-2}(X^{n-2}, Y^{n-2} \cup X^{n-3}) \arrow[r] \arrow[d] & 0 \\
 & \, & \, & \, &
\end{tikzcd} \tag{6.6.4}$$

in which the left and middle columns are the cellular chain complexes of Y and X respectively, and the arrows in the right hand column are obtained by diagram chasing. The homology of the right hand column is, by definition, the relative cellular homology $H_*(X, Y)$; it is isomorphic to the relative singular homology. The long exact sequence of homology obtained from this short exact sequence of complexes is just the cellular homology version of the long exact sequence of the pair (X, Y).

To avoid unpleasantly long expressions in what follows, it will be convenient sometimes to denote the group $H^\ell(X^\ell, X^{\ell-1} \cup Y^\ell)$ by $C_\ell^{\mathrm{CW}}(X, Y)$.

6.6.4 The alternating homology of $D^k(f)$ is the relative homology of (Y_k, Y_{k-1}).

We return to our discussion of the map of pairs

$$h_p : \Big(D^p(f) \times \Delta_{p-1}, \big(\text{diag } D^p(f) \times \Delta_{p-1}\big) \cup \big(D^p(f) \times \partial\,\Delta_{p-1}\big)\Big) \to (Y_p, Y_{p-1})$$

described in Subsection 6.6.2. Our requirement on the cell decomposition on $D^p(f)$ means that S_p acts on $D^p(f)$ by cellular homeomorphisms, and therefore acts on the cellular chain group $H_n\big((D^p(f))^\ell, (D^p(f))^{\ell-1}\big)$, where, as before, $(D^p(f))^\ell$ means the ℓ-skeleton of $D^p(f)$. We call a cellular chain ℓ-chain c *alternating* if for each $\sigma \in S_p$ we have $\sigma_*(c) = \operatorname{sign}\sigma \cdot c$.

Now give Δ_{p-1} a CW structure with a single $p-1$ cell, and with $(\Delta_{p-1})^{p-2} = \partial\,\Delta_{p-1}$. Then $H_{p-1}((\Delta_{p-1})^{p-1}, (\Delta_{p-1})^{p-2}) = H_{p-1}(\Delta_{p-1}, \partial\,\Delta_{p-1}) \simeq \mathbb{Z}$. The S_p- action on Δ_{p-1} induces the sign representation on $\mathbb{Z} = H_{p-1}(D_{p-1}, \partial D_{p-1})$: for each $\sigma \in S_p$, the degree of the induced map on ∂D_{p-1} is equal to the sign of σ, and thus the same is true for the map $H_{p-1}(\Delta_{p-1}, \partial\,\Delta_{p-1})$. Taking the product of the CW-structures on $D^p(f)$ and on Δ_{p-1} gives a CW structure on $D^p(f) \times \Delta_{p-1}$ on which, once again, S_p acts by cellular homeomorphisms, and in which $\big(\text{diag } D^p(f) \times \Delta_{p-1}\big) \cup \Big(D^p(f) \times \partial\,\Delta_{p-1}\big)\Big)$ is a subcomplex. The CW structure on $D^p(f) \times \Delta_{p-1}$ descends to a CW structure on the pair (Y_p, Y_{p-1}), with each cell of $Y_k \smallsetminus Y_{p-1}$ covered by an S_p-invariant sum of $p!$ cells in $\big(D^p \times \Delta_{p-1}\big) \smallsetminus \Big(\big(\text{diag}\, D^p(f) \times \Delta_{p-1}\big) \cup \big(\Delta^p(f) \times \partial\,\Delta_{p-1}\big)\Big)$.

Lemma 6.6.4. *The relative cellular chain complex $C_\bullet^{CW}(Y_p, Y_{p-1})$ is isomorphic to the complex of S_p-invariant relative chains $C_\bullet^{CW}\Big(D^p \times \Delta_{p-1}, \big(\mathit{diag}\, D^p(f) \times \Delta_{p-1}\big) \cup \big(\Delta^p(f) \times \partial\,\Delta_{p-1}\big)\Big)^{S_p}$.*

Proof. $H_\ell((Y_p)^\ell, (Y_p)^{\ell-1})$ is the free group on ℓ-cells in Y_ℓ which are not cells in $Y_{\ell-1}$. Each such cell c is covered by an S_p–orbit of cells

$$\sum_{\sigma \in S_p} \sigma_*(\tilde{c}), \tag{6.6.5}$$

where $\tilde{c}$ is any ℓ-cell such that $h_{p*}(\tilde{c}) = c$. The sum (6.6.5) is obviously S_p-invariant. Because $h_p \circ \sigma = h_k$, it follows that $h_{p*} \circ \sigma_* = h_{p*}$ and thus $(1/p!)h_{p*}(\sum_{\sigma \in S_p} \sigma_*(\tilde{c})) = c$. Thus the map $(1/p!)h_{p*}$ defines the required isomorphism of relative chain complexes. □

Because the interior of Δ_{p-1} consists of a single $(p-1)$-cell, every ℓ-cell in $D^p(f) \times \Delta_{p-1}$ is either the product of Δ_{p-1} with an $\ell - p + 1$ cell of $D^{p-1}(f)$, or lies in $D^p(f) \times \partial\, \Delta_{p-1}$. It follows that every relative cellular ℓ chain in $C_\ell^{\mathrm{CW}}\Big(D^p(f) \times \Delta_{p-1}, \big(\operatorname{diag} D^p(f) \times \Delta_{p-1}\big) \cup \big(D^p(f) \times \partial\, \Delta_{p-1}\big)\Big)$ is the product of a relative cellular chain in $C_{\ell-p+1}^{\mathrm{CW}}(D^p(f), \operatorname{diag} D^p(f))$ (possibly empty) with the cell Δ_{p-1}. That is,

$$\begin{aligned} C_\ell^{\mathrm{CW}}&\Big(D^p(f) \times \Delta_{p-1}, \big(\operatorname{diag} D^p(f) \times \Delta_{p-1}\big) \cup \big(D^p(f) \times \partial\, \Delta_{p-1}\big)\Big) \\ &\simeq C_{\ell-p+1}^{\mathrm{CW}}(D^k(f), \operatorname{diag} D^p(f)) \otimes_{\mathbb{Z}} C_{p-1}^{\mathrm{CW}}(\Delta_{p-1}, \partial\, \Delta_{p-1}) \\ &\simeq C_{\ell-p+1}^{\mathrm{CW}}(D^p(f), \operatorname{diag} D^p(f)) \otimes_{\mathbb{Z}} \mathbb{Z}. \end{aligned}$$

Moreover, the representation of S_p on the left is the tensor product of the two representations on the right. Because the representation on $\mathbb{Z} = C_{p-1}^{\mathrm{CW}}(\Delta_{p-1}, \partial\, \Delta_{p-1})$ is the sign representation, every relative S_p-invariant chain in $C_\ell^{\mathrm{CW}}\Big(D^p \times \Delta_{p-1}, \big(\operatorname{diag} D^p(f) \times \Delta_{p-1}\big) \cup \big(D^p(f) \times \partial\, \Delta_{p-1}\big)\Big)$ is the product of an alternating chain on D^p with the fundamental alternating $p-1$ chain Δ_{p-1} on Δ_{p-1}. Thus

$$H_j(Y_p, Y_{p-1}) \simeq H_j^{S_k}(D^p(f) \times \Delta_{p-1}, \operatorname{diag} D^p(f) \times \Delta_{p-1} \cup \Delta^p(f) \times \partial\, \Delta_{p-1}) \simeq H_j^{\mathrm{Alt}}(D^p(f), \operatorname{diag} D^p(f)),$$

where the middle group is the homology of the complex of symmetric chains. No alternating chain c can contain, in its support, any cell in $\operatorname{diag} D^p(f)$; such a cell would be fixed, pointwise, by at least one transposition, and thus its coefficient in c must be zero. The boundary of an alternating chain is itself an alternating chain. It follows that

$$H_j^{\mathrm{Alt}}(D^p(f), \operatorname{diag} D^p(f)) \simeq H_j^{\mathrm{Alt}}(D^p(f)).$$

We have proved

Lemma 6.6.5.

$$H_\ell(Y_p, Y_{p-1}) \simeq H_{\ell-p+1}^{Alt}(D^p(f))$$

□

Putting this together with the results of Subsection 6.6.1, we obtain the ICSS with integer coefficients:

Theorem 6.6.6. *There is a spectral sequence with $E^1_{p,q} = Alt\, H_{q+1}(D^p(f))$ and converging to $H^{p+q}(Y)$.* □

The differential

$$d_1 : H_{p+q}(Y_p, Y_{p-1}) \to H_{p+q-1}(Y_{p-1}, Y_{p-2})$$

becomes a morphism

$$\mathrm{Alt} H_{q+1}(D^p(f)) \to \mathrm{Alt} H_q(D^{p-1}(f)),$$

which we now identify. Recall the projection $\varepsilon^{j,p} : D^k(f) \to D^{p-1}(f)$ defined by $\varepsilon^{i,p}(x_1, \ldots, x_p) = (x_1, \ldots, \widehat{x_j}, \ldots, x_p)$. The key to the reappearance of the simplicial differential (**??**) (though now covariant rather than contravariant), is the fact that if $i_j : \Delta_{p-2} \to \Delta_{p-1}$ is the j'th face map, given by

$$i_j(t_1, \ldots, t_{p-1}) = (t_1, \ldots, t_{j-1}, 0, t_j, \ldots, t_{p-1})$$

then

$$h_{p-1}\big(\varepsilon^{j,p}(x_1, \ldots, x_p), (t_1, \ldots, t_{p-1})\big) = h_p\big((x_1, \ldots, x_p), i_j(t_1, \ldots, t_{p-1})\big). \tag{6.6.6}$$

We let $\delta_* : H^q(D^p(f)) \to H^q(D^{p-1}(f))$ denote the simplicial differential

$$\delta_* = \sum_{j=1}^{p} (-1)^{j+1} \varepsilon_*^{j,p}.$$

Let $\partial\,\Delta_{p-1}$ denote the cellular chain consisting of the boundary of the single $p-1$-cell Δ_{p-1} in our cellular decomposition of Δ_{p-1}. Then from (6.6.6) it follows that for any cellular chain c in $D^p(f)$, we have

$$h_{p*}\big(\partial(c \times \Delta_{p-1})\big) = h_{p*}\big((\partial c \times \Delta_{p-1}) + (c \times \partial\,\Delta_{p-1})\big) = h_{p*}\big((\partial c) \times \Delta_{p-1}\big) + h_{p-1*}(\delta_*(c) \times \Delta_{p-2}).$$

As we have seen, every closed S_p-invariant relative cellular chain in

$$C_\bullet^{\mathrm{CW}}(D^p(f) \times \Delta_{p-1}, (\operatorname{diag} D^p(f) \times \Delta_{p-1}) \cup D^p(f) \times \partial\,\Delta_{p-1})$$

can be written in the form $c \times \Delta_{p-1}$, where c is a closed alternating chain on $D^p(f)$. For such a chain, we therefore have

$$\partial h_{p*}(c \times \Delta_{p-1}) = h_{p*}\big(\partial(c \times \Delta_{p-1})\big) = h_{p-1*}(\delta_*(c) \times \Delta_{p-2}).$$

The differential

$$d_1 : E^1_{p,q} = H_{p+q}(Y_p, Y_{p-1}) \to H_{p+q-1}(Y_{p-1}, Y_{p-2}) = E^1_{p,q-1}$$

in the spectral sequence of the filtered space $(Y, Y_\bullet)$ is induced simply by the boundary map on relative chains. Therefore the following diagram is commutative.

$$\begin{tikzcd}
\mathrm{Alt}\, H_{q+1}(D^p(f)) \arrow[r, "\times \Delta_{p-1}"] \arrow[d, "\delta_*"'] & H^{S_p}_{p+q}(D^p(f) \times \Delta_{p-1}, B_p) \arrow[r, "h_{p*}"] \arrow[d] & H_{p+q}(Y_p, Y_{p-1}) \arrow[d, "d_1"] \\
\mathrm{Alt}\, H_{q+1}(D^{p-1}(f)) \arrow[r, "\times \Delta_{p-2}"'] & H^{S_{p-1}}_{p+q-1}(D^{p-1}(f) \times \Delta_{p-2}, B_{p-1}) \arrow[r, "h_{p-1*}"'] & H_{p+q-1}(Y_{p-1}, Y_{p-2})
\end{tikzcd} \tag{6.6.7}$$

where to save space, for each s we have denoted $(\mathrm{diag} D^s(f) \times \Delta_{s-1}) \cup (D^s(f) \times \partial\,\Delta_{s-1})$ by B_s. However, we wish to replace h_{p*} and h_{p-1*} in (6.6.7) by $\frac{1}{p!} h_{p*}$ and $\frac{1}{(p-1)!} h_{p-1*}$ respectively, in order that the horizontal arrows be isomorphisms. To retain commutativity, we therefore have to replace δ_* on the left by $\frac{1}{p}\delta_*$. Although this is not a well defined morphism on $H_{q+1}(D^p(f);\mathbb{Z})$, it is well defined on $\operatorname{Alt} H_{q+1}(D^p(f))$. For we have

$$\varepsilon^{j,p} = \varepsilon^{1,p} \circ (1,2) \circ \cdots \circ (j-1,j);$$

on the right there are $j-1$ transpositions, and therefore on *alternating* chains, $(-1)^{j-1}\varepsilon_*^{j,p} = \varepsilon_*^{1,p}$. From this it follows that on $\operatorname{Alt} H_{q+1}(D^p(f))$, $\delta_* = p\varepsilon_*^{1,p}$ and therefore $\frac{1}{p}\delta_*$ is well defined. Our description of the differential at E^1 is therefore complete: there is a commutative diagram whose rows are isomorphisms,

$$\begin{tikzcd}
\operatorname{Alt} H_{q+1}(D^p(f)) \arrow[r, "\times \Delta_{p-1}"] \arrow[d, "\frac{1}{p}\delta_*"'] & H^{S_p}_{p+q}(D^p(f) \times \Delta_{p-1}, B_p) \arrow[r, "\frac{1}{p!}h_{p*}"] \arrow[d] & H_{p+q}(Y_p, Y_{p-1}) \arrow[d, "d_1"] \\
\operatorname{Alt} H_{q+1}(D^{p-1}(f)) \arrow[r, "\times \Delta_{p-2}"'] & H^{S_{p-1}}_{p+q-1}(D^{p-1}(f) \times \Delta_{p-2}, B_{p-1}) \arrow[r, "\frac{1}{(p-1)!}h_{p-1*}"'] & H_{p+q-1}(Y_{p-1}, Y_{p-2})
\end{tikzcd} \tag{6.6.8}$$

and $d_1 : E^1_{p,q} \to E^1_{p,q-1}$ is given by $\frac{1}{p}\delta_* : \operatorname{Alt} H_{q+1}(D^p(f)) \to \operatorname{Alt} H_{q+1}(D^{p-1}(f))$.

6.7 Open questions:

1. Theorem **??** is proved by a rather complicated argument using equivariant stratified Morse theory. This remarkable theorem has not received the attention it deserves, in part because the published version is hard to read and suffers from some unfortunate typography. It would be a worthwhile project to write a clearer account. Houston's philosophical motivation for the theorem is worth describing because it is simple and illuminating. The difficulty in describing $D^k(f)$ is entirely due to the need to remove the diagonals, by which $D^k(f)$ differs from the simple minded scheme

   $$(X/Y)^k := X \times_Y X \times_Y \cdots \times_Y X := \{(x_1, \ldots, x_k) \in X^k : f(x_i) = f(x_j) \text{ for all } i, j\}.$$

   Away from these diagonals, $(X/Y)^k$ is a complete intersection, defined in X^k by the $(k-1)p$ equations $f_k(x_1) = f_k(x_i)$ for $1 \leq k \leq p$ and $2 \leq i \leq k$. Indeed, if f is finitely determined, then $(X/Y)^k$ is non-singular away from the diagonals, since at all genuine k-tuple points, which by the conic structure theorem occur away from 0, the corresponding multi-germ of f is stable. Now in the alternating chain complexes $C_\bullet^{\mathrm{Alt}}(f)$ and $C_\bullet^{\mathrm{Alt}}(D^k(f_t))$, the support of no chain can contain any simplex c lying entirely in any diagonal $\{x_i = x_j\}$, since, evidently, the transposition (i,j) leaves c fixed. It follows that for the alternating homology, $D^k(f)$ ought to behave like a complete intersection with isolated singularity, and new cycles should appear only in middle dimension. The extent to which this argument can be turned into a proof is not clear!

2. How can one compute the "alternating Milnor number" of $D^k(f)$ when f has corank > 1?

3. How can one compute the image Milnor number of a map-germ $(\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$? An answer to 1., together with Corollary 6.5.4, would provide a method; beyond this, there is

only a conjectural method which is part of the "Mond Conjecture", that

$$\mu_I(f) = \dim_{\mathbb{C}} \frac{\theta(i)}{ti(\theta_p) + i^*(\mathrm{Der}(-\log h))}$$

4. How can we find equations for $D^3(f)$, and higher multiple point spaces, when f has corank greater than 1?

Chapter 7

Multiple points in the target

By the Preparation Theorem, if $f : (\mathbb{C}^n, S) \to \mathbb{C}^{n+1}, 0)$ is a finite map-germ then $\mathcal{O}_{\mathbb{C}^n,0}$ is a finite module over $\mathcal{O}_{\mathbb{C}^{n+1},0}$. A *presentation* of $\mathcal{O}_{\mathbb{C}^n,S}$ as $\mathcal{O}_{\mathbb{C}^{n+1},0}$ -module is an exact sequence

$$\mathcal{O}^p_{\mathbb{C}^{n+1},0} \xrightarrow{\lambda} \mathcal{O}^q_{\mathbb{C}^{n+1},0} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{C}^n,S} \longrightarrow 0\,. \tag{7.0.1}$$

From a presentation one can learn a great deal about the geometry of the map f. Indeed in principle one can learn everything, since from the presentation one can obtain an equation for the image, and from this equation once can, in principle, determine the f itself, up to isomorphism, since it is the normalisation of its image. Other information, in the form of the Fitting Ideals, can be derived more immediately. We return to this after first developing an algorithm for finding a presentation.

Note that $\mathcal{O}_{\mathbb{C}^n,S} = \oplus_{x\in S}\,\mathcal{O}_{\mathbb{C}^n,x}$, and so if λ_x is the matrix of a presentation of $\mathcal{O}_{\mathbb{C}^n,x}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$, then the block diagonal matrix $\oplus_{x\in S}\lambda_x$ presents $\mathcal{O}_{\mathbb{C}^n,S}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$. So it is enough to develop a procedure to find each local presentation λ_x. In what follows we take $x = 0 \in \mathbb{C}^n$.

7.1 Procedure for finding a presentation:

Nakayama's Lemma tells us that if $g_1, \ldots, g_m \in \mathcal{O}_{\mathbb{C}^n,0}$ project to a $\mathbb{C}$-basis for $\mathcal{O}_{\mathbb{C}^n,0}\,/f^*\,\mathfrak{m}_{\mathbb{C}^{n+1},0}$, then $g_1, \ldots, g_m$ form a minimal set of generators for $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$. The structure of $\mathcal{O}_{\mathbb{C}^n,0}$ as $\mathcal{O}_{\mathbb{C}^{n+1},0}$-module is determined by the relations between these generators. The fact that the g_i generate $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$ is equivalent to the surjectivity of

$$\mathcal{O}^m_{\mathbb{C}^{n+1},0} \xrightarrow{\mathbf{g}} \mathcal{O}_{\mathbb{C}^n,0}\ ,$$

where $\mathbf{g}$ sends the i-th basis vector e_i to g_i. The module of relations between the g_i is the kernel of $\mathbf{g}$, and because $\mathcal{O}_{\mathbb{C}^{n+1},0}$ is Noetherian, it is finitely generated. Thus there is an $m \times r$ matrix λ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$ such that

$$\mathcal{O}^r_{\mathbb{C}^{n+1},0} \xrightarrow{\lambda} \mathcal{O}^m_{\mathbb{C}^{n+1},0} \xrightarrow{\mathbf{g}} \mathcal{O}_{\mathbb{C}^n,0} \longrightarrow 0 \tag{7.1.1}$$

is exact. Because the g_i form a minimal generating set for $\mathcal{O}_{\mathbb{C}^n,0}$, all entries in λ lie in the maximal ideal of $\mathcal{O}_{\mathbb{C}^{n+1},0}$. Thus (7.1.1) is the beginning of a minimal free resolution of $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$. The Auslander-Buchsbaum formula (see e.g. [Mat89, Chapter ?] or [Eis95, Chapter 19]) tells us that if p is the length of such a free resolution (the *projective dimension* of $\mathcal{O}_{\mathbb{C}^n,0}$ as $\mathcal{O}_{\mathbb{C}^{n+1},0}$-module)

, then $p + \operatorname{depth}_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0} = \operatorname{depth}_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^{n+1},0}$; it follows that $p = 1$. In other words, λ may be chosen injective. This forces r to be equal to m; for tensoring the exact sequence

$$0 \longrightarrow \mathcal{O}^r_{\mathbb{C}^{n+1},0} \xrightarrow{\lambda} \mathcal{O}^m_{\mathbb{C}^{n+1},0} \xrightarrow{\mathbf{g}} \mathcal{O}_{\mathbb{C}^n,0} \longrightarrow 0$$

with the field of fractions of $\mathcal{O}_{\mathbb{C}^{n+1},0}$ (the field $\mathcal{M} = \mathcal{M}_{\mathbb{C}^{n+1},0}$ of meromorphic functions), we retain exactness while killing $\mathcal{O}_{\mathbb{C}^n,0}$, and thus get an exact sequence $0 \longrightarrow \mathcal{M}^r \longrightarrow \mathcal{M}^m \longrightarrow 0$.

To find a matrix λ, one can use the following procedure:

1. Choose a projection $\pi : \mathbb{C}^{n+1} \to \mathbb{C}^n$ such that $\pi \circ f$ is finite. A suitable projection always exists. In practice this usually means selecting n of the $n+1$ component functions of f, though in principle it may be that none of these coordinate projections is finite. *In what follows we will assume that coordinates are chosen so that* $\pi(y_1, \ldots, y_{n+1}) = (y_1, \ldots, y_n)$.

2. Then $\mathcal{O}_{\mathbb{C}^n,0}$ (source) is free over $\mathcal{O}_{\mathbb{C}^n,0}$ (target); let $g_0, \ldots, g_d$ be a basis. Once again, by Nakayama's Lemma it is sufficient that the g_i form a $\mathbb{C}$-vector-space basis for $\mathcal{O}_{\mathbb{C}^n,0}/(\pi \circ f)^* \mathfrak{m}_{\mathbb{C}^n,0}$, which is finite dimensional by finiteness of $\pi \circ f$. *One of the* g_i *at least must be a unit in* $\mathcal{O}_{\mathbb{C}^n,0}$*; we take* $g_0 = 1$.

3. Find $\lambda^i_j \in \mathcal{O}_{\mathbb{C}^n,0}$ (target) such that

$$\begin{array}{rcccccc} f_{n+1} & = & \lambda^0_0 g_0 & + & \cdots & + & \lambda^m_0 g_m \\ g_1 f_{n+1} & = & \lambda^0_1 g_0 & + & \cdots & + & \lambda^m_1 g_m \\ \cdots & = & \cdots & \cdots & \cdots & \cdots & \cdots \\ g_m f_{n+1} & = & \lambda^0_m g_0 & + & \cdots & + & \lambda^m_m g_m \end{array} \tag{7.1.2}$$

Since $f_{n+1} = y_{n+1} \circ f$, (7.1.2) can be rewritten as

$$\begin{array}{rcccccccc} 0 & = & (\lambda^0_0 - y_{n+1}) g_0 & + & \cdots & + & \cdots & + & \lambda^m_0 g_m \\ 0 & = & \lambda^0_1 g_0 & + & (\lambda^1_1 - y_{n+1}) g_1 & + & \cdots & + & \lambda^m_1 g_m \\ \cdots & = & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & = & \lambda^0_m g_0 & + & \cdots & + & \cdots & + & (\lambda^m_m - y_{n+1}) g_m \end{array} \tag{7.1.3}$$

Thus the columns of the matrix

$$\begin{pmatrix} \lambda^0_0 - y_{n+1} & \lambda^0_1 & \cdots & \lambda^0_m \\ \lambda^1_0 & \lambda^1_1 - y_{n+1} & \cdots & \lambda^1_m \\ \cdots & \cdots & \cdots & \cdots \\ \lambda^m_0 & \lambda^m_1 & \cdots & \lambda^m_m - y_{n+1} \end{pmatrix} \tag{7.1.4}$$

are relations between the g_i.

Proposition 7.1.1. (7.1.4) *is the matrix of a presentation of* $\mathcal{O}_{\mathbb{C}^n,0}$ *over* $\mathcal{O}_{\mathbb{C}^{n+1},0}$. *In other words, the columns of* (7.1.4) *generate all the relations among the* g_i *over* $\mathcal{O}_{\mathbb{C}^{n+1},0}$.

Proof. A useful trick is described in [MP89, 2.2]: embed $\mathbb{C}^n$ as the hyperplane $\{t = 0\}$ in $\mathbb{C}^n \times \mathbb{C}$, and define $F : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^{n+1}$ by

$$F(x,t) = (f_1(x), \ldots, f_n(x), f_{n+1}(x) - t).$$

Write S for $\mathbb{C}^n \times \mathbb{C}$ (source) and T for $\mathbb{C}^{n+1}$ (target). Then

$$\mathcal{O}_{S,0} / F^* \mathfrak{m}_{T,0} = \frac{\mathcal{O}_{S,0}}{(f_1, \ldots, f_n, f_{n+1} - t)} \simeq \frac{\mathcal{O}_{\mathbb{C}^n,0}}{(f_1, \ldots, f_n)}$$

so that $g_0, \ldots, g_m$ form a free $\mathcal{O}_{T,0}$-basis for $\mathcal{O}_{S,0}$, and thus determine an $\mathcal{O}_T$-isomorphism $\mathcal{O}_{T,0}^{m+1} \xrightarrow{\varphi} \mathcal{O}_{S,0}$. In the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{S,0} \arrow[r, "t"] & \mathcal{O}_{S,0} \arrow[r] & \mathcal{O}_{\mathbb{C}^n,0} \arrow[r] & 0 \\
 & \mathcal{O}_{T,0}^{m+1} \arrow[u, "\varphi"'] \arrow[r, "{[t]_G^G}"] & \mathcal{O}_{T,0}^{m+1} \arrow[u, "\varphi"'] & &
\end{tikzcd} \tag{7.1.5}$$

$[t]_G^G$ denotes the matrix of the $\mathcal{O}_{T,0}$-linear map $\mathcal{O}_{S,0} \xrightarrow{t} \mathcal{O}_{S,0}$ (multiplication by t), with respect to the basis $g_0, \ldots, g_m$ of $\mathcal{O}_{S,0}$. We have

$$tg_i = (f_{n+1} - y_{n+1})g_i = \lambda_i^0 g_0 + \cdots + (\lambda_i^i - y_{n+1})g_i + \cdots + \lambda_i^m g_m$$

and thus $[t]_G^G$ is equal to the matrix (7.1.4). From the commutativity of (7.1.5) it follows that the cokernel of (7.1.4) is indeed isomorphic to $\mathcal{O}_{\mathbb{C}^n,0}$ as claimed. $\square$

The presentation obtained above is not necessarily minimal, since in general

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^n,0}}{f^* \mathfrak{m}_{\mathbb{C}^{n+1},0}} < \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^n,0}}{(\pi \circ f)^* \mathfrak{m}_{\mathbb{C}^n,0}}.$$

Nevertheless it is always injective, since the determinant of (7.1.4) is not zero – as can easily be seen, it is a monic polynomial of degree $m+1$ in $\mathbb{C}\{y_1, \ldots, y_n\}[y_{n+1}]$.

7.2 Fitting ideals

From the square matrix λ one can extract a great deal of information about the geometry of f.

Definition 7.2.1. Let $R^p \xrightarrow{\lambda} R^q \xrightarrow{g} M \longrightarrow 0$ be a presentation of the R-module M. The k'th *Fitting ideal of* M *as* R*-module*, $\mathrm{Fitt}_k^R(M)$, or simply $\mathrm{Fitt}_k(M)$ if it is clear which ring we are talking about, is the ideal generated by the $(q-k) \times (q-k)$ minors of λ, provided $p \geq q - k$, and is defined to be 0 if $p < q - k$ and R if $q - k \leq 0$.

Exercise 7.2.2. The Fitting ideals are independent of the choice of presentation of M. Prove this by showing

1. If $R^a \xrightarrow{\alpha} R^q \xrightarrow{g} M \longrightarrow 0$ and $R^b \xrightarrow{\beta} R^q \xrightarrow{g} M \longrightarrow 0$ are presentations of the same module with respect to the same set of generators, then

$$\min_{q-k}(\alpha) = \min_{q-k}(\beta).$$

2. If $R^s \xrightarrow{\mu} R^t \xrightarrow{h} M \longrightarrow 0$ is another presentation of the same module M, then $g + h : R^{q+t} \to M$ is surjective. For each basis vector e_i in R^t there exists $c_i \in R^q$ such that $g(c_i) = h(e_i)$, and thus $(c_i, -e_i) \in \ker(g+h)$. Show that the kernel of $g+h$ is generated by such pairs $(c_i, -e_i)$ together with pairs $(c, 0)$ with $c \in \ker g$, so that there is a presentation of the form
$$R^{p+t} \xrightarrow{\nu} R^{q+t} \xrightarrow{g+h} M \longrightarrow 0 \tag{7.2.1}$$
with
$$\nu = \begin{pmatrix} \lambda & -c \\ 0 & I_t \end{pmatrix}.$$
Clearly
$$\min_{q+t-k}(\nu) = \min_{q-k}(\lambda).$$
By symmetry, the kernel of $g+h$ is also generated by pairs $(0, d)$ with $d \in \ker h$ and pairs (e_j, d_j) where e_j is the j'th basis vector of R^p and $g(e_j) = -h(d_j)$. By 1, the ideals of $(q-k+t)$-minors are the same.

The Fitting ideals tell us a great deal about the geometry of f. We give two versions of this, first, one from algebraic geometry:

Proposition 7.2.3. *$V(Fitt_k^R(M)) = \{x \in Spec\ R : M_p$ needs more than k generators over $R\}$.*

In analytic geometry there are always two ways of looking at the same object. Let $\mathscr{S}$ be a coherent sheaf on the analytic space X. Define the ideal sheaf $\mathcal{F}_k(\mathscr{S})$ as the sheaf associated to the presheaf
$$U \mapsto \mathrm{Fitt}_k^{\Gamma(U,\mathcal{O}_X)}\Gamma(U, \mathscr{S});$$

Proposition 7.2.4.

$$V(\mathcal{F}_k(\mathscr{S})) = \{x \in X : \mathscr{S}_x \text{ needs more than } k \text{ generators over } \mathcal{O}_{X,x}\}.$$

Proof. From the presentation

$$\mathcal{O}_{X,x}^p \xrightarrow{\lambda} \mathcal{O}_{X,x}^q \longrightarrow \mathscr{S}_x \longrightarrow 0$$

tensoring with $\mathbb{C} = \mathcal{O}_{X,x} / \mathfrak{m}_{X,x}$ over $\mathcal{O}_{X,x}$ we obtain the exact sequence

$$\mathbb{C}^p \xrightarrow{\lambda(x)} \mathbb{C}^q \longrightarrow \mathscr{S}_x/\mathfrak{m}_{X,x}\,\mathscr{S}_x \longrightarrow 0\ ,$$

where $\lambda(x)$ is the $q \times p$ matrix over $\mathbb{C}$ obtained by evaluating λ at x. Now $\dim_{\mathbb{C}}\mathscr{S}_x/\mathfrak{m}_{X,x}\,\mathscr{S}_x$ is the minimum number of generators need by $\mathscr{S}_x$ as $\mathcal{O}_{X,x}$-module. If $x \in V(\mathrm{Fitt}_k(\mathscr{S}_x))$, then all $(q-k) \times (q-k)$ minors of $\lambda(x)$ vanish, and this means that the rank of $\lambda(x)$ is less than $q-k$, and, in turn, that $\dim_{\mathbb{C}}\mathscr{S}_x/\mathfrak{m}_{X,x}\,\mathscr{S}_x > k$. ∎

By coherence, we have

Proposition 7.2.5. $Fitt_k^{\mathcal{O}_{X,x}}(\mathscr{S}_x) = \big(\mathcal{F}_k(\mathscr{S})\big)_x$.

Corollary 7.2.6. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be finite and analytic. Then*

$$\begin{aligned} V(Fitt_k^{\mathcal{O}_{\mathbb{C}^{n+1},0}}(\mathcal{O}_{\mathbb{C}^n,0})) &= \{y \in \mathbb{C}^{n+1} : \textstyle\sum_{x\in f^{-1}(y)} mult_x(f) > k\} \\ &= \{y \in \mathbb{C}^{n+1} : y \textit{ has at least } k+1 \textit{ preimages, counting multiplicity}\} \end{aligned}$$

In particular, $\det \lambda$ defines the image of f, and the ideal of submaximal minors of λ defines the set of double points.

Definition 7.2.7. *The k'th target multiple point space of f, $M_k(f)$, is the space $V(Fitt_k^{\mathcal{O}_{\mathbb{C}^{n+1},0}}(\mathcal{O}_{\mathbb{C}^n,0}))$ with analytic structure given by $Fitt_k^{\mathcal{O}_{\mathbb{C}^{n+1},0}}(\mathcal{O}_{\mathbb{C}^n,0})$.*

Example 7.2.8.

1. Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be defined by

$$f(x, y) = (x, y^3, xy + y^5).$$

Take $\pi(Y_1, Y_2, Y_3) = (Y_1, Y_2)$; then $\mathcal{O}_{\mathbb{C}^2,0}$ (source) is generated over $\mathcal{O}_{\mathbb{C}^2,0}$ (target) by the classes of $1, y, y^2$. We have

$$\begin{array}{rcccrcrcr} f_3 &=& xy+y^5 &=& 01 &+& Y_1y &+& Y_2y^2 \\ g_1f_3 &=& xy^2+y^6 &=& Y_2^21 &+& 0y &+& Y_1y^2 \\ g_2f_3 &=& xy^3+y^7 &=& Y_1Y_21 &+& Y_2^2y &+& 0y^2 \end{array}$$

so as matrix of the presentation we obtain

$$\begin{pmatrix} -Y_3 & Y_2^2 & Y_1Y_2 \\ Y_1 & -Y_3 & Y_2^2 \\ Y_2 & Y_1 & -Y_3 \end{pmatrix}$$

2. Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be defined by $f(x_1, x_2) = (x_1^2, x_2^2, x_1x_2)$, and as before take $\pi(Y_1, Y_2, Y_3) = (Y_1, Y_2)$. Then $\mathcal{O}_{\mathbb{C}^2,0}$ (source) is generated over $\mathcal{O}_{\mathbb{C}^2,0}$ (target) by $1, x_1, x_2, x_1x_2$. We have

$$\begin{array}{rcccrcrcrcr} f_3 &=& x_1x_2 &=& 01 &+& 0x_1 &+& 0x_2 &+& 1x_1x_2 \\ g_1f_3 &=& x_1^2x_2 &=& 01 &+& 0x_1 &+& Y_1x_2 &+& 0x_1x_2 \\ g_2f_3 &=& x_1x_2^2 &=& 01 &+& Y_2x_1 &+& 0x_2 &+& 0x_1x_2 \\ g_3f_3 &=& x_1^2x_2^2 &=& Y_1Y_21 &+& 0x_1 &+& 0x_2 &+& 0x_1x_2 \end{array}$$

giving presentation matrix

$$\begin{pmatrix} -Y_3 & 0 & 0 & Y_1Y_2 \\ 0 & -Y_3 & Y_2 & 0 \\ 0 & Y_1 & -Y_3 & 0 \\ 1 & 0 & 0 & -Y_3 \end{pmatrix}.$$

Row and column operations transform this to

$$\begin{pmatrix} 0 & 0 & 0 & Y_3^2 - Y_1Y_2 \\ 0 & -Y_3 & Y_2 & 0 \\ 0 & Y_1 & -Y_3 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This is now the matrix of a presentation with respect to different set of generators (**Exercise:** which?), of which one is, according to the first column, superfluous. Deleting it gives the minimal presentation

$$\begin{pmatrix} 0 & 0 & Y_3^2 - Y_1Y_2 \\ -Y_3 & Y_2 & 0 \\ Y_1 & -Y_3 & 0 \end{pmatrix}$$

The determinant here is a square: this corresponds to the fact that f is a double covering of its image.

Exercise 7.2.9. Find a presentation for $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$ when

1. $f : (\mathbb{C}, 0) \to \mathbb{C}^2, 0)$ is defined by $f(x) = (x^2, x^5)$;
2. $f : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ is defined by $f(x = (x^2, x^{2k+1})$;
3. $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is defined by $f(x, y) = (x, y^2, yp(x, y^2))$;
4. $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is defined by $f(x, y) = (x, y^3, xy + y^{3k-2})$;
5. $f : (\mathbb{C}^6, 0) \to (\mathbb{C}^7, 0)$ is defined by $f(a, b, c, d, x_1, x_2) = (a, b, c, d, x_1^2 + ax_2, x_2^2 + bx_1, x_1x_2 + cx_1 + dx_2)$.

Exercise 7.2.10. Show that if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ is finite and generically k-to-1 onto its image, and if λ is the matrix of a presentation of $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$, then $\det \lambda$ is the k'th power of a reduced equation for the image.

Proposition 7.2.11. ([MP89]) *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be finite and generically 1-1, and let λ be the $(m+1) \times (m+1)$ matrix of a presentation of $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$, with respect to generators $g_0 = 1, g_1, \ldots, g_m$. Then the ideal $Fitt_1(\mathcal{O}_{\mathbb{C}^n,0})$ is generated by the $m \times m$ minors of the matrix λ' obtained from λ by deleting its first row.*

Proof. We continue with the notation of Proposition 7.1.1. Write $P := [t]_G^G$, and $h := \det P$. Then $\mathrm{Fitt}_1(\mathcal{O}_{\mathbb{C}^n,0})$ is generated by the entries in the adjugate matrix P^{adj}. Now

$$P^{\mathrm{adj}} = [ht^{-1}]_G^G$$

so the j'th column of P^{adj} is equal to $[ht^{-1}g_j]_E$ and in particular, since $g_0 = 1$, the first column of P^{adj} is $[ht^{-1}]_E$. Let $[g_j]_E^E$ denote the matrix of the $\mathcal{O}_S$-endomorphism determined by multiplication by g_j, with respect to basis E. The theorem is proved simply by observing that

$$[ht^{-1}g_j]_E = [g_j]_E^E[ht^{-1}]_E.$$

□

Exercise 7.2.12. Let m_j^i be the $m \times m$ minor determinant of λ obtained by omitting row i and column j.

1. Use Cramer's rule to show that for all i, j, k,

$$m_j^i g_k = m_j^k g_i \tag{7.2.2}$$

and in particular

$$m_j^i = m_j^0 g_i. \tag{7.2.3}$$

2. Because $g_i g_j$ lies in $\mathcal{O}_{\mathbb{C}^n,0}$ and $\mathcal{O}_{\mathbb{C}^n,0}$ is generated over $\mathcal{O}_{\mathbb{C}^{n+1},0}$ by the g_k, there exist $\Gamma_{ij}^k \in \mathcal{O}_{\mathbb{C}^{n+1},0}$ such that $g_i g_j = \sum_k \Gamma_{ij}^k g_k$,, with $\Gamma_{ij}^k \in \mathcal{O}_{\mathbb{C}^{n+1},0}$. Use 1. to show that

$$m_j^i g_k = \sum_\ell \Gamma_{ik}^\ell m_j^\ell.$$

Exercise 7.2.13. *1. Find equations for the double-point locus, C, of the image of the map-germ f of type H_2, given by $f(x,y) = (x, y^3, xy + y^5)$.*

2. Show that C is the image of the map $t \mapsto (t^4, t^3, t^5)$.

3. Check that $f^*(Fitt_1^{\mathcal{O}_{\mathbb{C}^{n+1},0}}(\mathcal{O}_{\mathbb{C}^n,0}))$ is a principal ideal in $\mathcal{O}_{\mathbb{C}^n,0}$.

4. Find the pre-image in $\mathbb{C}^2$ of C, and show that it has a singularity of type A_6 at 0.

5. Show that the set of real points on this curve is just 0.

6. Can you reconcile the conclusions of 2. and 5.?

The argument in the proof of 7.1.1 serves to prove another result:

Proposition 7.2.14. *([Cat84], [MP89]) The matrix λ of a presentation of $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$ can be chosen symmetric.*

Proof. We replace the diagram (7.1.5) by a second diagram in which the two isomorphisms of $\mathcal{O}_{\mathbb{C}^n\times\mathbb{C},0}$ (source) with $\mathcal{O}_{\mathbb{C}^{n+1},0}$ (target) are no longer assumed to be the same. Write $\mathcal{O}_{S,0} := \mathcal{O}_{\mathbb{C}^n\times\mathbb{C},0}$ (source), and $\mathcal{O}_{T,0} := \mathcal{O}_{\mathbb{C}^{n+1},0}$ (target). Because $\mathcal{O}_{S,0}$ is a Gorenstein ring, and is finite over $\mathcal{O}_{T,0}$ (target), there is a perfect symmetric $\mathcal{O}_{T,0}$-bilinear pairing $(\cdot,\cdot) : \mathcal{O}_{S,0} \times \mathcal{O}_{S,0} \to \mathcal{O}_{T,0}$. This is a consequence of local duality. It is proved by Scheja and Storch in [SS75], by showing that $\mathrm{Hom}_{\mathcal{O}_{T,0}}(\mathcal{O}_{S,0}, \mathcal{O}_{T,0})$ is cyclic as $\mathcal{O}_{S,0}$ module (where, for $s_1, s_2 \in \mathcal{O}_{S,0}$ and $\varphi \in \mathrm{Hom}_{\mathcal{O}_{T,0}}(\mathcal{O}_{S,0}, \mathcal{O}_{T,0})$, $s_1 \cdot \varphi(s_2) = \varphi(s_1 s_2)$), picking an $\mathcal{O}_{S,0}$-generator Φ for $\mathrm{Hom}_{\mathcal{O}_{T,0}}(\mathcal{O}_{S,0}, \mathcal{O}_{T,0})$, and setting

$$(s_1, s_2) = \Phi(s_1 s_2).$$

Because this gives a perfect pairing, for each basis $G := g_0, \ldots, g_m$ for $\mathcal{O}_{S,0}$ as $\mathcal{O}_{T,0}$ module there is a dual basis $\check{G} := \check{g}_0, \ldots, \check{g}_m$ with the property that $(\check{g}_i, g_j) = \delta_{ij}$. Let $\check{\varphi}$ be the $\mathcal{O}_{T,0}$ isomorphism $\mathcal{O}_{T,0}^{m+1} \to \mathcal{O}_{S,0}$ determined by the basis $\check{G}$. Then the matrix $[t]_G^{\check{G}}$ is symmetric (**Exercise**), and, by the argument of the proof of 7.1.1, is the matrix of a presentation of $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$. □

Corollary 7.2.15. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be finite and generically 1-1. Then $f^*Fitt_1(\mathcal{O}_{\mathbb{C}^n,0})$ is a principal ideal.*

Proof. Choose a symmetric presentation λ, with respect to generators $g_0 = 1, \ldots, g_m$. Then in the language of the proof of 7.2.11, $\mathrm{Fitt}_1(\mathcal{O}_{\mathbb{C}^n,0})$ is generated by $(m_0^0, \ldots, m_m^0)$, and so $f * \mathrm{Fitt}_1(\mathcal{O}_{\mathbb{C}^n,0})$ is generated by $f*(m_0^0), \ldots, f^*(m_m^0)$. It follows by (7.2.3) and the symmetry of λ that $f^*\mathrm{Fitt}_1(\mathcal{O}_{\mathbb{C}^n,0})$ is generated by $f*(m_0^0)$. □

Because $\mathrm{Fitt}_1(\mathcal{O}_{\mathbb{C}^n,0}) = \mathrm{Ann}_{\mathcal{O}_{\mathbb{C}^{n+1},0}}(\mathcal{O}_{\mathbb{C}^n,0} \,/\, \mathcal{O}_{D,0})$, the ideal $\mathrm{Fitt}_1(\mathcal{O}_{\mathbb{C}^n,0})\,\mathcal{O}_{D,0}$ is known as the *conductor* ideal of the ring homomorphism $\mathcal{O}_{D,0} \to \mathcal{O}_{\mathbb{C}^n,0}$. We denote it by $\mathscr{C}$. In fact $\mathscr{C}$ is also an ideal in $\mathcal{O}_{\mathbb{C}^n,0}$; it is the largest ideal of $\mathcal{O}_{D,0}$ which is also an ideal in $\mathcal{O}_{\mathbb{C}^n,0}$. The last corollary shows that as an ideal in $\mathcal{O}_{\mathbb{C}^n,0}$, $\mathscr{C}$ is principal. One can find a generator by picking a symmetric presentation λ, but there is an easier method , due, with a rather sophisticated proof, to Ragni Piene ([Pie79]), and, with a simpler proof, to Bill Bruce and Ton Marar ([BM96]):

Theorem 7.2.16. ([BM96]) *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be finite and generically 1-1. Let h be a reduced equation for its image, and let*

$$r_i := \frac{\partial(f_1, \ldots, \widehat{f_i}, \ldots, f_{n+1})}{\partial(x_1, \ldots, x_n)}$$

be the minor determinant of the matrix of the derivative df obtained by omitting row i. Then $(\partial h/\partial Y_i) \circ f$ is divisible by r_i in $\mathcal{O}_{\mathbb{C}^n,0}$, and the quotient generates the conductor ideal $\mathscr{C}$.

Exercise 7.2.17. Find a generator for the conductor when f is the map of Exercise 6.2.20(a). Show that $D^2_1(f)$ is isomorphic to the product $\mathbb{C} \times D_2$, where D_2 is the image of the stable map of Example 3.1.1. This has an explanation! What is it?

In a similar vein to 7.2.11:

Theorem 7.2.18. ([MP89, Theorem 4.1]) *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be finite and generically 1-1, and let λ be a symmetric presentation of $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$, with respect t generators $g_0 = 1, g_1, \ldots, g_m$. Then $\mathrm{Fitt}_2(\mathcal{O}_{\mathbb{C}^n,0})$ is generated by the $(m-1)\times(m-1)$ minors of the matrix obtained from λ by deleting its first row and column.*

The variety of zeros of the ideal of submaximal minors of an $m \times m$ matrix can have codimension no greater than 3, and if the codimension is 3 then the variety in question is Cohen Macaulay, by Theorem 1.6.1 and a theorem of Jozefiak ([Joz78]). Thus

Corollary 7.2.19. *Suppose, in the circumstances of 7.2.18, that in addition $\mathrm{codim}\,(V(\mathrm{Fitt}_2(\mathcal{O}_{\mathbb{C}^n,0})) = 2$. Then $V(\mathrm{Fitt}_2(\mathcal{O}_{\mathbb{C}^n,0}))$ is Cohen-Macaulay.*

Corollary 7.2.20. *If $n = 2$, and f satisfies the hypotheses of 7.2.19, then the number of triple points in the image of a stable perturbation of f is equal to $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^3,0} \,/\, \mathrm{Fitt}_2(\mathcal{O}_{\mathbb{C}^2,0})$.*

7.3 Open questions

1. Do the Fitting ideals give a reasonable analytic structure to the multiple point spaces? And are these spaces well-behaved in the case of finitely determined map-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$? How do they behave under deformation? In particular, if F is an unfolding of f on parameter space S, then is $M_k(F)$ Cohen Macaulay (and therefore flat over S)? Some partial answers are known, see [MP89],[KLU96], [KLU92], but for maps of corank greater than 1, nothing is known about the behaviour of $\mathrm{Fitt}_k^{\mathcal{O}_{\mathbb{C}^{n+1},0}}(\mathcal{O}_{\mathbb{C}^n,0})$ under deformation when $k > 3$. Recent improvements in computing power make more calculations possible, and new examples might clarify these questions. In particular, does a version of 7.2.20 hold for higher Fitting ideals? For example, is it true that if $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^4, 0)$ is finite and generically 1-1, and $\mathrm{codim}\,(V(\mathrm{Fitt}_3(\mathcal{O}_{\mathbb{C}^3,0})) = 4$, then the number of quadruple points in the image of a stable perturbation of f is equal to $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^4,0} \,/\mathrm{Fitt}_3(\mathcal{O}_{\mathbb{C}^3,0}))$?

2. One of the most famous open problems is the Lê Conjecture. Part of this conjecture says that if $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ has corank 2 then it cannot be injective. Do the Fitting ideals give any handle on this question? It seems not, since they do not distinguish between genuine double points, with two distinct preimages, and points with a non-immersive preimage. If there were some way of incorporating the involution on $D^2(f)$ into the picture, it might be possible to make some progress on this surprisingly intractable problem.

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