Lecture Notes for MA5NO
Cohomology, Connections, Curvature and Characteristic Classes

David Mond

October 27, 2006

1 Introduction

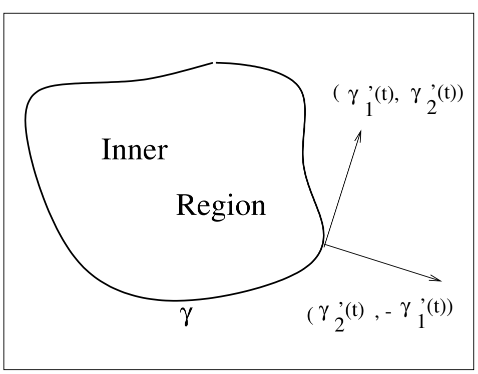
Let's begin with a little vector analysis (also known as "Physics").

Consider a point source of fluid, such as a burst water-main, on a perfectly uniform plane. The water spreads out uniformly from the source, with a uniform depth, and so we can measure the fuid flowing out from the source in units of area per second. The flow is regulated so that $2\pi K$ units of area flow out each second. Once a steady state has been reached, then if the source of the flow is taken as origin of coordinates, at the point with coordinates (x, y) the velocity vector $v(x, y)$ of the flow is

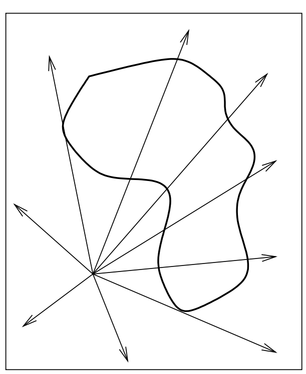
$$K\frac{(x,y)}{\|(x,y)\|^2}.$$

This is easy to see — by symmetry the flow is radial, and, assuming that a steady state has been reached, the amount of water crossing the circle of radius R centred at 0 is independent of R. This quantity is equal to the circumference $2\pi R$ times the norm of the velocity vector, and thus this norm is inversely proportional to R; to give $2\pi K$, the constant of proportionality must be K.

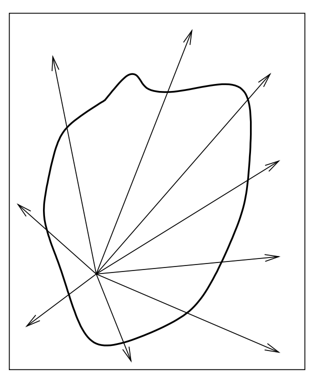
We can use this flow to derive a formula for the number of times a closed curve C in $\mathbb{R}^2 \setminus \{(0, 0)\}$ winds around $(0, 0)$. Suppose first that C is simple, i.e. does not cross itself. Then $(0, 0)$ is either outside C, or inside. In the first case, the net amount of water crossing the curve from inside to outside per second is $2\pi K$; in the second case, it is 0 (see the diagram).



Unit tangent and unit normal



Amount out = amount in



Amount out = 2 π C
Amount in = 0

The amount crossing C per second can also be calculated by an integral, namely the integral, along C, of the component of the flow normal to C, with respect to arc length (see Figure 2).

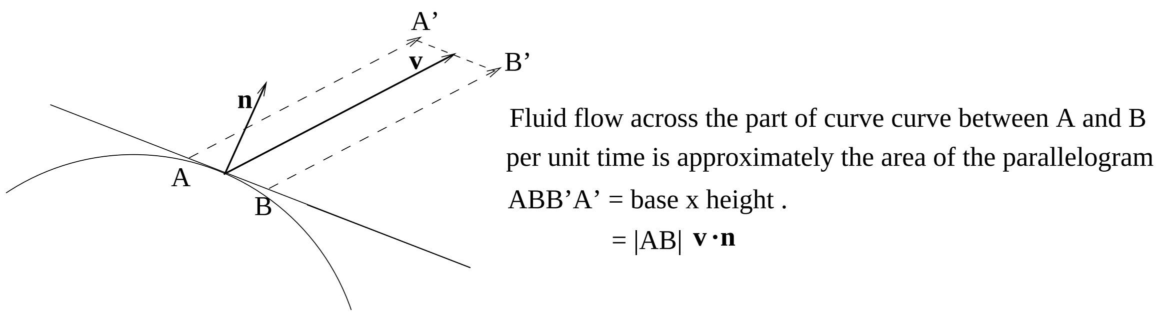


Figure 2

If $\gamma : [0, \ell] \to \mathbb{R}^2$ is a parametrisation of C by arc-length, then $(\gamma_1'(t), \gamma_2'(t))$ is a unit tangent, and so $(\gamma_2'(t), -\gamma_1'(t))$ is a unit vector normal to C. This vector is obtained from the tangent vector by rotating it by $\pi/2$ in a clockwise direction. If the area inside C is on the left, with respect to the sense in which the curve is parametrised, then this unit vector points outwards from this region. Thus, providing C is parametrised by arc length, the total flow out of the region is given by

$$\int_0^\ell v(\gamma(t)) \cdot \mathbf{n}(t)\ dt. \tag{1}$$

Again provided C is parametrised by arc length, this is equal to

$$K \int_0^\ell \frac{(x, y) \cdot (-\gamma_2'(t), \gamma_1'(t))}{\|(x, y)\|^2} dt, \tag{2}$$

which can be re-written as

$$K \int_0^\ell \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right) \cdot \gamma'(t) dt. \tag{3}$$

Note that even if we remove the requirement that C be parametrised by arc-length, (3) (but not (1)) *still gives the right answer*. Intuitively, if we go round C at twice the speed then γ', and hence the integrand, is multiplied by 2, but the domain of integration has half the length it had before, so the integral is unchanged.

Let us now fix the value of K to be $1/2\pi$. Then we have an operator on the set of simple closed curves in $\mathbb{R}^2 \setminus \{(0, 0)\}$, which gives the answer $0, 1$ or -1. If we now allow the curve C to cross itself, then the range of possible answers becomes all of $\mathbb{Z}$; one can see this by decomposing an arbitrary closed curve C into a sequence of simple closed curves, and observing that the integral is additive over disjoint domains. Whereas previously the integral gave ± 1 if the origin was inside the curve C and 0 if it is outside, now it is better simply to say that the integral measures the number of times C winds around the origin *in an anticlockwise direction.*

Exercise 1.1 Show that the integral really is unchanged under a reparametrisation

Exercise 1.2 Generalise this construction to 3 dimensions. That is, use a similar physical argument to devise an integral formula which, given a parametrisation of a closed surface in 3-space, gives the value ± 1 if the origin is in the region enclosed by the surface, and 0 if it is not.

Exercise 1.3 How can we *prove* that these formulae really do what we claim?

2 The first "C": Cohomology

I assume you have met the definitions of smooth manifold, differential form, the exterior derivative of a differential form, and the integral of a (compactly supported) differential form on an oriented manifold.

Given a smooth manifold M we denote by $\Omega^k(M)$ the space of smooth (C^∞) k-forms, and by $\Omega^\bullet(M)$ the complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0,$$

where $n = \dim M$ and d is the exterior derivative. "Complex" here means that $d^2 = 0$, and so

$$\mathrm{im}\{d : \Omega^{k-1}(M) \to \Omega^k(M)\} \subseteq \ker\{d : \Omega^k(M) \to \Omega^{k+1}(M)\};$$

the quotient of kernel by image is the *k'th de Rham cohomology group*, $H^k_{DR}(M)$. The subscript DR indicates de Rham, and when G. de Rham first studied this construction he thought he had a new invariant of smooth manifolds. This was not quite so: it turns out that de Rham cohomology coincides with cohomology defined by other means (e.g. singular cohomology) (with coefficients in $\mathbb{R}$), so we will drop the subscript.

Example 2.1 The differential form

$$\frac{-ydx}{x^2+y^2} + \frac{xdy}{x^2+y^2}$$

on $\mathbb{R}^2 \setminus \{(0,0)\}$ is known as $d\theta$. This is because in any region of the plane in which the polar coordinate θ (see the diagram below) is smooth and well-defined, then this form is indeed equal to the exterior derivative of θ, as you should check (perhaps by writing $\theta = \arctan(y/x)$ — some such expression is valid in a neighbourhood of each point in $\mathbb{R}^2 \setminus \{(0,0)\}$, and any two determinations of θ differ by a multiple of 2π (i.e. a constant), and thus have the same exterior derivative.)

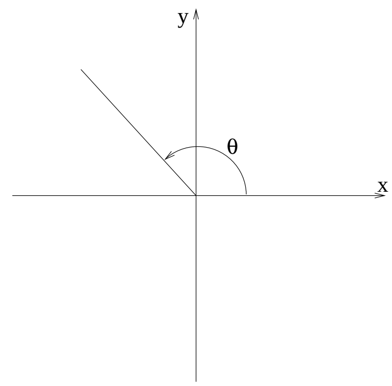


Figure 3

The name $d\theta$ is deliberately ironical (one might say, deliberately confusing), since the polar coordinate θ is not a smooth and single-valued function on all of $\mathbb{R}^2 \setminus \{(0,0)\}$. However, since $d\theta$ is at least *locally* the exterior derivative of a function, it is closed, given that $d^2 = 0$. On the other hand despite its name it is not exact: there can be no smooth function *defined on all of* $\mathbb{R}^2 \setminus \{(0,0)\}$ of which $d\theta$ is the exterior derivative. This follows, by Stokes's Theorem (2.9 below), from

Exercise 2.2 The integral in formula (3) of the Introduction is just $\int_C d\theta$.

If $d\theta$ were the exterior derivative of a function defined on all of $\mathbb{R}^2 \setminus \{(0,0)\}$, then by Stokes's theorem its integral over any simple closed curve in $\mathbb{R}^2 \setminus \{(0,0)\}$ would be zero, which, by the discussion in the Introduction, we know not to be the case for $d\theta$. The fact that $d\theta$ is closed but not exact means that its class in $H^1(\mathbb{R}^2 \setminus \{(0,0)\})$ is not 0. We shall see later that the class of $d\theta$ generates (is a basis of) $H^1(\mathbb{R}^2 \setminus \{(0,0)\})$.

We shall also see (in 2.20 below) that *every* closed k-form ω on a smooth manifold is, in some neighbourhood U_x of every point x, the exterior derivative of some $(k-1)$-form σ_x, so that our form $d\theta$ is not special in this regard. The point is whether or not these local $(k-1)$-forms piece together to form a global $(k-1)$-form σ such that $\omega = d\sigma$.

Example 2.1 continued In fact, if we break up C into pieces C_i, each one contained in some region in which θ can be represented as a smooth, single- valued function θ_i, then

$$\int_C d\theta = \sum_i \int_{C_i} d\theta = \sum_i \int_{C_i} d\theta_i.$$

By Stokes's theorem (which in the case of curves is just the fundamental theorem of calculus), if P_{i-1} and P_i are the end-points of C_i, so that $\partial C_i = P_i - P_{i-1}$, then

$$\int_{C_i} d\theta_i = \theta_i(P_i) - \theta_i(P_{i-1}).$$

Note that each function θ_i really is well defined and single-valued (unlike θ), and so we really can apply Stokes's theorem after we have broken up the curve into bits. The choice of definition of θ_i makes no difference, of course, since any two distinct versions differ by a constant. In the curve shown here, we might define the functions

$$\begin{array}{lll} \theta_1 & : \mathbb{R}^2 \setminus \text{Ray } 1 & \rightarrow & (-\pi/4, 7\pi/4) \\ \theta_2 & : \mathbb{R}^2 \setminus \text{Ray } 2 & \rightarrow & (-3\pi/4, 5\pi/4) \\ \theta_3 & : \mathbb{R}^2 \setminus \text{Ray } 3 & \rightarrow & (-3\pi/2, \pi/2). \end{array}$$

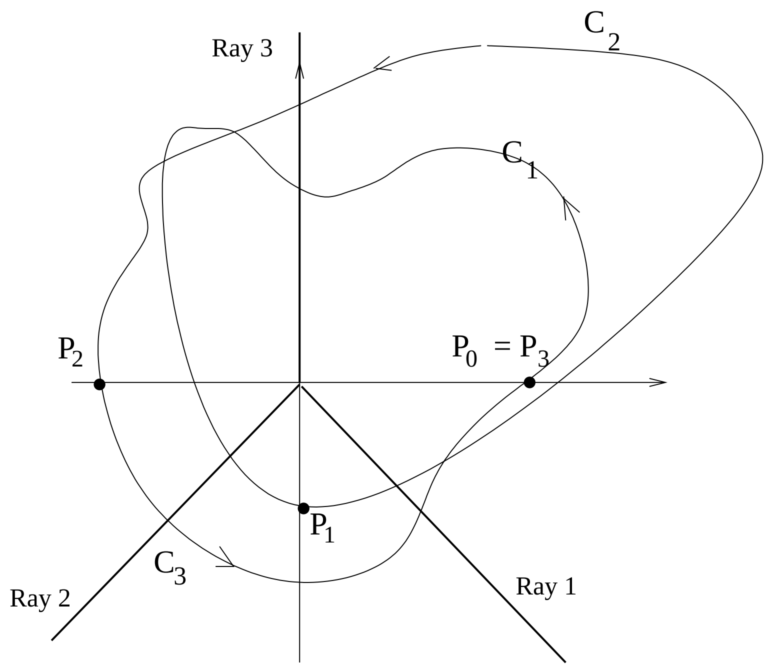


Figure 4

Observe that C_i is contained in the domain of θ_i for $i = 1, 2, 3$. We have

$$\int_C d\theta = \int_{C_1} d\theta_1 + \int_{C_2} d\theta_2 + \int_{C_3} d\theta_3$$

$$= (\theta_1(P_1) - \theta_1(P_0)) + (\theta_2(P_2) - \theta_2(P_1)) + (\theta_3(P_3) - \theta_3(P_2))$$

$$= (3\pi/2 - 0) + (\pi - (-\pi/2)) + (0 - (-\pi)) = 4\pi.$$

Exercise 2.3 Calculate $H^0(\mathbb{R})$ and $H^1(\mathbb{R})$. This is easy to do directly from the definition.

De Rham cohomology is *functorial*: if $f : M \rightarrow N$ is a smooth map, then pull-back of forms gives a map $f^* : \Omega^k(N) \rightarrow \Omega^k(M)$ for each k, which commutes with the exterior derivative d; it follows that f^* passes to the quotient to give a map $H^k(N) \rightarrow H^k(M)$, also denoted f^*, or $H^k(f)$ is you are punctilious. We have $(g \circ f)^* = f^* \circ g^*$, and $(\mathrm{id}_M)^*$ is the identity, so H^k is a *contravariant functor*.

The wedge product of forms, $\Omega^j(M) \times \Omega^k(M) \rightarrow \Omega^{j+k}(M)$, also passes to the quotient (because $d(\omega \wedge \rho) = d\omega \wedge \rho + (-1)^k \omega \wedge d\rho$) to give a product $H^j(M) \times H^k(M) \rightarrow H^{j+k}(M)$, and this gives the direct sum $\oplus_k H^k(M)$ a ring structure, which is also functorial.

Exercise 2.4 Show that the wedge product of forms passes to the quotient to define a product $H^j(M) \times H^k(M) \to H^{j+k}(M)$

But what is a differential form? I assume you know the definition, but is this an adequate peg on which to hang a concept? Here are some more examples.

Example 2.5 1. Suppose that M is an n-dimensional oriented manifold equipped with a Riemannian metric (an inner product on each tangent space T_pM, varying smoothly with p). Then we have the notion of *orthonormal basis* for T_pM. An n-form which takes the value 1 on any (and therefore every) positively oriented orthonormal basis is called a *volume form.* Indeed, if $U \subset M$ is a region with compact closure then the integral $\int_U \mathrm{vol}_M$ is the (oriented) volume of M.

Exercise 2.6 If $M \subset \mathbb{R}^2$ or $\mathbb{R}^3$, we have a pre-existing notion of "volume" (called "length" if M is 1-dimensional and "area" if M is 2-dimensional). Give a heuristic argument that if we give M the Riemann metric it inherits naturally from $\mathbb{R}^n$, then $\int_U \mathrm{vol}_M$ agrees with this pre-existing notion.

Exercise 2.7 Suppose that $M \subset \mathbb{R}^{n+1}$ is a smooth oriented hypersurface, with positively oriented unit normal vector field $u(x) = (u_1(x), \dots, u_{n+1}(x))$ (i.e. for each $x \in M$, a basis for T_xM is positive if this basis, preceded by $n(x)$, is a positive basis for $\mathbb{R}^{n+1}$). Show that

$$\sum_i (-1)^{i-1} u_i(x) dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}$$

is the volume form. Hint: A positive orthonormal basis $v_1, \dots, v_n$ of T_xM gives rise to a positive orthonormal basis $u(x), v_1, \dots, v_n$ for $\mathbb{R}^{n+1}$.
In particular, write down an explicit volume form on the n-sphere S^n.

Exercise 2.8 Show that if M is a compact oriented manifold (without boundary) and vol_M its volume form, then $\mathrm{vol}_M \neq 0$ in $H^n(M)$.

2. If $M^n \subset \mathbb{R}^{n+1}$ is an oriented hypersurface, there is a *Gauss map* $\gamma : M \to S^n$, $\gamma(x) =$ the positively oriented unit normal to M at x. Gauss used this to define a the curvature $\kappa(x)$ of M at x:

$$\kappa(x) = \lim_{U \searrow \{x\}} \frac{\mathrm{volume}(\gamma(U))}{\mathrm{volume}(U)}.$$

Make a drawing to see that this is reasonable! With a bit of extra effort this can be given a sign: $+1$ if locally γ preserves orientation, -1 if it reverses it. If γ is not a local diffeomorphism at x, then $\kappa(x) = 0$. Can you find a heuristic argument for this? It is closely related to the proof of *Sard's theorem*, that the set of critical values of a smooth map has measure zero.

In fact, if vol_M and vol_{S^n} are volume forms, then because the space of alternating n-tensors on an n-dimensional vector-space is 1-dimensional, $\gamma^*(\mathrm{vol}_{S^n})$ must at each point on M be a scalar multiple of vol_M; and of course by continuity we find that the scalar in question is precisely the Gauss curvature κ of M.

3. Let X be a vector field on $\mathbb{R}^3$, and imagine that it is the velocity field of a fluid flow. For each point $x \in \mathbb{R}^3$, each pair of vectors v_1, v_2, and each positive real ε, imagine a parallelogram P_ε based at x and spanned by εv_1 and εv_2. Note that the area of P_ε, divided by ε^2, is independent of ε. Let

$$\omega(v_1, v_2) = lim_{\varepsilon \to 0} \frac{\text{fluid flow through } P_\varepsilon \text{ per unit time}}{\varepsilon^2}.$$

We can give this a sign: flow through P_ε is positive if its projection to the normal direction agrees with $v_1 \times v_2$, and negative if it disagrees. Then ω is a smooth 2-form on $\mathbb{R}^3$.

When is ω closed?

4. The vector field X also defines a 1-form ρ: $\rho(v) = v \cdot X$ (where the dot means ordinary scalar product). What is $d\rho$?

What are forms for? One should, of course, ask this kind of question about every new object one meets in mathematics. Asking it and trying to answer helps to free up intellectual energy to devote to the topic in question, even when one does not find a clear answer at first.

Example 2.9 *Stokes's Theorem* says that for an oriented manifold M^n with boundary ∂M, and for any compactly supported form $\omega \in \Omega^{n-1}(M)$,

$$\int_M d\omega = \int_{\partial M} \omega.$$

This is also valid on a manifold with corners, such as a k-simplex, and leads to an interesting pairing between de Rham cohomology and singular homology, which I now describe.

A *singular k-simplex* in M is a smooth map to M from the standard k-simplex Δ_k, defined by

$$\Delta_k = \{(x_1, \ldots, x_{k+1}) \in \mathbb{R}_+^{k+1} : \sum_i x_i = 1\}.$$

If s is a singular k-simplex and $\omega \in \Omega^k(M)$, we can integrate ω over s — that is, the integral $\int_{\Delta_k} s^*(\omega)$ is defined.

A singular k-chain is a "formal sum" $\sum_i n_i s_i$, where the n_i are integers and s_i is a singular k-simplex. "Formal" means you don't have to worry about what it is, only about what it does. And what is does, is that you can integrate a k-form over it: if c is the k-chain $\sum_i n_i s_i$, then we define

$$\int_c \omega = \sum_i n_i \int_{s_i} \omega.$$

The collection of all singular k-chains forms a $\mathbb{Z}$-module, $C_k(M)$. If s is a singular k-simplex, one can think of the singular chain $-1 \cdot s$ (or $-s$) simply as s with the opposite orientation. Certainly, that is how it behaves in integration: by definition of the integral over a k-chain,

$$\int_{-s} \omega = -\int_s w.$$

There are inclusions $i_j : \Delta^{k-1} \hookrightarrow \Delta^k$ for $j = 1, \ldots, k+1$: $\quad i_j(x_1, \ldots, x_k) = (x_1, \ldots, 0, \ldots, x_k)$ with the zero in the j-th position. Using these we define a boundary operator $C_k(M) \to C_{k-1}(M)$, by setting

$$\delta(s) = \sum_j (-1)^{j-1}(s \circ i_j)$$

on a singular k-simplex s and extending linearly to formal sums of k-simplices. The coefficient $(-1)^{j-1}$ guarantees that the $k-1$-simplex $s \circ i_j$ appears with the correct orientation (i.e. as part of the boundary of s).

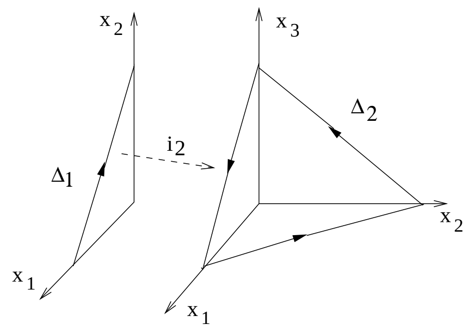


Figure 5: The diffeomorphism i_2 gives the second edge of Δ_2 the opposite orientation to its boundary orientation

Abusing our notation slightly and denoting by $\delta\Delta_k$ the singular $(k-1)$-chain in Δ_k equal to $\sum_(-1)^{j-1} i_j$, it follows that for any $k-1$-form defined on the standard k-simplex Δ_k,

$$\int_{\partial\Delta_k} \omega = \int_{\delta\Delta_k} \omega \tag{4}$$

From this and Stoke's theorem we get the following "simplicial" version of Stokes's Theorem:

Theorem 2.10 *If $c = \sum_i m_i \sigma_i$ is a singular k-chain in the smooth manifold M, and $\omega \in \Omega^{k-1}(M)$, then*

$$\int_c d\omega = \int_{\delta c} \omega.$$

□

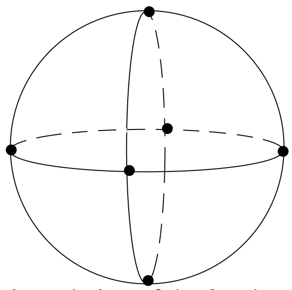
Exercise 2.11 Prove this.

We find that $\delta^2 = 0$ (Exercise). Note that this doesn't mean that the double boundary is empty, merely that in its expression as formal sum, all the coefficients are 0. We define the k-th homology group $H_k(M)$ to be

$$\frac{\ker\{\delta : C_k(M) \to C_{k-1}(M)\}}{\text{im}\{\delta : C_{k+1}(M) \to C_k(M)\}}.^1$$

[1] Every continuous singular k-simplex in a smooth manifold can be well-enough approximated by a smooth singular k-simplex, so this group coincides with the group defined in terms of continuous singular k-simplices

Every compact smooth manifold M can be finitely triangulated; that is, can be subdivided into a finite number of simplices. Any such triangulation $\mathbf{\Delta}$ gives rise to a singular chain $\sigma_{\mathbf{\Delta}} \in C_k(M)$, and in $C_k(N)$ if M is a smooth submanifold of N. If M is a manifold without boundary, then $\delta(\sigma_{\mathbf{\Delta}}) = 0$.



Triangulation of the 2-sphere with eight 2-simplices. The black dots are vertices of the 2-simplices, and the arcs are their edges.

Exercise: Regarding the torus as a square with opposite sides identified, find a triangulation of the 2-torus, in which each 2-simplex is isometric to a right-angled isosceles plane triangle.

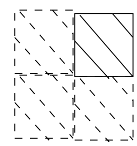


Figure 6

Exercise 2.12 Check that in each of these two cases, the boundary of the singular 1-chain $\sigma_{\mathbf{\Delta}}$ is equal to 0.

Two k-chains which differ by a boundary are said to be *homologous*. Stokes's theorem tells us that for any $\omega \in \Omega^k(M)$, $\int_{\delta c} \omega = \int_c d\omega$, and therefore

1. if $d\omega = 0$ then $\int_{\delta c} \omega = 0$. and
2. if $\delta c = 0$ then $\int_c (d\omega) = 0$,

As a consequence, if ω is closed and c_1 and c_2 are homologous then $\int_{c_1} \omega = \int_{c_2} \omega$, and if $\delta c = 0$ and ω_1 and ω_2 are cohomologous (i.e. differ by $d\rho$ for some ρ) then $\int_c \omega_1 = \int_c \omega_2$. Thus, integration descends to a pairing

$$H_k(M) \times H^k(M) \to \mathbb{R}, \quad ([c], [\omega]) \mapsto \int_c \omega.$$

The *de Rham Theorem* (which we will not prove directly) says that by means of this pairing

$$H^k(M) = \mathrm{Hom}_{\mathbb{Z}}(H_k(M), \mathbb{R})$$

and

$$H_k(M) \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{Hom}_{\mathbb{R}}(H^k(M), \mathbb{R}).$$

Note that if our singular chains have coefficients in $\mathbb{R}$ instead of in $\mathbb{Z}$, these two formulae simplify slightly.

Remark 2.13 If $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$ are any two triangulations of a manifold M^n, then the two n-chains $\sigma_{\mathbf{\Delta}_1}$ and $\sigma_{\mathbf{\Delta}_2}$ are homologous. Thus they represent the same homology class in $H^n(M)$ (and in $H^n(N)$ if $M \subset N$), the *fundamental class* of M.

Exercise 2.14 The integral of a holomorphic function along a closed curve in $\mathbb{C}$ can be viewed, in real terms (i.e. interpreting $\mathbb{C}$ as $\mathbb{R}^2$) as the integral of a pair of 1-forms along the curve. Use Stokes's Theorem, and the Cauchy-Riemann equations, to show that
(i) if C is a simple closed curve enclosing a domain $U \subset \mathbb{C}$ in which the function f is holomorphic, then $\int_C f(z)dz = 0$;
(ii) if C_1 and C_2 are simple closed curves which together make up the boundary of a region within which f is holomorphic, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

What is the most general statement of this type that you can make, using Stokes's Theorem and the Cauchy-Riemann equations?

The Poincaré Lemma

We now veer from the impressionistic to the technical, and prove a result which turns out to explain, in some sense, why de Rham cohomology is the same as other standard cohomology theories, such as singular cohomology.

Theorem 2.15 *(The Poincaré Lemma)*

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof We use a lemma:

Lemma 2.16 *Let $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be projection, and let $s : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}$ be the inclusion $s(x) = (x, 0)$. Then for each k, $s^* : H^k(\mathbb{R}^n \times \mathbb{R}) \to H^k(\mathbb{R}^n)$ and $\pi^* : H^k(\mathbb{R}^n) \to H^k(\mathbb{R}^n \times \mathbb{R})$ are mutually inverse isomorphisms.*

Proof Since $\pi \circ s = \mathrm{id}_{\mathbb{R}^n}$, it follows from functoriality that $s^* \circ \pi^*$ is the identity on $H^k(\mathbb{R}^n)$. The other equality, $\pi^* \circ s^* = \mathrm{id}^*_{\mathbb{R}^n \times \mathbb{R}}$ is not obvious. To prove it, we construct a *chain homotopy* between the maps (of complexes) $\pi^* \circ s^* : \Omega^\bullet(\mathbb{R}^n \times \mathbb{R}) \to \Omega^\bullet(\mathbb{R}^n \times \mathbb{R})$ and the identity map on the same complex. That is, we construct a family of maps $K : \Omega^k(\mathbb{R}^n \times \mathbb{R}) \to \Omega^{k-1}(\mathbb{R}^n \times \mathbb{R})$ such that

$$1 - \pi^* \circ s^* = \pm(dK \pm Kd).$$

For in that case, if $\omega \in \Omega^k(M)$ is closed then $(1 - \pi^* \circ s^*)(\omega) = \pm d(K(\omega))$ and so is zero in cohomology, and thus $1 - \pi^* \circ s^*$ is the zero map on $H^k(\mathbb{R}^n \times \mathbb{R})$.

To construct K, note that every form on $\mathbb{R}^n \times \mathbb{R}$ is a linear combination (over $\mathbb{R}$) of forms of the following two kinds:
(1) $\pi^*(\phi)f(x, t)$

(2) $\pi^*(\phi)f(x,t) \wedge dt$
where ϕ is a form on $\mathbb{R}^n$. We define K by
(1) $K(\omega) = 0$ if ω is of type (1), and
(2) if $\omega = \pi^*(\phi)f(x,t) \wedge dt$ is a form of type (2), then $K(\omega) = \pi * (\phi) \int_0^t f(x,u)du$
and extending by $\mathbb{R}$-linearity.

If ω is a q-form of type (1), we have $(1 - \pi^* \circ s^*)(\omega) = \pi^*(\phi)f(x,t) - \pi^*(\phi)f(x,0)$, while

$$(dK - Kd)(\omega) = -Kd\omega = -K\left\{d\pi^*(\phi)f(x,t) + (-1)^q\pi^*(\phi) \wedge \left\{\sum_i \frac{\partial f}{\partial x_i}dx_i + \frac{\partial f}{\partial t}dt\right\}\right\}$$

$$= -K\left\{\pi^*(d\phi)f(x,t) + (-1)^q \sum_i \pi^*(\phi \wedge dx_i)\frac{\partial f}{\partial x_i} + (-1)^q\pi^*(\phi)\frac{\partial f}{\partial t} \wedge dt\right\}$$

$$= (-1)^q pi^*(\phi)\int_0^t \frac{\partial f}{\partial u}du$$

$$= (-1)^q\pi^*(\phi)\{f(x,t) - f(x,0)\}.$$

Thus, up to sign, $dK - Kd$ and $1 - \pi^* \circ s^*$ agree on forms of type (1).

An equally straightforward calculation (Exercise) proves the result also for forms of type (2). □

The Poincaré Lemma now follows by induction, from e.g. a calculation of $H^*(\mathbb{R})$. □

The same method of proof shows

Corollary 2.17 *For any manifold M, the projection $\pi : M \times \mathbb{R} \to M$ and the zero-section $M \to M \times \mathbb{R}$ induce mutually inverse isomorphisms on cohomology.* □

Corollary 2.18 *(Homotopy invariance of de Rham cohomology) If f and g are smoothly homotopic smooth maps from M to N, then f^* and g^* agree on cohomology.*

Proof Given a smooth homotopy $F : M \times \mathbb{R} \to N$, with $f(x) = F(x,0)$ and $g(x) = F(x,1)$, it follows that $f^* = s_0^* \circ F^*$ and $g^* = s_1^* \circ F^*$, where $s_0(x) = (x,0)$ and $s_1(x) = (x,1)$. Both s_0^* and s_1^* are inverse isomorphisms to π^*, and thus coincide. Hence $f^* = g^*$. □

Corollary 2.19 *If $f : M \to N$ is a smooth homotopy equivalence, then f^* is an isomorphism on cohomology.* □

Remark 2.20 By functoriality, de Rham cohomology is a diffeomorphism- invariant. Every point on an n-manifold M has arbitrarily small neighbourhoods U diffeomorphic to $\mathbb{R}^n$. It follows from the Poincaré Lemma that inside such a neighbourhood, every closed form is exact, or, in other words, the complex

$$0 \to \mathbb{R} \to \Omega^0(U) \to \Omega^1(U) \to \cdots \to \Omega^n(U) \to 0$$

is exact. In case you are familiar with sheaf theory, this means that the complex of sheaves of germs of differential forms on M, $\Omega^\bullet_M$, is a *resolution* of the sheaf $\mathbb{R}_M$:

$$0 \to \mathbb{R}_M \to \Omega^0_M \to \Omega^1_M \to \cdots \to \Omega^n_M \to 0$$

is an exact sequence of sheaves.

The Mayer Vietoris Sequence

Every n-manifold can be put together from pieces diffeomorphic to $\mathbb{R}^n$. Where the number of these pieces is finite, the Mayer-Vietoris sequence in principle gives a way of calculating $H^*(M)$. To obtain it, we begin by constructing a short exact sequence of complexes.

Suppose that U_1 and U_2 are open subsets of the n-manifold M. We have a commutative diagram

$$\begin{array}{ccccc} & & U_1 & & \\ & i_1 \nearrow & & \searrow j_1 & \\ U_1 \cap U_2 & & & & U_1 \cup U_2 \\ & i_2 \searrow & & \nearrow j_2 & \\ & & U_2 & & \end{array}$$

from which we derive an evidently exact sequence

$$0 \to \Omega^k(U_1 \cup U_2) \stackrel{(j_1^*, j_2^*)}{\longrightarrow} \Omega^k(U_1) \oplus \Omega^k(U_2) \stackrel{i_1^* - i_2^*}{\longrightarrow} \Omega^k(U_1 \cap U_2).$$

Lemma 2.21 $i_1^* - i_2^*$ *is surjective.*

Proof Choose a partition of unity subordinate to the open cover $\{U_1, U_2\}$ of $U_1 \cup U_2$. That is, choose ϕ_1 and ϕ_2, smooth functions on $U_1 \cup U_2$, such that supp $\phi_i \subset U_i$ for $i = 1, 2$, and such that $\phi_1 + \phi_2$ is identically equal to 1 on $U_1 \cup U_2$. Given a form $\omega \in \Omega^k(U_1 \cap U_2)$, the form $\phi_1\omega$ can be smoothly extended to give a form on all of U_2 by declaring it equal to 0 on $U_2 \setminus U_1 \cap U_2$, and similarly $\phi_2\omega$ extends to a form on U_1. Regarding $\phi_1\omega$ and $\phi_2\omega$ in this way as forms on U_2 and U_1, we have $\omega = \phi_1\omega + \phi_2\omega = i_1^*(\phi_2\omega) - i_2^*(-\phi_1\omega)$, and this proves surjectivity. □

So the exact sequence can be augmented by adding a $\to 0$ on the right. We thus have a short exact sequence for each k; as is easily seen, these piece together to give a short exact sequence of *complexes*

$$0 \to \Omega^\bullet(U_1 \cup U_2) \stackrel{(j_1^*, j_2^*)}{\longrightarrow} \Omega^\bullet(U_1) \oplus \Omega^\bullet(U_2) \stackrel{i_1^* - i_2^*}{\longrightarrow} \Omega^\bullet(U_1 \cap U_2) \to 0$$

(that is, the morphisms shown commute with the exterior derivatives in the complexes).

Proposition 2.22 *A short exact sequence* $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ *of complexes gives rise to a long exact sequence of cohomology*

$$\cdots \to H^k(A^\bullet) \to H^k(B^\bullet) \to H^k(\mathcal{C}^\bullet) \stackrel{d^*}{\longrightarrow} H^{k+1}(A^\bullet) \to \cdots.$$

Proof The morphisms $A^k \to B^k$ and $B^k \to C^k$ descend to morphisms of cohomology $H^k(A^\bullet) \to H^k(B^\bullet)$ and $H^k(B^\bullet) \to H^k(C^\bullet)$ because they commute with the differentials in the three complexes. And exactness at $H^k(B)$ is easy to prove.

The only mystery is the definition of d^*, which is constructed by a diagram chase. Begin with an element of $H^k(C^\bullet)$, which you can represent by an element c_k of of C^k such that $d_C(c_k) = 0$. By the surjectivity of the map $B^k \to C^k$, there is a $b_k \in B_k$ mapping to c_k. By commutativity, the image in C^{k+1} of $d_B(b_k)$ is 0; by exactness of the sequence of complexes, there exists $a_{k+1} \in A^{k+1}$ mapping to $d_B(b_k)$. One proves:

- $d_A(a_{k+1}) = 0$, so a_{k+1} defines a cohomology class in $H^{k+1}(A^\bullet)$, and
- this class is independent of all of the choices made in its construction, so we have a well-defined map $d^* : H^k(C^\bullet) \to H^{k+1}(A^\bullet)$.
- d^* is linear, because the choices made in the construction of a_{k+1} can be made linearly.

The proof of exactness at the other spots in the sequence is then straightforward, and fun. □

The long exact sequence arising from the short exact sequence of complexes

$$0 \to \Omega^\bullet(U_1 \cup U_2) \xrightarrow{(j_1^*, j_2^*)} \Omega^\bullet(U_1) \oplus \Omega^\bullet(U_2) \xrightarrow{i_1^* - i_2^*} \Omega^\bullet(U_1 \cap U_2) \to 0$$

is called the *Mayer-Vietoris sequence.*

Exercise 2.23 Go through the construction of d^* in the Mayer-Vietoris sequence.

Exercise 2.24 Prove the rank theorem: if $0 \to A \to B \to C \to 0$ is an exact sequence of finite dimensional vector spaces and linear maps, then $\dim B = \dim A + \dim C$. Hint: choose bases $a_1, \ldots, a_n$ for A and $c_1, \ldots, c_m$ for C. Use them to get a basis for B.

Exercise 2.25 Show that if $0 \to A^1 \to \cdots \to A^n \to 0$ is an exact sequence of finite-dimensional vector spaces and linear maps, then $\sum_i (-1)^i \dim A^i = 0$. Hint: break the long exact sequence up into a collection of short exact sequences, and apply the rank theorem.

Exercise 2.26 Use the Mayer-Vietoris sequence to calculate the cohomology of the circle S^1 (Hint: Cover S^1 with open sets U_1 and U_2 each diffeomrphic to $\mathbb{R}$). By following the construction of d^* in Mayer- Vietoris, give as precise a description as you can of a non-zero element of $H^1(S^1)$.

Exercise 2.27 Use the Mayer-Vietoris sequence, and the previous exercise, to calculate the cohomology of the sphere S^2, and, inductively, of the sphere S^n.

Exercise 2.28 Show that the inclusion $S^n \hookrightarrow \mathbb{R}^n \setminus \{0\}$ is a homotopy-equivalence and deduce that it induces an isomorphism in cohomology.

Exercise 2.29 Calculate the cohomology of $\mathbb{R}^n \setminus \{P_1, P_2\}$, where P_1 and P_2 are any two points. Generalise to k points.

Exercise 2.30 For future use we need the following statement, the *Five Lemma*: suppose that in the following diagram of abelian groups and homomorphisms, the rows are exact and all the vertical maps except possibly for ϕ_k are ismorphisms.

$$\begin{array}{ccccccccccccc}
\cdots \to & A^{k-2} & \to & A^{k-1} & \to & A^k & \to & A^{k+1} & \to & A^{k+2} & \to \cdots \\
& \uparrow \phi_{k-2} & & \uparrow \phi_{k-1} & & \uparrow \phi_k & & \uparrow \phi_{k+1} & & \uparrow \phi_{k+2} & \\
\cdots \to & B^{k-2} & \to & B^{k-1} & \to & B^k & \to & B^{k+1} & \to & B^{k+2} & \to \cdots
\end{array}$$

Then ϕ_k also is an isomorphism.

Two Applications

Theorem 2.31 *On the sphere S^n there is a nowhere-vanishing vector field if and only if n is odd.*

Proof
Step 1: If v is a nowhere vanishing vector field on S^n then after dividing by its length, we can assume $\|v(x)\| = 1$ for all x. The map

$$F(x,t) = \cos(\pi t)x + \sin(\pi t)v(x)$$

is then a homotopy between the identity map and the antipodal map $a(x) = -x$.
Step 2: In $H^n(S^n)$, the class of the volume form is not zero, by Stokes's Theorem (Exercise). But if n is even, $a^*(\mathrm{vol}_{S^n}) = -\mathrm{vol}_{S^n}$ (Exercise). This means that a cannot be homotopic to the identity, since homotopic maps induce the same morphism on cohomology. □

Theorem 2.32 *Brouwer's Fixed Point Theorem: Any continuous map of the unit ball $D^{n+1} \subset \mathbb{R}^{n+1}$ to itself has a fixed point.*

Proof First assume $f : D^{n+1} \to D^{n+1}$ is smooth, and has no fixed point. Define a smooth map $r : D^{n+1} \to S^n$ by mapping each x to the point where the line-segment $f(x)$ to x, continued, meets the boundary, S^n. Clearly r is the identity map on S^n; in other words, if $i : S^n \to D^{n+1}$ is inclusion, we have $r \circ i = \mathrm{id}_{S^n}$. It follows that the composite

$$H^n(S^n) \xrightarrow{r^*} H^n(D^{n+1}) \xrightarrow{i^*} H^n(S^n)$$

is the identity on $H^n(S^n)$. But $H^n(\mathbb{R}^{n+1}) = 0$ and $H^n(S^n) \neq 0$ so this is impossible.

If we assume that the *continuous* map $f : D^{n+1} \to D^{n+1}$ has no fixed point, then by compactness there exists $\varepsilon > 0$ such that for all $x \in D^{n+1}$, $\|f(x) - x\| \geq \varepsilon$. We can approximate f by a smooth map $g : D^{n+1} \to D^{n+1}$ such that $\|f(x) - g(x)\| < \varepsilon$ for all x (how?); it follows that g also has no fixed point, a contradiction. □

3 Compactly Supported Cohomology and Poincaré Duality

Heuristic Introduction

If C_1 and C_2 are two oriented closed curves on the oriented 2-torus T^2, we can assign them an *intersection index* $C_1 \cdot C_2 \in \mathbb{Z}$ (after slightly shifting one to make them transverse to one another, if they are not transverse to start with), and then counting their intersection points, with sign, as follows: let $\hat{c}_i$ be a positive basis for T_xC_i, $i = 1, 2$; then

$$(C_1 \cdot C_2)_x = \begin{cases} \;\;1 & \text{if } \hat{c}_1, \hat{c}_2 \text{ is a positive basis for } T_xT^2 \\ -1 & \text{if } \hat{c}_1, \hat{c}_2 \text{ is a negative basis for } T_xT^2 \end{cases}$$

(This "slight shifting" needs some justification: we really mean a homotopy of the embedding $C_i \hookrightarrow T^2$. Any two "slight shiftings" of the same curve C_i are homotopic to one another, and thus have the same intersection index with any curve they are both transverse to.)

The set of all oriented closed curves can be made into an abelian group $\mathcal{C}(T^2)$ by taking formal sums with integer coefficients. The pairing on curves extends to a pairing on formal sums of curves, in the obvious way:

$$\sum_i n_i C_i \;\cdot\; \sum_j m_j D_j = \sum_{i,j} n_i m_j C_i \cdot D - j.$$

One checks that it is skew-symmetric.

Like any bilinear pairing, this pairing gives rise to a *duality map*

$$\mathcal{C}(T^2) \to \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(T^2), \mathbb{Z}),$$

sending C to $C\cdot$ and $\sum_i C_i$ to $\sum_i C_i\cdot$. This map is not injective, however. **Exercise** Find a closed curve $C \subset T^2$ such that for every closed curve $C' \subset T^2$, $C \cdot C' = 0$

In order to get an injective duality map, we have to kill elements of $\mathcal{C}(T^2)$ whose intersection index with every curve is 0. We can do this by imposing an equivalence relation on formal sums of closed curves: for example, homotopy. However, we get a better result if we impose a still weaker equivalence relation, that of homology. We may as well go straight to the point: instead of $\mathcal{C}(T^2)$ we consider the group $H_1(T^2; \mathbb{Z})$.

Poincaré observed that the duality map

$$H_1(T^2; \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_1(T^2; \mathbb{Z}), \mathbb{Z})$$

induced by the intersection pairing is an isomorphism, and for that reason it, and its generalisation to other compact manifolds and other dimensions (i.e. not just curves), is known as *Poincaré duality*.

If M is a compact n-dimensional manifold, there is a well-defined intersection pairing

$$H_k(M; \mathbb{Z}) \times H_{n-k}(M; \mathbb{Z}) \to \mathbb{Z};$$

given homology classes $[c_k]$ and $[c_{n-k}]$ it is possible to represent them by chains c_k and c_{n-k} which are in "general position" with respect to one another, and then count intersection points, with sign, much as we did for closed curves on the torus. This pairing induces a duality map

$$H_k(M; \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_{n-k}(M; \mathbb{Z}), \mathbb{Z});$$

but in general this is not an isomorphism. The problem is the existence of *torsion elements* in homology. A torsion element is a non-zero homology class $[c]$ such that for some integer

$m \neq 0$, $m[c] = 0$. **Exercise** Suppose that $[c] \in H_k(M)$ is a torsion element. Show that for any $[c_{n-k}] \in H_{n-k}(M;\mathbb{Z})$, we have $[c_k] \cdot [c_{n-k}] = 0$.

A familiar example of torsion element can be found in $H_1(\mathbb{RP}^2;\mathbb{Z})$; if we think of $\mathbb{RP}^2$ as the quotient of the 2-sphere S^2 by the equivalence relation identifying antipodal points, then we can represent a non-zero element of $H_1(\mathbb{RP}^2;\mathbb{Z})$ by a half-circle in S^2 joining a pair of antipodal points (so that it becomes a closed curve in $\mathbb{RP}^2$). One can think of this curve as the central circle of the Möbius strip; it is well known that cutting the strip along its central circle does not disconnect it, but that cutting it along a curve which winds twice around the strip does disconnect it. This means that this twice-winding curve is the boundary of a 2-chain — either one of the two halves into which it disconnects the Möbius strip will do.

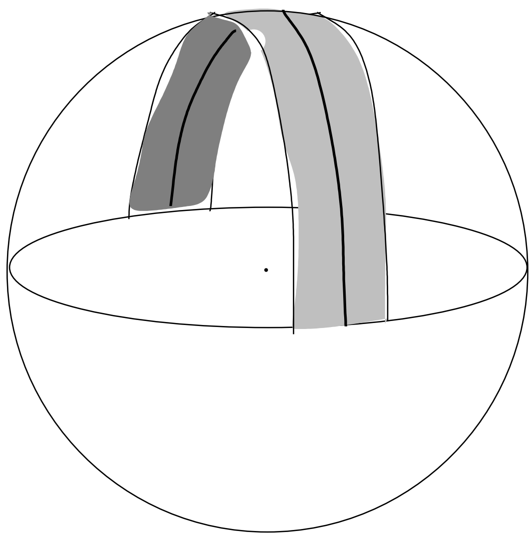


Figure 7: The image of the shaded band in $\mathbb{RP}^2$ is a Mobius strip; its central circle (the image of the thick black curve) is non-zero in $H_1(\mathbb{RP}^2$ - in fact it's a generator.

In order for the duality map to be an isomorphism, it turns out to be necessary to kill torsion, by taking coefficients in $\mathbb{Q}$ rather than in $\mathbb{Z}$, or, equivalently, by tensoring the homology groups $H_*(M;\mathbb{Z})$ with $\mathbb{Q}$.

Theorem 3.1 *Homological Poincaré Duality: if M^n is a compact oriented manifold then the intersection pairing $H_k(M;\mathbb{Q}) \times H_{n-k}(M;\mathbb{Q}) \to \mathbb{Q}$ gives rise to an isomorphism*

$$H_k(M;\mathbb{Q}) \simeq Hom(H_{n-k}(M;\mathbb{Q}),\mathbb{Q}).$$

□

Poincaré Duality in de Rham Cohomology

An apparently quite different duality arises from the wedge product of differential forms: if M^n is compact and oriented, we get a pairing

$$H^k(M) \times H^{n-k}(M) \to \mathbb{R}$$

$$([\omega_1], [\omega_2]) \mapsto \int_M \omega_1 \wedge \omega_2.$$

Exercise Show that this pairing is well-defined.
It turns out that this is closely related to the intersection form in homology. It is this version of Poincaré Duality that we will study in detail. Later we will see how it corresponds to homological Poincaré duality.

Our proof of (cohomological) Poincaré duality for compact manifolds will go by induction on the number of contractible open sets necessary to cover the manifold, using the Mayer Vietoris sequence. However, in the course of assembling a compact manifold from open sets, one has a non-compact manifold until the final step. In order for an inductive proof to be possible, we therefore need a version of Poincaré duality which holds on non-compact manifolds. The key is to consider a special class of differential forms, the class of *compactly supported* differential forms. A form $\omega \in \Omega^k(M)$ is compactly supported if outside some compact set $X \subset M$ it is identically zero. Even if M is not compact, one can integrate a compactly supported form over it; thus there is a pairing

$$\Omega^k(M) \times \Omega_c^{n-k}(M) \to \mathbb{R}$$

$$(\omega_1, \omega_2) \mapsto \int_M \omega_1 \wedge \omega_2.$$

Here $\Omega_c^{n-k}(M)$ denotes the vector space of all compactly supported $n-k$-forms.

If $\omega \in \Omega_c^j(M)$ then $d\omega \in \Omega_c^{j+1}(M)$, and thus $(\Omega_c^\bullet(M), d)$, is a subcomplex of the de Rham complex $(\Omega^\bullet(M), d)$. Its cohomology groups are the *compactly supported cohomology groups* of M, and are denoted $H_c^k(M)$. Note that although $(\Omega_c^\bullet(M), d)$ is a subcomplex of $(\Omega^\bullet(M), d)$, it is *not* in general the case that the compactly supported cohomology space $H_c^k(M)$ is a subspace of $H^k(M)$.

Exercise Why not?

Exercise Compute $H_c^0(\mathbb{R})$ and $H_c^1(\mathbb{R})$

Of course, if M is compact then every form is compactly supported, so the rings $H_c^*(M)$ and $H^*(M)$ coincide.

Theorem 3.2 *Poincaré Duality in de Rham Cohomology. The integration pairing $H^k(M) \times H_c^{n-k}(M) \to \mathbb{R}$ induces an isomorphism $H^k(M) \to (H_c^{n-k}(M))^*$.* □

Warning: the pairing also induces a map $H_c^{n-k}(M) \to (H^k(M))^*$; however, this is not necessarily an isomorphism.

Exercise (i) Show that if a bilinear pairing $V_1 \times V_2 \to \mathbb{R}$ of finite dimensional vector-spaces induces an isomorphism $V_1 \to V_2^*$ then it also induces an isomorphism $V_2 \to V_1^*$
(ii) Conclude that if $H^k(M)$ is finite dimensional then the integration pairing induces an isomorphism $H_c^{n-k}(M) \to (H^k(M))^*$.
(iii) Give an example of vector spaces and a pairing $V_1 \times V_2 \to \mathbb{R}$ inducing one isomorphism but not the other. Hint: take V_1 to be a suitable infinite dimensional vector space and V_2 to be V_1^*.
(iv) Give an example of a manifold M such that $H_c^{n-k}(M) \to H^k(M)^*$ is not an isomorphism.

Before beginning the proof of 3.2, we note an important consequence:

Corollary 3.3 *If M is a connected oriented n-manifold, then $H_c^n(M) \simeq \mathbb{R}$. If M is also compact, then $H^n(M) \simeq \mathbb{R}$.*

Proof $H^0(M) = \mathbb{R}$. □

Exercise 3.4 Show that if M is connected, oriented and n-dimensional but is not compact then $H^n(M) = 0$.

Compactly Supported Forms and Cohomology

Our proof of Theorem 3.2 will be by induction on the number of open sets in a cover, using the Mayer-Vietoris sequence in the inductive step. Our first step will be to prove it for $M = \mathbb{R}^n$. This is another reason why it is useful to introduce compactly supported cohomology: if we were trying to prove an assertion valid only for compact manifolds, we couldn't use induction beginning with something non-compact like $\mathbb{R}^n$.

Exercise 3.5 Suppose that M is an oriented n-dimensional manifold without boundary and let $\omega \in \Omega_c^n(M)$ be a compactly supported n-form on M, such that $\int_M \omega \neq 0$. Show that $[\omega] \neq 0$ in $H_c^n(M)$.

Since $H^k(\mathbb{R}^n) = \mathbb{R}$ if $k = 0$ and is 0 otherwise, we have to prove that $H_c^k(\mathbb{R}^n) = \mathbb{R}$ if $k = n$ and is 0 otherwise. Not surprisingly, we need a version of the Poincaré Lemma for compactly supported cohomology. This will be somewhat different from the previous version. For a start, the dimension in which the compactly supported cohomology of $\mathbb{R}^n$ is non-zero changes with n (it *is* n) and so in place of the isomorphism $H^*(\mathbb{R}^n) \simeq H^*(\mathbb{R}^n \times \mathbb{R})$, we look for an ismorphism $H_c^*(\mathbb{R}^n) \simeq H_c^{*+1}(\mathbb{R}^n \times \mathbb{R})$. In fact at no extra cost we can prove this for an arbitrary oriented manifold M.

We construct a map $H_c^{k-1}(M) \to H_c^k(M \times \mathbb{R})$ as follows: let e be a compactly supported 1-form on $\mathbb{R}$ such that $\int_{\mathbb{R}} e = 1$. By the exercise above, $[e] \neq 0$ in $H_c^1(\mathbb{R})$ (and in fact it generates $H_c^1(\mathbb{R})$). Let $\pi : M \times \mathbb{R} \to M$ and $\rho : M \times \mathbb{R} \to \mathbb{R}$ be projections; we define $e_* : \Omega_c^{k-1}(M) \to \Omega_c^k(M \times \mathbb{R})$ by

$$e_*(\omega) = \pi^*(\omega) \wedge \rho^*(e)$$

(which we will write simply as $e_*(\omega) = \omega \wedge e$.) This map descends to a map on cohomology (**Exercise**), which we also denote e_*, $H_c^{k-1}(M) \to H_c^k(M \times \mathbb{R})$. We will show that it is an isomorphism.

Exercise Show, using iterated integration, that if $\int_M \omega \neq 0$ then $\int_{M\times\mathbb{R}} e_*(\omega) \neq 0$. In fact we will not need to use the result of this exercise in our proof.

We construct the cohomological inverse to e_* by *integration along the fibre*. For each k define

$$\pi_* : \Omega_c^k(M \times \mathbb{R}) \to \Omega_c^{k-1}(M)$$

by :

(1) if $\omega = \pi^*(\phi) f(x,t)$ we set $\pi_*(\omega) = 0$,
(2) if $\omega = \pi^*(\phi) f(x,t) \wedge dt$, then $\pi_*(\omega) = \phi \int_{-\infty}^{\infty} f(x,t)dt$.

Every form is a sum of forms of these two types; we extend π_* linearly. Note that in $\Omega_c^0(M\times\mathbb{R})$ there are only forms of type (1), and so here $\pi_* = 0$.

Exercise 3.6 Show that $d\pi_* = \pi_* d$ (so that π_* induces a morphism $H_c^k(M\times\mathbb{R}) \to H_c^{k-1}(M)$).

It is easy to see that $\pi_* \circ e_* = 1$ on $H_c^{k-1}(M)$. To show that $e_* \circ \pi_* = 1$ on $H_c^k(M \times \mathbb{R})$, as in the proof of the previous version of the Poincaré Lemma we construct a homotopy operator, $K : \Omega_c^\bullet(M \times \mathbb{R}) \to \Omega_c^{\bullet-1}(M)$.

It is defined as follows:

(1) If $\omega = \pi^*(\phi) f(x,t)$ then $K(\omega) = 0$
(2) If $\omega = \pi^*(\phi) f(x,t) \wedge dt$ then

$$K(\omega) = \pi^*(\phi)\left\{\int_{-\infty}^{t} f(x,u)du - A(t)\int_{-\infty}^{\infty} f(x,u)du\right\}$$

where $A(t) = \int_{-\infty}^{t} e$.

Lemma 3.7 *$1 - e_* \circ \pi_* = (-1)^{k-1}(dK - Kd)$ on $\Omega_c^k(M \times \mathbb{R})$.*

Proof **Exercise** (see Bott and Tu pages 38-39). □

Proposition 3.8 *The maps $\pi_* : H_c^k(M \times \mathbb{R}) \to H_c^{k-1}(M)$ and $e_* : H_c^{k-1}(M) \to H_c^k(M \times \mathbb{R})$ are mutually inverse isomorphisms.* □

Corollary 3.9

$$H_c^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

□

Exercise (i) Find a generator for $H^n_c(\mathbb{R}^n)$.
(ii) Show that if $U \subset \mathbb{R}^n$ is any open set, it is possible to choose a generator ω for $H^n_c(\mathbb{R}^n)$ with $\mathrm{supp}(\omega) \subset U$.

Exercise Prove Poincaré Duality (3.2) for $\mathbb{R}^n$.

Good Covers

An open cover $\{U_\alpha\}_{\alpha \in A}$ of the n-manifold M is *good* if for every choice $\alpha_0, \ldots, \alpha_k \in A$, $U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$ is diffeomorphic to $\mathbb{R}^n$. Such covers are also sometimes called *acyclic* covers. They are important in all cohomology theories; one can think of them as a way of assembling M out of cohomologically trivial pieces, so that in computing the cohomology of M, only the combinatorics of the cover (i.e. which open sets intersect which) intervenes.

Lemma 3.10 *If $\{V_\beta\}_{\beta \in B}$ is any open cover of M, there is a good cover $\{U_\alpha\}_{\alpha \in A}$ such that each U_α is contained in some V_β.*

The cover $\{U_\alpha\}_{\alpha \in A}$ is a *refinement* of the cover $\{V_\beta\}_{\beta \in B}$ if every U_α is contained in some V_β; so the lemma says that every cover has a good refinement. This property of open covers is described by saying that they are *cofinal* in the set of all open covers of M, partially ordered by refinement.
Proof of 3.10 If $M = \mathbb{R}^n$, any cover all of whose members is convex is good since the intersection of an arbitrary family of convex open sets is a convex open set, and hence diffeomorphic to $\mathbb{R}^n$; such covers are evidently cofinal in the set of all covers. Convexity is a metric property — it involves the notion of straight line — and to use the same idea on a manifold, we endow it with a Riemannian metric, which allows us to speak of *geodesics*, the Riemannian equivalent of straight lines. A set U in a Riemannian manifold is *geodesically convex* if for every two points in U there is a unique geodesic in M of minimal length joining them, and moreover this geodesic is entirely contained in U.

The intersection of any collection of geodesically convex sets is also geodesically convex. So to obtain a good refinement of an open cover $\{V_\beta\}_{\beta \in B}$, it is enough to choose, for every point in M, some geodesically convex neighbourhood, small enough to be contained in one of the V_β. A theorem of Riemannian geometry assures us that this can be done: if $x \in M$ is any point and V is any open neighbourhood of x, there is a geodesically convex open neighbourhood U of x contained in V. □

Note that any sub-cover of a good cover is also good. Thus, every compact manifold has a finite good cover.

From 3.10 we easily deduce (by way of practice in the technique):

Theorem 3.11 *If the manifold M has a finite good cover, then its de Rham cohomology spaces are all finite dimensional.*

Proof Induction, using Mayer Vietoris: let $U_1, \ldots, U_m$ be a finite good cover of M, and for each $k = 1, \ldots, m$ let $M_k = U_1 \cup \cdots \cup U_k$. Clearly the de Rham cohomology of any manifold diffoemorphic to $\mathbb{R}^n$ is finite dimensional; this is the start of our induction.

Suppose that the de Rham cohomology of M_k is finite dimensional; we now compare it with the de Rham cohomology of $M_{k+1} = M_k \cup U_{k+1}$ using the Mayer-Vietoris long exact sequence. This made up of 3-term segments

$$H^q(M_k \cap U_{k+1}) \xrightarrow{d^*} H^{q+1}(M_{k+1}) \xrightarrow{(j_1^*, j_2^*)} H^{q+1}(M_k) \oplus H^{q+1}(U_{k+1}).$$

From this we obtain the short exact sequence

$$0 \to \ker d^* \to H^{q+1}(M_{k+1}) \to \mathrm{im}(j_1^* + j_2^*) \to 0.$$

In a short exact sequence, finite dimnesionality of any two of the three non-trivial spaces implies finite dimensionality of the third.

Besides M_{k+1}, the short exact sequence here involves the three spaces M_k, U_{k+1} and $M_k \cap U_{k+1}$. Do we know enough about the cohomology of all three to conclude that the middle term is finite dimensional? In order to reach this conclusion, we must be careful to choose the right inductive hypothesis: it is not simply that the de Rham cohomology of M_k is finite dimensional, but that *every manifold having a good cover consisting of no more than k open sets has finite dimensional de Rham cohomology*, since this implies that the cohomology of M_k *and of* $M_k \cap U_{k+1}$ is finite-dimensional. This is evidently true when $k = 1$; in view of the short exact sequence we derived from Mayer Vietoris, if true for k it is true for $k+1$. So the proof is complete. □

Now we proceed with the proof of Poincaré duality, for manifolds having a finite good cover. This involves comparing two Mayer-Vietoris sequences, one for ordinary de Rham cohomology and one for compactly supported cohomology.

First we develop the Mayer-Vietoris sequence for compactly supported cohomology. Recall the commutative diagram

$$\begin{array}{ccccc} & & U_1 & & \\ & i_1 \nearrow & & \searrow j_1 & \\ U_1 \cap U_2 & & & & U_1 \cup U_2 \\ & i_2 \searrow & & \nearrow j_2 & \\ & & U_2 & & \end{array}$$

Inclusion of *open* sets makes possible a new morphism: push-forward. If $i : U \to V$ is inclusion, define $i_* : \Omega^k(U) \to \Omega^k(V)$ (note: this is covariant, not contravariant) by "extending by zero". That is, if $\omega \in \Omega^k_c(U)$ then outside some compact $K \subset U$, ω is identically zero. Thus we can define a form $i_*(\omega)$ on the bigger open set V by setting $i_*(\omega)$ to be 0 at every point in $V \setminus U$.

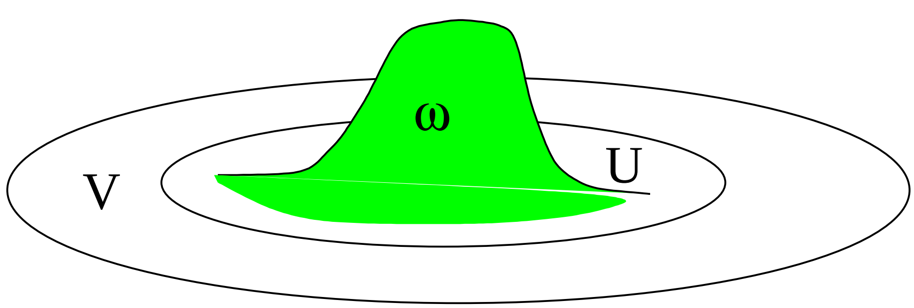


Figure 8

It is now easy to see, as in the case of non-compactly supported forms, that the following sequence is exact:

$$0 \to \Omega_c^k(U_1 \cap U_2) \stackrel{(-i_{1*}, i_{2*})}{\longrightarrow} \Omega_c^k(U_1) \oplus \Omega_c^k(U_2) \stackrel{j_{1*}+j_{2*}}{\longrightarrow} \Omega_c^k(U_1 \cup U_2) \to 0.$$

The last arrow needs a little clarification: if $\omega \in \Omega_c^k(U_1 \cup U_2)$, choose $\phi_1 \in \Omega_c^0(U_1)$ and $\phi_2 \in \Omega_c^0(U_2)$ such that (if we extend each of them by 0 to functions on all of $U_1 \cup U_2$)) $\phi_1 + \phi_2 = 1$ everywhere on $U_1 \cup U_2$. Then $\phi_i\omega \in \Omega_c^k(U_i)$ for $i = 1, 2$ and

$$\omega = j_{1*}(\phi_1\omega) + j_{2*}(\phi_2\omega)$$

As before, the maps in this short exact sequence commute with the exterior derivatives, and thus we get a short exact sequence of complexes, and hence a long exact sequence of cohomology, the *Mayer-Vietoris sequence for compactly supported cohomology.*

$$0 \to H_c^0(U_1 \cap U_2) \to H_c^0(U_1) \oplus H_c^0(U_2) \to H_c^0(U_1 \cup U_2) \stackrel{d_*}{\longrightarrow}$$
$$\to H_c^1(U_1 \cap U_2) \to H_c^1(U_1) \oplus H_c^1(U_2) \to H_c^1(U_1 \cup U_2) \stackrel{d_*}{\longrightarrow} \cdots$$
$$\cdots \to H_c^n(U_1 \cap U_2) \to H_c^n(U_1) \oplus H_c^n(U_2) \to H_c^n(U_1 \cup U_2) \to 0$$

Exercise Use Mayer Vietoris for compactly supported cohomology to compute the compactly supported cohomology of the circle S^1, and, inductively, of the sphere S^n.

Now suppose that M is a manifold with a finite good cover $U_1, \ldots, U_r$, and write $M_k = U_1 \cup \cdots \cup U_k$. By an earlier exercise, we know that the Poincaré Duality morphism $H^k(M_1) \to H_c^{n-k}(M_1)^*$ is an isomorphism; as inductive hypothesis we assume that it is an isomorphism for all manifolds having a good cover consisting of no more than k open sets (and in particular for M_k and $M_k \cap U_{k+1}$). If we dualise the Mayer-Vietoris sequence for compactly supported cohomology, it remains exact.
(**Exercise:** (i) Prove that if

$$\cdots \to A^{k-1} \stackrel{\phi_{k-1}}{\longrightarrow} A^k \stackrel{\phi_k}{\longrightarrow} A^{k+1} \to \cdots$$

is an exact sequence of vector spaces and linear maps then so is

$$\cdots \leftarrow A^{k-1} \stackrel{\phi_{k-1}^*}{\longleftarrow} A^k \stackrel{\phi_k^*}{\longleftarrow} A^{k+1} \leftarrow \cdots$$

(ii) Give an example of a short exact sequence of Abelian groups where the corresponding statement (in which A^* is replaced by $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z})$) fails. (Hint: try practically any s.e.s.))

The dualised Mayer-Vietoris for compactly supported cohomology plays to just the right rhythm for us to compare it with Mayer-Vietoris for ordinary cohomlogy via the Poincaré duality morphism, and we get a large (and hard to typeset) diagram:

$$\begin{array}{ccccccccc}
\to & H^q(M_{k+1}) & \to & H^q(M_k) \oplus H^q(U_{k+1}) & \to & H^q(M_k \cap U_{k+1}) & \stackrel{d^*}{\longrightarrow} & H^{q+1}(M_{k+1}) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to & H_c^{n-q}(M_{k+1})^* & \to & H_c^{n-q}(M_k)^* \oplus H_c^{n-q}(U_{k+1})^* & \to & H_c^{n-q}(M_k \cap U_{k+1})^* & \stackrel{(d_*)^*}{\longrightarrow} & H_c^{n-q-1}(M_{k+1})^* & \to \cdots
\end{array}$$

in which the vertical maps are Poincaré duality morphisms. The induction step uses the 5-lemma (cf. Exercise at the foot of page 8) to deduce the statement for M_{k+1} from the statements for M_k, U_{k+1} and $M_k \cap U_{k+1}$. After all, the vertical map $H^q(M_{k+1}) \to H_c^{n-q}(M_{k+1})^*$ is flanked by two maps on each side, each one of which is an isomorphism, by the induction hypothesis. However, despite its naturality, the diagram is not obviously commutative, and we need to check that in fact it is. Well, in fact it's not: it's only *sign commutative.* That is, any two compositions of arrows starting at the same point and finishing at the same point agree up to multiplication by -1. We establish this below. Fortunately, the proof of the 5-Lemma survives this slight weakening of its hypotheses, and so the induction is complete. This completes the proof of 3.2, except for

Lemma 3.12 *The above diagram is sign-commutative.*

Proof The diagram shows three squares. The left-most one, in more detail, is

$$\begin{array}{ccc} H^q(M_{k+1}) & \stackrel{j_1^*+j_2^*}{\longrightarrow} & H^q(M_k) \oplus H^q(U_{k+1}) \\ \downarrow & & \downarrow \\ H_c^{n-q}(M_{k+1})^* & \stackrel{(j_{1*})^*+(j_{2*})^*}{\longrightarrow} & H_c^{n-q}(M_k)^* \oplus H_c^{n-q}(U_{k+1})^* \end{array}$$

Checking that this is commutative is easy, provided you don't lose your grip of what the bottom arrow is. In what follows we denote both vertical maps by PD.

Let $[\omega] \in H^q(M_{k+1})$; then $\mathrm{PD} \circ (j_1^* + j_2^*)(\omega)$ is a linear map from $H^{n-q}(M_k) \oplus H^{n-q}(U_{k+1})$ to $\mathbb{R}$; it takes $([\rho], [\sigma])$ to

$$(\int_{M_k} j_1^*(\omega) \wedge \rho, \int_{U_{k+1}} j_2^*(\omega) \wedge \sigma).$$

Meanwhile, $(j_{1*})^* + (j_{2*})^* \circ \mathrm{PD}([\omega])$ is also a linear map from $H^{n-q}(M_k) \oplus H^{n-q}(U_{k+1})$ to $\mathbb{R}$; it takes $([\rho], [\sigma])$ to

$$(\int_{M_{k+1}} \omega \wedge j_{1*}(\rho), \int_{M_{k+1}} \omega \wedge -j_{2*}(\sigma).$$

It is straightforward to see that the two pairs of integrals are equal.

Commutativity of the middle square is equally straightforward, and you should check it yourself. The only difficulty comes with two square involving d^* and $(d_*)^*$,

$$\begin{array}{ccc} H^q(M_k \cap U_{k+1}) & \stackrel{d^*}{\longrightarrow} & H^{q+1}(M_{k+1}) \\ \downarrow & & \downarrow \\ H_c^{n-q}(M_k \cap U_{k+1})^* & \stackrel{(d_*)^*}{\longrightarrow} & H_c^{n-q-1}(M_{k+1})^* \end{array}$$

Recall the definition of d^*: choose functions ϕ_1, ϕ_2 such that $\mathrm{supp}(\phi_1) \subset M_k$, $\mathrm{supp}(\phi_2) \subset U_{k+1}$ and $\phi_1 + \phi_2 = 1$ on M_{k+1}. Then given $[\omega] \in H^q(M_k \cap U_{k+1})$, $d^*[\omega]$ is the cohomology class of any form $d^*\omega \in \Omega^{q+1}(M_k \cup U_{k+1})$ such that

$$d^*\omega = \begin{cases} -d(\phi_2\omega) & \text{on } M_k \\ d(\phi_1\omega) & \text{on } U_{k+1}. \end{cases}$$

Note that $\mathrm{supp}(d^*\omega) \subset M_k \cap U_{k+1}$.

Similarly, for $[\sigma] \in H_c^{n-q-1}(M_{k+1})$, $d_*([\sigma])$ is the cohomology class of any form $d_*\sigma \in \Omega_c^{n-q}(M_k \cap U_{k+1})$ such that

$$\begin{aligned} -i_{1*}(d_*\sigma) &= d(\phi_1\sigma) \quad \text{on} \quad M_k \\ i_{2*}(d_*\sigma) &= d(\phi_2\sigma) \quad \text{on} \quad U_{k+1} \end{aligned}$$

Thus, for $[\omega] \in H^q(M_k \cap U_{k+1})$ and $[\sigma] \in H_c^{n-q-1}(M_{k+1})$,

$$\mathrm{PD} \circ d^*([\omega])([\sigma]) = \int_{M_{k+1}} d^*\omega \wedge \sigma = \int_{M_k \cap U_{k+1}} d^*\omega \wedge \sigma$$

(as $\mathrm{supp}(d^*\omega) \subset M_k \cap U_{k+1}$)

$$= \int_{M_k \cap U_{k+1}} -d(\phi_1\omega) \wedge \sigma = \int_{M_k \cap U_{k+1}} (d\phi_1) \wedge \omega \wedge \sigma$$

as ω is closed.

Meanwhile,

$$(d_*)^* \circ \mathrm{PD}([\omega])([\sigma]) = \int_{M_k \cap U_{k+1}} \omega \wedge d_*\sigma$$

$$= \int_{M_k \cap U_{k+1}} \omega \wedge -d(\phi_1\sigma) = -\int_{M_k \cap U_{k+1}} \omega \wedge d\phi_1 \wedge \sigma,$$

as σ is closed. It follows that up to sign, the two integrals coincide. □

Our proof of Cohomological Poincaré Duality, Theorem 3.2, is now complete, at least, that is, for manifolds having a finite good cover. In fact it's true even without the hypothesis on the existence of a finite good cover - a careful proof can be found in Madsen and Tornehave, Chapter 13. Their proof contains just one extra step, the following theorem on "induction on open sets":

Theorem 3.13 *Let M^n be a smooth n-manifold with an open cover $\{U_\alpha\}_{\alpha \in A}$. Suppose that there is a collection $\mathcal{C}$ of open sets of M such that*
(1) $\emptyset \in \mathcal{C}$.
(2) Any open set V diffeomorphic to $\mathbb{R}^n$ and contained in some open set U_α of the cover belongs to $\mathcal{C}$.
(3) If V_1, V_2 and $V_1 \cap V_2$ belong to $\mathcal{C}$ then so does $V_1 \cup V_2$.
(4) If $V_1, V_2, \ldots$ is a sequence of pairwise disjoint open sets all belonging to $\mathcal{C}$ then their union $\cup_i V_i$ belongs to $\mathcal{C}$.

Then $M \in \mathcal{C}$. □

Exercise Assuming this theorem, prove 3.2 for an arbitrary smooth oriented manifold without boundary.

Recall that the *Euler Characteristic* of a topological space X, $\chi(X)$, is defined to be

$$\sum_k (-1)^k \dim H^k(X; \mathbb{R}).$$

Exercise 3.14 *Suppose that M is a compact oriented manifold without boundary, of dimension n.*
(i) Prove that if n is odd then $\chi(M) = 0$;
(ii) Prove that if n is of the form $4k+2$ then $\chi(M)$ is even. (Hint: the intersection form on $H^{2k+1}(M)$ must be symplectic, i.e. non-degenerate and skew-symmetric.)

3.1 The Poincaré dual of a submanifold

Suppose that M is an oriented n-dimensional manifold. If Y is an oriented $(n-k)$-dimensional submanifold (without boundary) integration over Y defines a linear map $\Omega_c^{n-k}(M) \to \mathbb{R}$ sending ω to $\int_Y \omega$:

$$\int_Y : \Omega_c^{n-k}(M) \to \mathbb{R}, \qquad \omega \mapsto \int_Y \omega.$$

(We ought to be writing the integral as $\int_Y i^*(\omega)$, where $i : Y \to M$ is inclusion, but will usually omit the i^* in this context.) This map passes to the quotient in the usual way, to define a map

$$\int_Y : H_c^{n-k}(M) \to \mathbb{R};$$

for if $\omega = d\sigma$ then

$$\int_Y \omega = \int_{\partial Y} \sigma = \int_{\emptyset} \sigma = 0.$$

The map $\int_Y : H_c^{n-k}(M) \to \mathbb{R}$ is clearly linear. In other words, it belongs to $H_c^{n-k}(M)^*$. Poincaré duality 3.2 tells us that the map

$$H^k(M) \to H_c^{n-k}(M)^*$$

sending $[\omega_1]$ to the linear map

$$\int_M \omega_1 \wedge \ , \qquad [\omega] \mapsto \int_M \omega_1 \wedge \omega$$

is an isomorphism. Unwinding the definitions, this means that there exists a closed k-form $\omega_Y \in \Omega^k(M)$ such that for all closed compactly supported forms $\omega \in \Omega_c^{n-k}(M)$,

$$\int_Y \omega = \int_M \omega_Y \wedge \omega.$$

Also, although this form ω_Y is not unique, its cohomology class is unique; in other words any two such forms ω_Y and ω'_Y differ by an exact form $d\sigma$. The cohomology class of ω_Y is called the *Poincaré dual* of the submanifold Y. By abuse of notation the form ω_Y itself is also sometimes called the Poincaré dual of Y. It is also useful sometimes to denote the cohomology class dual to the submanifold Y by $\mathrm{PD}(Y)$.

Exercise 3.15 Suppose that Y_0 and Y_1 are k-dimensional oriented submanifolds of M, and that Y_0 is homotopic to Y_1 in the sense that there is an oriented manifold Y and a map $F : Y \times [0,1] \to M$ such that $F(Y \times \{0\}) = Y, F(Y \times \{1\}) = Y_1$, and $F_0 : Y \to Y_0$ and $F_1 : Y \to Y_1$ are orientation-preserving diffeomorphisms.
(i) Show that for every closed form $\omega \in \Omega_c^k(M)$, $\int_{Y_0} \omega = \int_{Y_1} \omega$.
(ii) Deduce that $\mathrm{PD}(Y_0) = \mathrm{PD}(Y_1)$.

How can we construct such a form ω_Y? In some cases it is clear. Suppose that $Y = \{y_0\}$ is a single point. Here an orientation is just a sign, $+1$ or -1. Then $\int_Y : H^0_c(M) \to \mathbb{R}$. If M is not compact then $H^0_c(M) = 0 = H^n(M)$, so $\mathrm{PD}(Y) = 0$. On the other hand, if M is compact then $H^0_c(M) \simeq \mathbb{R}$ (we assume M connected). The closed 0-forms are just constant functions. For any function $f \in \Omega^0_c(M)$ we have $\int_Y f = \pm f(y_0)$, where the sign is the orientation of Y. In order that $\int_Y f = \int_M \omega_Y \wedge f$ for all constant functions f, as required for ω_Y to be the Poincaré dual of Y, we must thus have

$$\pm f(y_0) = \int_M \omega_Y \wedge f = f(y_0) \int_M \omega_Y$$

(recall that f is constant). Hence ω_Y must be an n-form whose integral over M is either 1 (if the orientation of Y is $+1$) or -1, if the orientation of Y is -1.

Exercise 3.16 *What is the Poincaré dual of M itself?*

The relation between cohomological Poincaré duality and the intersection of submanifolds is neatly expressed by the following theorem.

Theorem 3.17 *If X^k and Y^{n-k} are compact oriented submanifolds of the compact oriented manifold M^n, then*

$$\int_M PD(X) \wedge PD(Y) = (X \cdot Y)_M.$$

This is really a special case of the following "naturality" property of Poincaré duality.

Theorem 3.18 *Suppose that X^k and M^n are smooth oriented manifolds and that $f : X \to M$ is a smooth map. Let Y^ℓ be an oriented submanifold of M with f transverse to Y, and give $f^{-1}(Y)$ the transverse preimage orientation. Then*

$$f^*(PD_M(Y)) = PD_X(f^{-1}(Y)).$$

(Here we use subscripts to distinguish between Poincaré duality on M and on X.)
Proof of 3.17 from 3.18 Assume first that X and Y are transverse, and denote the inclusion of X in M by i. Give $i^{-1}(Y) = X \cap Y$ its transverse preimage orientation. Note that

$$(X \cdot Y)_M = \sum_{x \in i^{-1}(Y)} \mathrm{sign}(x).$$

By what we observed before concerning the Poincaré dual of a point, we have

$$\mathrm{PD}_X(i^{-1}(Y)) = \sum_{x \in X \cap Y} \mathrm{sign}(x)\omega_0, \tag{5}$$

where ω_0 is a k-form on X such that $\int_X \omega_0 = 1$. By 3.18,

$$i^*(\mathrm{PD}_M(Y)) = \mathrm{PD}_X(i^{-1}(Y)). \tag{6}$$

By definition of PD, we have

$$\int_M \mathrm{PD}_M(X) \wedge \mathrm{PD}_M(Y) = \int_X i^*(\mathrm{PD}_M(Y))$$

and by (6) this is equal to

$$\int_X \mathrm{PD}_X(i^{-1}(Y))$$

which, by (5), is equal to

$$\int_X \sum_{x \in X \cap Y} \mathrm{sign}(x)\omega_0.$$

The right hand side evaluates to $\sum_{x\in X\cap Y} \mathrm{sign}(x)$, i.e. to $(X \cdot Y)_M$.

If X and Y are not transverse, we can nevertheless deform the embedding of X in M in a homotopy, so that it becomes transverse to Y. Denote the deformed X by X'. By Exercise 3.15, $\mathrm{PD}_M(X') = \mathrm{PD}_M(X)$. Hence

$$\int_M \mathrm{PD}_M(X) \wedge \mathrm{PD}_M(Y) = \int_M \mathrm{PD}_M(X') \wedge \mathrm{PD}_M(Y),$$

and this is equal to $(X' \cdot Y)_M$ by what we have proved for the transverse case. Finally, $(X \cdot Y)_M = (X' \cdot Y)_M$; indeed, we *define* $(X \cdot Y)_M$ by perturbing X so that it becomes transverse to Y and then counting intersection points with their signs. □

We will not give a complete proof of 3.18, but instead consider some special cases and give an overview of the proof, which can be found e.g. in Bott and Tu, pages 65-67. Consider first the very simple case of a cylinder $M = \mathbb{R} \times S^1$ and let $Y = S^1 \times \{0\}$. Locally we can take coordinates t, θ on M, although of course θ is not well-defined globally. Since different branches of θ differ by a constant, their exterior derivatives coincide, and define a (global) 1-form $d\theta$. If $\omega = f d\theta \in \Omega^1(M)$, then $d\omega = 0$ if and only if $\partial f/\partial t = 0$, i.e. if $f = f(\theta)$ is independent of t. For such a form ω there is no mystery in finding a form ω_Y such that $\int_Y \omega = \int_M \omega_Y \wedge \omega$: simply take any smooth compactly supported function $c(t)$ such that $\int_{\mathbb{R}} c(t)dt = 1$, and let $\omega_Y = c(t)dt$. Then

$$\int_M o_Y \wedge \omega = \int_M f(\theta)c(t)dt \wedge d\theta = \int_{\mathbb{R}} c(t)dt \int_Y f(\theta)d\theta = \int_Y \omega.$$

The second equality here is obtained simply by expressing the integral over M as an iterated integral. [2]

If $\omega = f(\theta, t)\, d\theta + g(\theta, t)dt$ is a more general closed form, the situation is scarcely more difficult: if $i : S^1 \to M$ is the inclusion $\theta \mapsto (\theta, 0)$ and $p : M \to S^1$ is projection, then we know by 2.17

[2]This may seem more familiar if we parametrise M by the obvious map $[0, 2\pi] \times \mathbb{R} \to M$, which is a diffeomorphism off a set of measure 0. Then

$$\int_M c(t)dt \wedge f(\theta)d\theta = \int_{\mathbb{R}\times[0,2\pi]} c(t)f(\theta)d\theta dt = \int_0^{2\pi} \int_{\mathbb{R}} c(t)f(\theta)dt\, d\theta.$$

that $p^* : H^1(Y) \to H^1(M)$ and $i^* : H^1(Y) \to H^1(M)$ are mutually inverse. That is, ω and $p^*i^*\omega$ differ by an exact form $d\sigma$. Now $p^*i^*\omega$ is a 1-form of the type we considered in the previous paragraph. It follows that with the same ω_Y as before, we have

$$\int_M \omega_Y \wedge \omega = \int_M \omega_Y \wedge p^*i^*\omega = \int_Y p^*i^*\omega = \int_Y \omega$$

(note that on Y, ω is equal to $p^*i^*\omega$).

Now suppose Y_0 is any oriented manifold and $M = \mathbb{R}^k \times Y_0$, and let $Y = \{0\} \times Y_0 \subset M$. Esssentially the same argument as above, using Fubini's Theorem (evaluation of a multiple integral by iterated integration) shows that we can take, as ω_Y, the pull-back to M of a k form $c = c(t)dt_1 \wedge \cdots \wedge dt_k$ on $\mathbb{R}^k$ such that $\int_{\mathbb{R}^k} c = 1$. For

$$\int_M c \wedge \omega = \int_M c \wedge p^*i^*\omega = \int_{\mathbb{R}^k} c(t)dt_1 \wedge \cdots \wedge dt_k \int_{Y_0} p^*i^*\omega = \int_{Y_0} \omega.$$

The proof of 3.18 consists of two steps. Both involve vector bundles, and it you are not familiar with them it may be best to postpone reading the remainder of this subsection until you have gained some familiarity with them (Chapter 15 of Madsen and Tornehave, which you will read later in the course, is concerned with vector bundles).

The first step is to generalise the previous observation to the situation where E is the total space of an oriented vector bundle of rank k and Y is its zero section (in the previous paragraph, $M = \mathbb{R}^k \times Y$ is the total space of a *trivial* vector bundle). Since now E is no longer *globally* a product, we have to work a little to recreate in this new situation the form ω_Y we used in the previous paragraphs. In fact it is not hard, by piecing together local constructions, to find a form $\omega_Y \in \Omega^k(E)$ such that the integral of ω_Y over each fibre of the vector bundle $E \to Y$ is equal to 1. It is a little harder to translate to this new context the property of being "independent of the Y-variables" which we used to reduce the integral of $\omega_Y \wedge \omega$ over M to an iterated integral. To do this, we make use of the idea of "integration in the fibre direction" which appeared briefly in the proof of the Poincaré Lemma. This is a well-defined morphism $I : \Omega^\bullet(E) \to \Omega^{\bullet - k}(Y)$ defined simply by integrating out the fibre variables.

Definition 3.19 *The Thom class of the oriented vector bundle $E \to Y$ is (the cohomology class of) a k-form $\Phi_E \in \Omega^k_{cv}(E)$ such that $I(\Phi_E) = 1_M$, where 1_M is the function on M with contant value 1.*

Here the subscript cv means compactly supported in the vertical (i.e. fibre) direction.

Lemma 3.20 *(i) The Thom class exists — that is, there always is a form Φ_Y with the property described in Definition 3.19.*
(ii) If E is an oriented vector bundle over the oriented manifold Y, and we identify Y with the zero section of E, then the Poincaré dual ω_Y of Y is equal to Φ_E. □

I omit the proof, though it amounts to little more than using a partition of unity to piece together local constructions like those in the special case dealt with above where M was a trivial vector bundle.

The second step uses the fact that an oriented submanifold Y of the oriented manifold M has a neighbourhood V_0 in M which is diffeomorphic to the total space of its normal bundle $\nu(Y, M)$ in M. The standard proof of the tubular neighbourhood theorem (see e.g. Guillemin and Pollack, or my Manifolds lecture notes, page 34) can easily be adapted to show this. The tubular neighbourhood theorem is proved in the simplest case where $Y \subset \mathbb{R}^N = M$ by considering the map

$$F : \nu(Y, \mathbb{R}^N) \to \mathbb{R}^N$$

defined by $F(y, v) = y + v$, and showing that there is a neighbourhood U of the zero section of $\nu(Y, \mathbb{R}^N)$ on which F is a diffeomorphism onto a neighbourhood V of Y in $\mathbb{R}^N$. It is easy to see that U contains a neighbourhood U_0 of the zero section which is diffeomorphic to the total space $\nu(Y, \mathbb{R}^N)$; it follows that V also contains a neighbourhoood of Y also diffeomorphic to $\nu(Y, \mathbb{R}^N)$.

We obtain the Poincaré dual ω_Y of Y by identifying this neighbourhood V_0 of Y in M with $\nu(Y, M)$ and pushing forward the Thom class of $\nu(Y, M)$ to all of M by extending it by zero (as described on page 21). It follows easily from Lemma 3.20 that the cohomology class of this extension of the Thom class of $\nu(Y, M)$ is the Poincaré dual of Y.

Now that we have an effective construction of ω_Y, we can easily prove 3.18.

Proposition 3.21 *(i) If $f : X \to M$ is transverse to the submanifold Y of M then the normal bundle of $f^{-1}(Y)$ in X is isomorphic to the pull-back by f of the normal bundle of Y in M:*

$$\nu(f^{-1}(Y), X) \simeq f^*\nu(Y, M).$$

(ii) If $E \to M$ is a vector bundle and $f : X \to M$ is a smooth map, then the Thom class of the vector bundle $f^*(E)$ over X is equal to the pull-back by f of the Thom class of E:

$$\Phi_{f^*(E))} = f^*(\Phi_E).$$

□

The proofs of both parts of the proposition are straightforward. The proof of 3.18 follows.

3.2 The degree of a smooth map

An immediate consequence of Poincaré Duality is that for any oriented n-manifold M without boundary, $H^n_c(M) \simeq \mathbb{R}$, with the isomorphism given by integration over M.

Let M and N be smooth connected boundaryless oriented n-manifolds, and let $f : M \to N$ be a proper map (that is, the preimage in M of every compact set in N is compact). Then f induces a pull-back map on compactly supported cohomology, $f^* : H^n_c(N) \to H^n_c(M)$. The degree of f, $\deg(f)$ is defined by the commutative diagram

$$\begin{array}{ccc} H^n_c(N) & \xrightarrow{f^*} & H^n_c(M) \\ \int_N \downarrow & & \int_M \downarrow \\ \mathbb{R} & \xrightarrow{\deg(f)} & \mathbb{R} \end{array}$$

Proposition 3.22 *If $y \in N$ is a regular value of f then*

$$deg(f) = \sum_{x \in f^{-1}(y)} sign_x(f),$$

where $sign_x(f)$ is $+1$ if f preserves orientation at x and -1 if it reverses it.

Proof We need the "stack of records lemma":

Lemma 3.23 *If $y \in N$ is a regular value of the proper map $f : M^n \to N^n$, there is a neighbourhood V of y in N such that $f^{-1}(y) = \cup_i^r U_i$, such that $U_i \cap U_j = \emptyset$ for $i \neq j$ and $f_| : U_i \to V$ is a diffeomorphism for all i.*

Proof See e.g. Madesen and Tornehave, Lemma 11.8 page 100, or Guillemin and Pollack page ??. □

Exercise Prove 3.22 from the stack of records lemma. □

Corollary 3.24 *$deg(f)$ is an integer.*

If M is compact, every map $f : M \to N$ is of course proper.

Exercise Show that if $f : M \to N$ is not surjective then $\deg(f) = 0$.

Exercise Suppose that W is an oriented $n+1$-manifold, N is an oriented boundaryless n-manifold, and $F : W \to N$ is a proper map. Let $f : \partial W \to N$ be the restriction of F. Show that $\deg(f) = 0$.

Given disjoint oriented simple closed curves $C_1, C_2 \subset \mathbb{R}^3$, define the *linking number* $\ell(C_1, C_2)$ as follows: $\ell(C_1, C_2)$ is the degree of the smooth map $f : C_1 \times C_2 \to S^2$ sending (x_1, x_2) to $(x_1 - x_2)/\|x_1 - x_2\|$. Here $C_1 \times C_2$ is given the product orientation.

Exercise Show that if we deform C_1 to C_1' in a family $C_{1,t}$ of closed curves such that $C_{1,t}$ and C_2 are always disjoint, then $\ell(C_1, C_2) = \ell(C_1', C_2)$.

Exercise Show that if C_1 and C_2 can be "untangled from one another" — i.e. if it is possible to deform them to new curves C_1', C_2' lying on opposite sides of some hyperplane $H \subset \mathbb{R}^3$ (with the two curves disjoint from one another at all times in the deformation), then $\ell(C_1, C_2) = 0$.

Exercise Find the linking numbers of the pairs of curves shown:

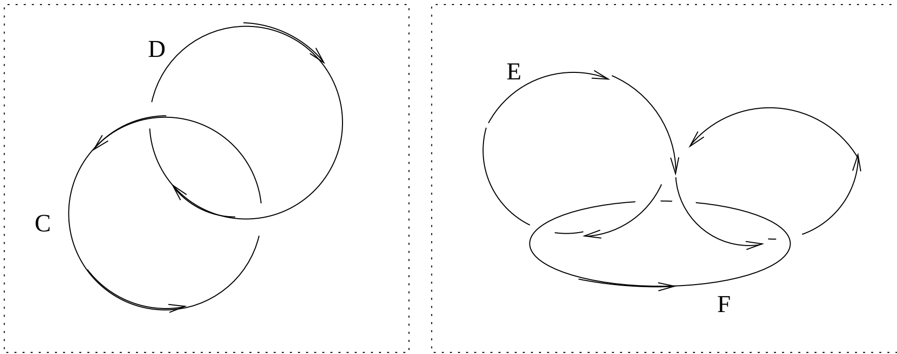


Figure 9

Hint: count preimages (under $f : C_1 \times C_2 \to S^2$) of the unit vector in S^2 pointing out of the plane of the paper towards you.

Mayer-Vietoris for Singular Homology* [3]

If X is any topological space and U_1, U_2 are open subsets, there is a Mayer-Vietoris long exact sequence

$$\cdots \to H_k(U_1 \cap U_2) \xrightarrow{(i_{1*}, -i_{2*})} H_k(U_1) \oplus H_k(U_2) \xrightarrow{j_{1*}+j_{2*}} H_k(U_1 \cup U_2) \xrightarrow{\partial_*} H_{k-1}(U_1 \cap U_2) \to \cdots .$$

Here the morphisms i_*, j_* are induced by the inclusions in the obvious way: if $U \xrightarrow{i} V$ is any continuous map and $s : \Delta_k \to U$ is a singular k-simplex, $i_*(s) = i \circ s$ is a singular k simplex in V.

This long exact sequence can be constructed by almost the same procedure used for the Mayer Vietoris sequence of cohomology. As before, there is an exact sequence of complexes

$$0 \to C_\bullet(U_1 \cap U_2) \xrightarrow{(i_{1*}, i_{2*})} C_\bullet(U_1) \oplus C_\bullet(U_2) \xrightarrow{j_{1*}-j_{2*}} C_\bullet(U_1 \cup U_2);$$

however this time the the arrow $C_k(U_1) \oplus C_k(U_2) \to C_k(U_1 \cup U_2)$ is plainly not surjective for $k > 0$ (see for example the diagram below). However, this lack of surjectivity can easily be remedied by *subdivision*: any singular k-chain in $U_1 \cup U_2$ can be subdivided into a singular chain of the form $j_{1*}(c_1) + j_{2*}(c_2)$, where $c_1 \in C_k(U_1)$ and $c_2 \in C_k(U_2)$. This is schematically shown in the diagram on the right, where the 2-simplex c is subdivided into $c_1 \subset U_1$ and $c_2 \subset U_2$.

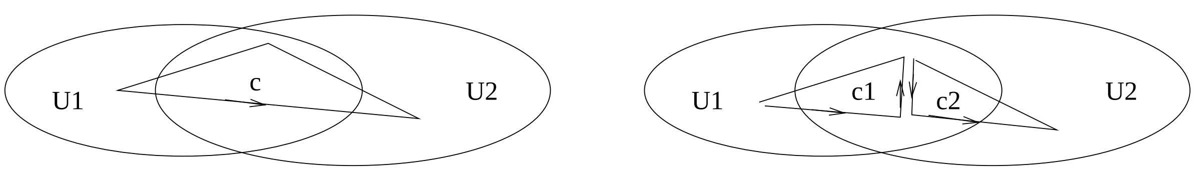


Figure 10

[3]This and anything else with a star is here for entertainment only, and is not examinable

Building on this idea, it is possible to prove the following Subdivision Lemma:

Lemma 3.25 *Suppose that U_1 and U_2 are open sets in the toplogical space X. If $c \in C_k(U_1 \cup U_2)$ is a k-cycle then there is a k-chain $c' \in C_k(U_1 \cup U_2)$ such that*
(i) $c - c' \in \delta(C_{k+1}(U_1 \cup U_2))$
(ii) every one of the singular simplices s making up c' lies either in U_1 or in U_2; thus

$$c' \in j_{1*}(C_k(U_1)) + j_{2*}(C_k(U_2)).$$

□

Exercise Using the Subdivision Lemma to make up for the failure of surjectivity of $j_{1*} + j_{2*} : C_k(U_1) \oplus C_k(U_2) \to C_k(U_1 \cup U_2)$, prove Mayer-Vietoris for singular homology

Exercise* Can you find an inductive proof of the de Rham theorem using Mayer-Vietoris for homology and cohomology?

The Künneth Formula for de Rham Cohomology

The Künneth Formula gives us a way of computing the cohomolgy of a product $M \times N$ in terms of the cohomology of the two factors M and N:

Theorem 3.26 $H^*(M \times N) \simeq H^*(M) \otimes_{\mathbb{R}} H^*(N)$

This means the following:
(1) For each q,

$$H^q(M \times N) \simeq \bigoplus_{i+j=q} H^i(M) \otimes H^j(N)$$

(2) Something about the ring structure, which as yet I do not attempt to state.

In fact, where finite-dimensional vector-spaces are concerned, the mere existence of an isomorpphism is a pretty feeble statement; it is equivalent to their having the same dimension. Much more interesting is a statement describing the isomorphism in concrete terms. Here it is:

For each q, the projections $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ induce linear maps $\pi_M^* : \Omega^q(M) \to \Omega^q(M \times N)$ and $\pi_N^* : \Omega^q(N) \to \Omega^q(M \times N)$, and hence a bilinear map

$$\pi_M^* : \Omega^p(M) \times \Omega^q(N) \to \Omega^{p+q}(M \times N)$$

given by

$$(\omega_p, \tau_q) \mapsto \pi_M^*(\omega_p) \wedge \pi_N^*(\tau_q).$$

This passes to the quotient to give a bilinear map $H^p(M) \times H^q(N) \to H^{p+q}(M \times N)$. By definition of tensor product, there is a unique *linear* map $H^p(M) \otimes H^q(N) \to H^{p+q}(M \times N)$ making the diagram

$$\begin{array}{ccc} H^p(M) \times H^q(N) & & \\ \downarrow & \searrow & \\ H^p(M) \otimes H^q(N) & \to & H^{p+q}(M \times N) \end{array}$$

(in which the vertical map is the canonical projection $(u, v) \mapsto u \otimes v$) commutative. The Künneth theorem asserts that for each q the sum

$$\bigoplus_{i+j=q} H^i(M) \otimes H^j(N) \to H^q(M \times N)$$

of all these maps is an isomorphism.

Example 3.27 The torus T^2 is diffeomorphic to $S^1 \times S^1$; so $H^1(T^2) \simeq H^0(S^1) \otimes H^1(S^1) \oplus H^1(S^1) \otimes H^0(S^1)$, and thus is 2-dimensional, and $H^2(T^2) = H^1(S^1) \otimes H^1(S^1)$.

I leave you to decide what the natural (and correct) statement about multiplication is.

To prove the Künneth Theorem, we use Mayer Vietoris and induction on the number of open sets in a good cover of M; (we prove it only when either M or N has a finite good cover). Let $\{U_1, \ldots, U_N\}$ be a good cover of M, and write $U_1 \cup \cdots \cup U_k = M_k$.

Step 1 It's true if M itself is diffeomorphic to $\mathbb{R}^n$ for some n, for then $M \times N \simeq \mathbb{R}^n \times N$, and $\pi_N^* : H^q(N) \to H^q(\mathbb{R}^n \times N)$ is an isomorphism, so that $\pi_M^* \otimes \pi_N^* : H^0(\mathbb{R}^n) \times H^q(N) \to H^q(\mathbb{R}^n \times N)$ is also an isomorphism.

Step 2: the induction step Assume the theorem is true whenever M is any manifold having a good cover consisting of no more than k open sets. We use Mayer Vietoris for the pair M_k, U_{k+1} of open sets.

One of the goood things about exact sequences of vector spaces (as opposed to exact sequences of abelian groups, or of modules over a more general ring) is that their exactness is extremely robust. Not only does dualising leave them exact, as we saw in the proof of Poincaré Duality, but so does tensoring with another vector space: if $A^\bullet$,

$$\cdots \to A^{k-1} \xrightarrow{\psi^{k-1}} A^k \xrightarrow{\psi^k} A^{k+1} \to \cdots$$

is an exact sequence of vector spaces and linear maps, and V is any other vector space, then

$$\cdots \to A^{k-1} \otimes V \xrightarrow{\psi^{k-1} \otimes 1_V} A^k \otimes V \xrightarrow{\psi^k \otimes 1_V} A^{k+1} \otimes V \to \cdots,$$

which we denote by $A^\bullet \otimes V$, is exact too. Here $\psi^k \otimes 1_V$ is the linear map sending $a_k \otimes v$ to $\psi^k(a_k) \otimes v$.

Exercise (i) Prove this.
(ii) What happens when you tensor the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$$

of abelian groups, with the abelian group $\mathbb{Z}_2$? Here tensor product is over $\mathbb{Z}$, of course - we're looking at abelian grops (i.e. $\mathbb{Z}$-modules) instead of vector spaces (i.e. k-modules, where k is a field).

Remark 3.28 It may be that at this point in your studies you haven't really used tensor products before. Tensor product is one of those notions which it is best simply to use without worrying too much, at first, about what it means. You have to remember only that if V and W are vector spaces over the field k then $V \otimes_k W$ is the vector space generated by elements $v \otimes w$, where $v \in V, w \in W$, subject only to the rules

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$$

$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$$

$$\lambda(v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w).$$

It is "being able to pass scalars across from one factor in the tensor product to the other" (the third rule) that makes this the tensor product over k rather than over anything else. If, for example, k is a subfield of K, then any K-vector space is also a k-vector space; and if V and W are two such, then

$$V \otimes_K W \quad \text{and} \quad V \otimes_k W$$

are different spaces. Which one is bigger?

Denote by $\mathrm{MV}_k^\bullet$ the Mayer-Vietoris long exact sequence of the pair M_k, U_{k+1} of open sets. Tensoring with $H^j(N)$ we get the exact sequence $MV_k^\bullet \otimes H^j(N)$.

The direct sum of any number of exact sequences is also exact. However one needs to be a little careful to interpret this statement correctly. For example, if

$$0 \to A^1 \to A^2 \to A^3 \to A^4 \to 0$$

and

$$0 \to B^1 \to B^2 \to B^3 \to 0$$

are both exact, then so are

$$0 \to A^1 \oplus B^1 \to A^2 \oplus B^2 \to A^3 \oplus B^3 \to A^4 \to 0$$

and

$$0 \to A^1 \to A^2 \oplus B^1 \to A^3 \oplus B^2 \to A^4 \oplus B^3 \to 0.$$

(Check it!). We want to sum the exact sequences $\mathrm{MV}_k^\bullet \otimes H^j(N)$ over j, but in such a way that each spot in the resulting exact sequence, the sum $i + j$ of the indices on the tensor products $H^i(\text{something}) \otimes H^j(N)$ are all the same. To make clear how to do this, imagine extending each sequence $\mathrm{MV}_k^\bullet \otimes H^j(N)$ by an infinite sequence of zeros on each end. Place them in vertical array, with $\mathrm{MV}_k^\bullet \otimes H^0(N)$ at the top and $\mathrm{MV}_k^\bullet \otimes H^n(N)$ (where $n = \dim N$) at the bottom, and then slide each row three spots to the right with respect to the one above it. This is indicated schematically in the following diagram:

$$\begin{array}{ccccccccccccc}
H^0 \otimes H^0 & \to & (H^0 \oplus H^0) \otimes H^0 & \to & H^0 \otimes H^0 & \to & H^1 \otimes H^0 & \to & (H^1 \oplus H^1) \otimes H^0 & \to & H^1 \otimes H^0 & \to & H^2 \otimes H^0 \\
0 & \to & 0 & \to & 0 & \to & H^0 \otimes H^1 & \to & (H^0 \oplus H^0) \otimes H^1 & \to & H^0 \otimes H^1 & \to & H^1 \otimes H^1 \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 & \to & H^0 \otimes H^2
\end{array}$$

Now consider the "grand total " exact sequence you get by summing over columns. This has the form

$$\cdots \to \bigoplus_{i=0}^{q} H^i(M_{k+1}) \otimes H^{q-i}(N)$$

$$\to \bigoplus_{i=0}^{q} (H^i(M_k) \oplus H^i(U_{k+1})) \otimes H^{q-i}(N)$$

$$\to \bigoplus_{i=0}^{q} H^i(M_k \cap U_{k+1}) \otimes H^{q-i}(N) \to \cdots$$

Each of the sums of spaces here is precisely what is called for in the Künneth Formula, and maps, via $\pi_M^* \wedge \pi_N^*$, to $H^q(M_{k+1} \times N)$, to $H^q(M_k \times N) \oplus H^q(U_{k+1} \times N)$ or to $H^q(M_k \cap U_{k+1} \times N)$, respectively. Thus, from the grand total exact sequence we have a sequence of Künneth maps to the Mayer Vietoris sequence for the pair $M_k \times N, U_{k+1} \times N$ of open sets in $M \times N$. By induction, we can assume that those mapping to $H^q(M_k \times N)$, to $H^q(U_{k+1} \times N)$ and to $H^q(M_k \cap U_{k+1}) \times N$ are all isomorphisms. It will follow from the 5-Lemma that the same is true for the map

$$\bigoplus_{i=0}^{q} H^i(M_{k+1}) \otimes H^{q-i}(N) \to H^q(M_{k+1} \times N),$$

provided we can show that the diagram is commutative. This I leave to you as an **Exercise**.□

Exercise Devise a form of notation which makes it possible to describe the argument involving "sliding the exact sequences 3 spots to the right" without drawing such large diagrams. (I am serious).

Exercise Formulate and prove the correct statement of the Künneth formula for the multiplicative operation (i.e. wedge product) on cohomology.

Exercise Can you use Induction on Open Sets (3.13) to prove the Künneth Theorem without the hypothesis that M or N has a finite good cover?

The Leray-Hirsch Theorem

The map $\pi : E \to M$ is smooth manifolds is a *locally trivial fibre bundle* with fibre F if every point $x \in M$ has a neighbourhood U such that there is a commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi_U} & U \times F \\ \pi \searrow & & \swarrow \pi_U \\ & U & \end{array}$$

in which ϕ_U is a diffeomorphism and $\pi_U : U \times F \to U$ is simply projection. Commutativity of the diagram means that ϕ_U restricts to give a map (also a diffeomorphism, of course) from $\pi^{-1}(y)$ to $\{y\} \times F$ (which we identify with F) for each $y \in U$.

The diffeomorphism ϕ_U is a *local trivialisation* of $\pi : E \to M$ over U.

Example 3.29 1. If $E = M \times F$ then the projection $\pi : E \to M$ is a locally trivial fibre bundle. Ineed, it is *globally* trivial.

2. If M is a smooth n-dimensional manifold, its tangent bundle is a locally trivial fibre bundle with fibre $\mathbb{R}^n$. Of course, it has additional structure: it is possible to choose the diffeomorphisms ϕ_U so that their restriction to the fibre, $T_yM = \pi^{-1}(y) \to F = \mathbb{R}^n$ is a linear isomorphism for every $y \in U$.

3. If $\pi : E \to M$ is a locally trivial fibre bundle then π is evidently a submersion. Not every submersion is a locally trivial fibre bundle, but we have

   Theorem 3.30 The Ehresmann Fibration Theorem*: If $\pi : E \to M$ is a proper submersion, then it is a locally trivial fibre bundle.*

4. **Exercise** The map $S^1 \times S^1 \to S^1$ sending (z_1, z_2) to z_1z_2 is a locally trivial fibre bundle. What is its fibre? What about $S^1 \times S^1 \times S^1 \to S^1$ defined by $(z_1, z_2, z_3) \mapsto z_1z_2z_3$?

5. **Exercise** Find an example of a submersion $\pi : E \to M$ which is *not* a locally trivial fibre bundle. Hint: cut a hole in the source of a locally trivial fibre bundle.

Suppose that $f : E \to M$ is any smooth map (not necessarily a locally trivial fibre bundle). The morphism $f^* : H^*(M) \to H^*(E)$ of cohomology rings makes makes $H^*(E)$ into a module over the ring $H^*(M)$: if $[\omega] \in H^q(M)$ and $[\sigma] \in H^p(E)$ then we can wedge $f^*(\omega)$ with σ and get a new cohomology class on E. This extends linearly in an obvious way, and gives a pairing $H^*(M) \times H^*(E) \to H^*(E)$.

The structure of $H^*(E)$ as a module over $H^*(M)$ can reflect the differential topology of the map f in an interesting way.

The most uninteresting modules are the free modules: theirs is the most transparent of structures.

Definition 3.31 Let R be a ring and M an R-module: M is a *free* R-module if there exists some collection $m_{\lambda\{\lambda\in\Lambda\}}$ of elements of M such that every $m \in M$ can be written uniquely as a linear combination $m = \sum_\lambda r_\lambda m_\lambda$, with all except finitely many of the r_λ equal to 0. In this case $m_{\lambda\{\lambda\in\Lambda\}}$ is called a *free basis*, or *free R-basis* to be more precise, for M.

Every vector space is a free k-module, where k is the field of scalars; this freeness reflects the fact that as a ring, k is pretty uninteresting.

On the other hand, an abelian group with torsion elements is not a free $\mathbb{Z}$-module. Nor, in fact, is $\mathbb{Q}$, even though it has no torsion (**Exercise**, if you enjoy this kind of thing).

If $E = M \times N$ then $H^*(E)$ is a free module over $H^*(M)$: for suppose that $c_1, \ldots, c_r \in H^*(N)$ form an $\mathbb{R}$-basis. Let $e_1, \ldots, e_r$ be the cohomology classes on $M \times N$ obtained by pulling back the c_i, $e_i = \pi_N^*(c_i)$ for $i = 1, \ldots, r$. Then I claim that the e_i form a free $H^*(M)$-basis for $H^*(M \times N)$.

In fact this is simply another way of stating the Künneth Theorem, and I leave it to you to prove it - it amounts to no more than unravelling definitions.

Now let $\pi : E \to M$ be a locally trivial fibre bundle, and suppose that there are cohomology classes $e_1, \ldots, e_m \in H^*(E)$ whose restriction to each fibre generates the cohomology of the fibre (as vector space over $\mathbb{R}$) (note that this is certainly the case if $E = M \times N$; the classes e_i described two paragraphs above have this property). Then we can define a map

$$\psi : H^*(M) \otimes \oplus_{i=1}^r \mathbb{R} \cdot e_i \to H^*(E)$$

by

$$(\sum_j \omega_j) \otimes (\sum_i \alpha_i e_i) \mapsto \sum_{i,j} \alpha_i \omega_j \wedge e_i.$$

Here $\oplus_{i=1}^r \mathbb{R} \cdot e_i$ means the vector space of formal linear combinations $\sum_i \alpha_i e_i$ with real coefficients (i.e. $\alpha_1 e_1 + \cdots + \alpha_r e_r = \beta_1 e_1 + \cdots + \beta_r e_r$ if and only if $\alpha_i = \beta_i$ for each i). It should be distinguished from the subspace of $H^*(E)$ generated over $\mathbb{R}$ by the cohomology classes $e_1, \ldots, e_r$, which we denote $\mathbb{R}\{e_1, \ldots, e_r\}$. In this space, two disctinct linear combinations of the e_i may be equal.

There is an obvious map

$$\bigoplus_{i=1}^r \mathbb{R} \cdot e_i \to \mathbb{R}\{e_1, \ldots, e_r\}$$

sending a formal linear combination $\lambda_1 e_1 + \cdots + \lambda_r e_r$ to the informal linear combination $\lambda_1 e_1 + \cdots + \lambda_r e_r \in H^*(E)$.

Theorem 3.32 *Let $\pi : E \to M$ be a locally trivial fibre bundle with fibre F. If there are cohomology classes $e_1, \ldots, e_r$ on E whose restriction to each fibre is an R-basis for its cohomology, then $H^*(E)$ is a free module over $H^*(M)$ with free basis $e_1, \ldots, e_r$. Thus,*

$$H^*(E) \simeq H^*(M) \otimes \mathbb{R}\{e_1, ..., e_r\} \simeq H^*(M) \otimes H^*(F).$$

Proof The hypothesis is preserved "under restriction of the base": that is, if $U \subset M$ and we write E_U for $\pi^{-1}(U)$, and denote by i the inclusion $E_U \hookrightarrow E$, then the classes $i^*(e_1), \ldots, i^*(e_r)$ in $H^*(E_U)$ have the same property as the e_i: their restrictions to each fibre of $\pi : E_U \to U$ form a basis for its cohomology. If $U \subset M$ is an open set over which E is trivial, then the Künneth Theorem tells us that the map

$$H^*(U) \otimes \mathbb{R}\{i^*(e_1), \ldots, i^*(e_r)\} \to H^*(E_U)$$

is an isomorphism.

Exercise Complete the proof, using Mayer Vietoris. □

4 Morse Theory

We begin with a proof of the Ehresmann Fibration Theorem. Recall that we are assuming $f : E \to M$ is a proper submersion; for the moment, we assume also that E and M are manifolds without boundary, and that M is connected.

Step 1 At each point $e \in E$, choose a complement H_e in $T_e E$ to $\ker d_e f$, in such a way that the H_e "vary smoothly" with e. This can be done, for example, by giving E a Riemann metric and setting $H_e = (\ker d_e f)^\perp$. As f is a submersion, the restriction of $d_e f$ gives an isomorphism $H_e \to T_{f(e)}M$. The "H" in H_e stands for "horizontal", of course; in pictures of fibre bundles one usually shows the fibre as arrayed vertically above the base M, which is drawn horizontal. The choice of this field of horizontal subspaces H_e is called an Ehresmann connection.

Step 2 Suppose that p and q are two points in M, and are joined by a simple smooth curve C parametrised by $\gamma : [0, a] \to M$. We use an Ehresmann connection to define a diffeomorphism $E_p \to E_q$ as follows: first, γ' defines a vector field on C, which we denote by $\partial/\partial t$. We lift this to a smooth vector field X on $f^{-1}(C)$ by setting

$$X(e) = \text{unique } v \text{ in } H_e \text{ such that } d_e f(v) = \partial/\partial t.$$

The theory of ODE's assures us that for every point $e \in f^{-1}(C)$, there is an integral curve of X passing through e. Note that if γ_e is an integral curve of X, then for all $t \in [0, a]$ and $e \in f^{-1}(\gamma(t))$,

$$d_e f(\gamma_e'(t)) = \gamma'(t).$$

Because f is proper, for each $t \in [0, a]$ there exists $\varepsilon_t > 0$ such that for all $e \in f^{-1}(\gamma(t))$ there is an integral curve γ_e through e satisfying $\gamma_e(t) = e$ and defined on the interval $[t - \varepsilon_t, t + \varepsilon_t] \cap [0, a]$.

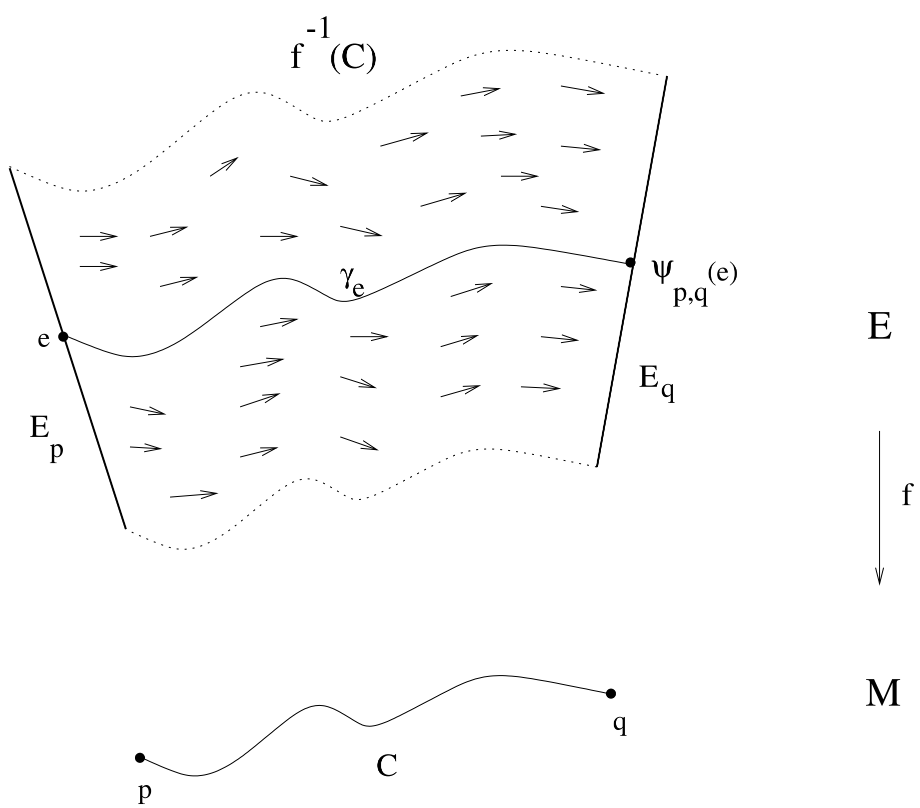


Figure 11

By compactness, a finite number of the intervals $[t - \varepsilon_t, t + \varepsilon_t]$ cover $[a, b]$, and it follows that any integral curve through a point $e \in f^{-1}(p)$ can be extended (by integral curves) to a curve defined on all of $[0, a]$. Let $e \in f^{-1}(p)$ and let $\gamma_e : [0, a] \to f^{-1}(C)$ be the (unique) integral curves of X satisfying $\gamma_e(0) = e$. Then we define a map $\psi_{p,q} : E_p \to E_q$ by $e \mapsto \gamma_e(a)$. General theory of ODE's tells us that $\psi_{p,q}$ is smooth – varying $e \in f^{-1}(p)$ corresponds to varying the intial values of the solution of an ODE, and the solution of an ODE depends smoothly on the initial conditions. Moreover, $\psi_{p,q}$ is a diffeomorphism: its inverse is $\psi_{q,p}$.

Step 3 Given $p \in M$, choose a coordinate chart centred on x and by means of the chart identify some neighbourhood U of x with the open unit ball in $\mathbb{R}^n$. Each point $q \in U$ is then joined to p by a unique radial segment. These segments will play the role of the curve C in the previous step. The maps $\psi_{p,q}$ fit together to give a diffeomorphism $\psi_U : E_p \times U \to f^{-1}(U)$ sending (e, q) to $\psi_{p,q}(e)$, and clearly the diagram

$$\begin{array}{ccc} f^{-1}(U) & \overset{\psi_U}{\longleftarrow} & U \times E_p \\ f \searrow & & \swarrow \pi_U \\ & U & \end{array}$$

is commutative.

The inverse of ψ_U is the diffeomorphism ϕ_U required by the definition of "local trivialisation".

If M is connected then it is path-connected (**Exercise**) and it follows from Step 2 that all of the fibres E_p are diffeomorphic to one another. If we call any one of them F, then it follows from what we have just done that $f : E \to M$ is a locally trivial fibre bundle with fibre F. □

Remark 4.1 If we allow E to have boundary, then we must insist that not only f but also its restriction to ∂E be proper submersions. Assuming this is the case, we take care to choose our Ehresmann connection so that at each point $e \in \partial E$, $H_e \subset T_e \partial E$. Once this is done, the integral curve γ_e through any point $e \in \partial E$ remains in ∂E at all times, and thus the diffeomorphism $\psi_{p,q} : E_p \to E_q$ maps $\partial(E_p) = E_p \cap \partial E$ to $\partial(E_q) = E_q \cap \partial E$.

Example 4.2 The argument used in the proof of the Ehresmann fibration theorem can easily be used to define the *monodromy* associated to a loop in the target of a proper submersion $f : E \to B$. That is, if $p \in B$ and γ is a loop based at p (i.e. a smooth map $\gamma : [0, 1] \to B$ with $\gamma(0) = \gamma(1) = p$), then our construction gives us a diffeomorphism

$$\psi_{p,p} : E_p \to E_p,$$

the *geometric monodromy* associated to γ. To emphasize its dependence on the choice of γ, we denote it by ψ_γ. Of course, ψ_γ depends on the choice of Ehresmann connection, but it can quite easily be shown that any two different Ehresmann connections give rise to homotopic, indeed isotopic, diffeomorphisms. Thus, for example, the induced morphism of cohomology

$$\psi_\gamma^* : H^k(E_p) \to H^k(E_p)$$

(the *cohomologoical monodromy*) is independent of the choice of connection. Indeed, it is also easy to show that the homotopy class of ψ_γ depends only on the homotopy class of γ (as a loop based at p).

Exercise (i) Let E be the Möbius strip, which you can think of as the rectangle $[-R, R] \times [-1, 1]$ subject to the equivalence relation

$$(-R, t) \sim (R, -t).$$

There is an obvious map from E to its central circle B=$[-R, R] \times \{0\}$, in which each fibre is the interval $[-1, 1]$. Let $\gamma : [0, 1] \to B$ be a parametrisation of the central circle. Find the geometric monodromy diffeomorphism determined by a sensible choice of Ehresmann connection.

Exercise If $p : E \to B$ is a locally trivial fibre bundle, and if $f : S \to B$ is any smooth map, then there is a locally trivial fibre bundle $f^*(E \to B)$ over S with total space $S \times_B E := \{(s, e) \in S \times E : f(s) = p(e)\}$ and projection $(s, e) \mapsto s$.

1. Prove local triviality of $f^*(E \to B)$.
2. Given a horizontal distribution $\{H_e\}$ on E (as in the proof of the Ehresmann fibration theorem), construct a pull-back horizontal distribution on $S \times_B E$.

Although locally trivial fibre bundles are extremely important, maps from manifolds to $\mathbb{R}$ are rarely submersions. In particular, if M is compact then any smooth map $f : M \to \mathbb{R}$ must have a global maximum and a global minimum, and these are of necessity critical points. Morse Theory is concerned with what happens for "generic" smooth maps $f : M \to \mathbb{R}$ where M is compact; that is, with how the presence of critical points of the simplest sort alters the description of the map.

Definition 4.3 Suppose that $x \in M$ is a critical point of the smooth map $f : M \to \mathbb{R}$, and let $\phi : U \to V \subset \mathbb{R}^n$ be a chart arround x. The *Hessian matrix* of f at x, with respect to ϕ, is the matrix of second order partial derivatives

$$[\partial^2(f \circ \phi^{-1})/\partial x_i \partial x_j].$$

evaluated at $\phi(x)$.

Lemma 4.4 *The following properties of the Hessian matrix of the function f at a critical point are independent of the choice of chart: its rank, the number of negative eigenvalues, the number of positive eigenvalues.*

Proof **Exercise**. Note that as the Hessian is a symmetric matrix, all of its eigenvalues are real. □

Definition 4.5 (1) The critical point x of $f : M \to \mathbb{R}$ is *non-degenerate*, or a *Morse* critical point, if the Hessian at x is a non-singular matrix. In this case the *index* of the critical point is the number of negative eigenvalues it has.

(2) The function $f : M \to \mathbb{R}$ is a *Morse function* if all of its critical points are non-degenerate, and if for no two distinct critical points p_1, p_2 are the critical values $f(p_1), f(p_2)$ equal.

We remark that the condition that x be a non-degenerate critical point is a "transversality condition" on $f \circ \phi^{-1}$: it is equivalent to the map $d(f \circ \phi^{-1}) : V \to M_{1,n}(\mathbb{R})$ (sending x to the $1 \times n$ real matrix $[d_x(f \circ \phi^{-1}])$ being transverse to 0. With a bit of effort, this condition can be rephrased without reference to charts, as a property of f, and it is a general fact (the Thom Transversality Theorem) that "most" maps will satisfy any given transversality condition. The rephrasing is as follows: any function $f : M \to \mathbb{R}$ defines a section of the cotangent bundle T^*M, the vector bundle whose fibre over $x \in M$ is $T^*_x M := (T_x M)^*$, simply sending x to $d_x f \in T^*_x M$. A critical point of f is a point x such that $d_x f$ lies in the zero section $M \times \{0\} \subset T^*M$; then

Lemma 4.6 *The critical point x of $f : M \to \mathbb{R}$ is non-degenerate if and only if $df : M \to T^*M$ is transverse to the zero-section at x.*

Proof The association $M \mapsto T^*M$ is a functor on the category of smooth manifolds and diffeomorphisms, in the sense that a diffeomorphism $\phi : M \to N$ induces a diffeomorphism $T^*(\phi) : T^*M \to T^*N$ by $(x, \alpha) \mapsto (\phi(x), \alpha \circ (d_x\phi)^{-1})$. In fact, of course, this is how one proves in the first place that T^*M is a manifold (and a vector bundle over M). The diffeomorphism $T^*(\phi)$ maps zero section to zero-section; it follows that to prove the lemma, it is enough to use a chart $\phi : U \to \mathbb{R}^n$, where U is a chart on M around x. For the diagram

$$\begin{array}{ccc} T^*U & \xrightarrow{T^*(\phi)} & T^*\mathbb{R}^n \\ df \uparrow & & \uparrow d(f \circ \phi^{-1}) \\ U & \xrightarrow{\phi} & \mathbb{R}^n \end{array}$$

commutes, and thus $df \pitchfork M \times \{0\}$ if and only if $d(f \circ \phi^{-1}) \pitchfork \mathbb{R}^n \times \{0\}$. To lighten notation, write $f \circ \phi^{-1}$ simply as $h : \mathbb{R}^n \to \mathbb{R}$. The map $dh : \mathbb{R}^n \to T^*(\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{R}^n$ maps x to $(x, \sum \partial h/\partial x_i)$ (derivatives evaluated at x). It is transverse to $\mathbb{R}^n \times \{0\}$ if and only the projection to the second copy of $\mathbb{R}^n$ is a submersion — i.e. if and only if the matrix $[\partial^2 h/\partial x_i \partial x_j]$ is non-singular. This proves the lemma. $\square$

Morse functions are in fact open and dense (i.e. form an open and dense set) in the space of all smooth functions $M \to \mathbb{R}$, equipped with a sensible topology (e.g. the Whitney C^∞ topology). We do not show that here, but we will show the following simpler result:

Theorem 4.7 *Suppose that $M \subset \mathbb{R}^N$ is a smooth submanifold, and let $f : M \to \mathbb{R}$ be any smooth map. Then for almost all $a \in \mathbb{R}^N$, the function $f_a : M \to \mathbb{R}$ defined by $f_a(x) = f(x) + a \cdot x$ has only non-degenerate critical points.*

Proof Consider the map $F : M \times \mathbb{R}^N \to \mathbb{R}$ defined by $F(x, a) = f(x) + a \cdot x$. I claim that its differential *with respect to* $x \in M$, i.e. the map $d_M F : M \times \mathbb{R}^N \to T^*M$ sending (x, a) to $(x, d_x f_a)$, is transverse to the zero section of T^*M. The point is that $d_x f_a(\hat{x}) = df_x(\hat{x}) + a \cdot \hat{x}$; this can better be written as $d_x f_a = d_x f + a \cdot$. Now *every* linear map $T_x M \to \mathbb{R}$ is equal to $a \cdot$ for suitable $a \in \mathbb{R}^N$ — indeed, for suitable $a \in T_x M$ if so desired. The map

$$\begin{cases} \{x\} \times \mathbb{R}^N \to T^*_x M \\ (x, a) \mapsto d_x f + a \cdot \end{cases}$$

is just a translate of the linear epimorphism

$$\begin{cases} \mathbb{R}^N \to T_x^*M \\ a \mapsto a\cdot \end{cases}$$

and the derivative of this last map is the map itself, and thus an epimorphism; it follows that the derivative of the preceding map is also an epimorphism. And this implies that $d_M F \pitchfork M \times \{0\}$, as claimed.

Now we use a well-known elementary lemma due to René Thom:

Lemma 4.8 *Suppose that X, Y and Z are smooth manifolds and that W is a submanifold of Z. If $G : X \times Y \to Z$ is transverse to W, then for almost all $y \in Y$, the map $G_y : X \to Z$ defined by $G_y(x) = G(x, y)$, is transverse to W.*

Proof One checks (**Exercise**) that

$$f_y \pitchfork W \quad \text{if and only if } y \text{ is a regular value of } \pi : F^{-1}(W) \to Y$$

and then applies Sard's Theorem. □

We now use the lemma, taking $X = M, Y = \mathbb{R}^N, Z = T^*M$ and $W =$ the zero section of T^*M. The map G, of course, is $d_M F : M \times \mathbb{R}^N \to T^*M$.

Applying the lemma, we deduce that for almost all $a \in \mathbb{R}^n$, the map $(d_M F)_a : M \to T^*M$ is transverse to the zero section. But $(d_M F)_a$ is just the derivative of f_a; thus, we have shown that for almost all $a \in \mathbb{R}^N$, $df_a : M \to T^*M$ is transverse to $M \times \{0\}$ in T^*M, and thus that f_a has only non-degenerate critical points. □

We will refer to a function with only non-degenerate critical points as *locally Morse*.

Exercise Suppose that in the previous theorem M is compact. Show that
(i) If f_a has only non-degenerate critical points then there are only finitely many of them.
(ii) The set $\{a \in \mathbb{R}^N : f_a \text{ is locally Morse}\}$ is open and dense in $\mathbb{R}^N$; indeed, if f_a is locally Morse and b is close enough to a then f_b has the same number of critical points as f_a.
(iii) If f_a is locally Morse, there exist b arbitrarily close to a in $\mathbb{R}^N$ such that f_b is (*globally*) Morse (i.e. such that f_b is locally Morse and has all its critical values distinct).
(iv) Is the set of such b open and dense in $\mathbb{R}^N$?

One way of viewing this result is that *any* smooth function $f : M \to \mathbb{R}$ can be perturbed an arbitrarily small amount and become a Morse function. This is in fact the key step in showing that "Morse functions are dense".

Exercise If $a \in \mathbb{R}^N$ is a unit vector, the function $h_a : M \to \mathbb{R}$ defined simply by $x \mapsto a \cdot x$ is called a *height function*. Show that for almost all unit vectors the height function h_a is locally Morse.

Example 4.9 1. If $S^n \subset \mathbb{R}^{n+1}$ then every height function is Morse, having just two critical points.

2. If $T^2 \subset \mathbb{R}^3$ is the standard picture of a 2-torus (i.e. a doughnut lying on a table) then the height function h_{e_3} is not locally Morse, but h_{e_1} and h_{e_2} are Morse.

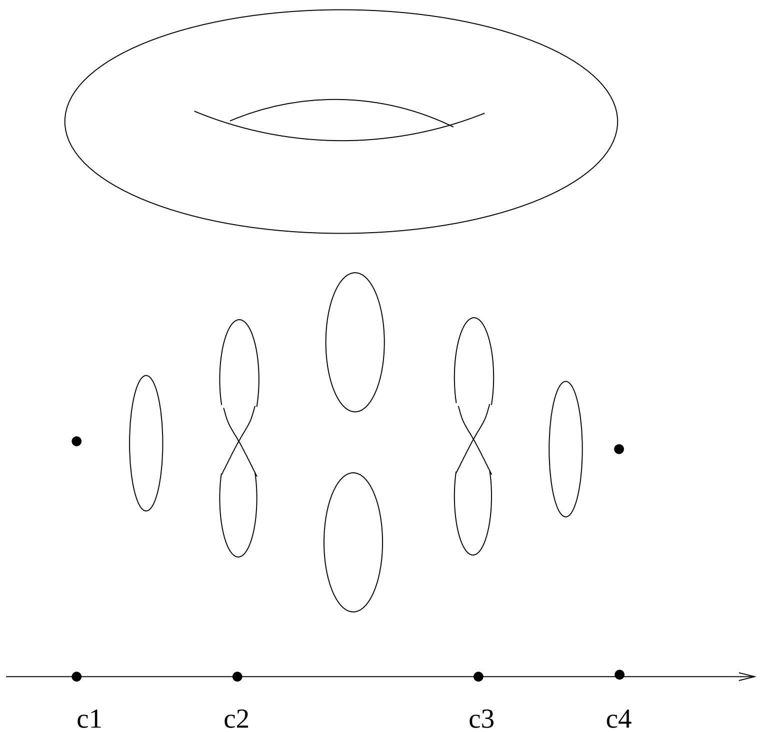


Figure 12: Over each interval (c_i, c_{i+1}) the height function on the torus is a locally trivial fibre bundle. As t passes through a critical value c_i, the fibre $f^{-1}(t)$ changes.

3. If $M \subset \mathbb{R}^N$ is a smooth submanifold and $p \in \mathbb{R}^N$, the function

$$\begin{cases} f_p : M \to \mathbb{R} \\ x \mapsto \|p - x\|^2 \end{cases}$$

is a called a "distance squared" function. Madsen and Tornehave show in Theorem 12.4 on page 114 that for almost all $p \in \mathbb{R}^N$, f_p is a locally Morse function. In fact a great deal of information about the differential geometry of the embedding $M \hookrightarrow \mathbb{R}^N$ can be obtained from a study of the critical points of the distance squared functions (Morse and non-Morse).

Suppose that $f : M \to \mathbb{R}$ is a Morse function. Morse theory is concerned with how the fibre $f^{-1}(t)$ and the "sub-level set" — the manifold with boundary $f^{-1}((-\infty, t])$ — change as t passes through a critical value. A first step is to describe them in the neighbourhood of the critical point.

Theorem 4.10 The Morse Lemma: *Suppose that the function $f : M \to \mathbb{R}$ has a non-degenerate critical point of index k at the point $p \in M$. Then there is a chart ϕ on M around p, with $\phi(p) = 0$, such that $f \circ \phi^{-1}$ takes the form*

$$f \circ \phi^{-1}(x_1, \ldots, x_n) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2.$$

Proof A detailed proof can be found in e.g. Madsen and Tornehave, pages 117-118. However, it is complicated and in my opinion not very illuminating.

Singularity Theory provides a much more interesting approach to this kind of question. It falls into two parts, one of which is easy:

Step 1 A suitable chart can be found in which $f \circ \phi^{-1}$ has the required form *modulo a remainder term of order 3.*

This is just the classification of quadratic forms: by means of any chart mapping x to 0, we can identify some neighbourhood of x in M with $\mathbb{R}^n$, so forget M and think of f as being defined on some neighbourhood of 0 in $\mathbb{R}^n$. Let H be the Hessian matrix of f at 0.

If $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism, then the Hessian matrix of $f \circ L$ at 0 is just

$$L^t H L$$

(we are assuming that f has a critical point at 0). The theory of real quadratic forms assures us that we can choose appropriate L so that $L^t H L$ is a diagonal matrix with each diagonal entry equal to 1, to zero or to -1. As f has a non-degenerate critical point, there can be no zeros. Now re-order the coordinates to get all the -1's at the start. This proves Step 1.

Step 2 I claim that any function with a non-degenerate critical point is *2-determined*: that is, if I add to it *any* function g with vanishing first and second order partials at 0, then there exists a diffeomorphism ϕ, defined on some neighbourhood of 0 in $\mathbb{R}^n$ and mapping 0 to 0, such that $(f+g) \circ \phi = f$. The way to prove this, due to John Mather, is to show that in fact there is a *family* of diffeomorphisms, ϕ_t, depending smoothly on t, such that $(f+tg) \circ \phi_t = f$, with $\phi_0 = \mathrm{id}_{\mathbb{R}^n}$. To construct such a family of diffeomorphisms, we integrate a certain vector field X constructed on a neighbourhood of $(0,0)$ in $\mathbb{R}^n \times \mathbb{R}$.

Think of the functions $f_t = f + tg$, as t varies, as together making a function $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $\quad F(x,t) = f(x) + tg(x)$. We have

$$\partial F/\partial t = g;$$

thus, if we can find $X_1, \ldots, X_n$ (functions of x and t) such that

$$g = X_1 \partial F/\partial x_1 + \cdots + X_n \partial F/\partial x_n$$

and we define a vector field X by

$$X = \partial/\partial t - X_1 \partial/\partial x_1 \ - \cdots - \ X_n \partial/\partial x_n$$

then $X \cdot F = 0$. This means that F *is constant along the integral curves of* X. Let $\Gamma_x : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$ be the integral curve of X satisfying $\Gamma_x(0) = (x,0)$. Because the component of X in the t direction has constant length 1, $\Gamma_x(t)$ has the form

$$\Gamma_x(t) = (\gamma_x(t), t)$$

for some smooth curve γ_x. By what we have just said,

$$F(\Gamma_x(t)) = F(\Gamma_x(0)) = F(x,0) = f(x).$$

Moreover

$$F(\Gamma_x(t)) = F(\gamma_x(t), t) = f(\gamma_x(t)) + tg(\gamma_x(t)) = (f + tg)(\gamma_x(t)).$$

As in the proof of the Ehresmann fibration theorem, the map

$$\phi_t : x \mapsto \gamma_x(t)$$

is a diffeomorphism; the previous two equations show that

$$(f + tg) \circ \phi_t = f$$

as required.

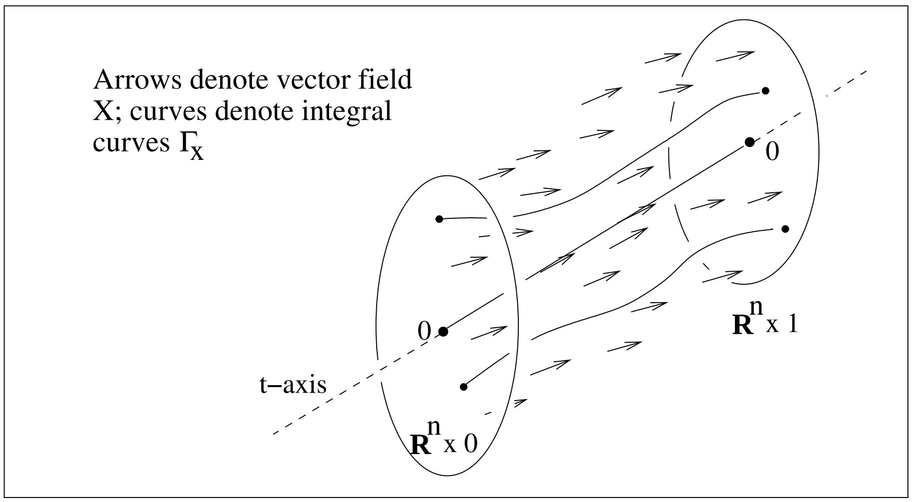


Figure 13

In order to guarantee that $\phi_t(0) = 0$, we make the additional requirement on X that it be tangent to the t-axis; that is, that $X_i(0, t) = 0$ for $i = 1, \ldots, n$.

We have yet to describe the construction of the vector field X. I will only give a sketch. It begins with the observation that invertibility of the Hessian matrix $H(f)(0)$ is equivalent to the solvability of the system of linear equations in unknown functions $a_{i,j}(x)$

$$\begin{array}{rcl} x_1 & = & a_{1,1}\frac{\partial f}{\partial x_1} + \cdots + a_{1,n}\frac{\partial f}{\partial x_n} \\ \cdots & & \quad\quad \cdots \quad \cdots \\ x_n & = & a_{n,1}\frac{\partial f}{\partial x_1} + \cdots + a_{1,n}\frac{\partial f}{\partial x_n} \end{array} \tag{7}$$

in some neighbourhood of 0 in $\mathbb{R}^n$ (by Cramer's rule). For by Taylor's Theorem, modulo terms of order ≥ 2 we have

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \cdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = H(f)(0) \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}$$

where $H(f)(0)$ is the Hessian matrix of f at 0; since $H(f)(0)$ is invertible, we get

$$(H(f)(0))^{-1}\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \cdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}$$

(again modulo terms of order ≥ 2). That is, we can take the $a_{i,j}$ in the system of equations (7) to be the entries of $(H(f)(0)^{-1}$. A result from elementary commutative algebra (Nakayama's lemma) shows that this is good enough: if we can solve the equation (1) to first order (i.e. ignoring terms of order 2 and higher), then we can solve it precisely.

Techniques from elementary commutative algebra (in particular Nakayama's Lemma), plus a patching argument (for (3)) show that this implies that

1. any function g, all of whose first and second order partials vanish at 0, can be written as a linear combination
$$g = a_1\frac{\partial f}{\partial x_1} + \cdots + a_n\frac{\partial f}{\partial x_n},$$
where the a_i are functions of x, vanishing at 0, and defined on some neighbourhood of 0;

2. the same is true if f is replaced by $f + tg$;

3. the same function g can be written as a linear combination
$$g = A_1\frac{\partial F}{\partial x_1} + \cdots + A_n\frac{\partial F}{\partial x_n},$$
where now the A_i are functions of x and t, defined on some neighbourhood of $\{0\} \times [0, 1]$ in $\mathbb{R}^n \times \mathbb{R}$ and vanishing when $x = 0$.

Note that (3) is, more or less, the construction of the vector field X (just take $X_i = -A_i$ for $i = 1, \ldots, n$).

Step 3 By Step 1, we have brought f to the form

$$f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2 + h$$

where h has order ≥ 3 at 0. Now apply Step 2, taking $g = -h$. □

Remark 4.11 The same method of proof gives a criterion for a function f to be *k-determined* in some neighbourhood of a critical point: if every equation

$$x_1^{i_1} \cdots x_n^{i_n} = a_1\frac{\partial f}{\partial x_1} + \cdots + a_n\frac{\partial f}{\partial x_n}$$

in which $i_1 + \cdots + i_n = k - 1$ can be solved for the unknown functions a_i in some neighbourhood of 0, then for every function g of order $k + 1$ at 0, $f + g$ can be transformed to f by a suitable change of coordinates.

The definition of k-determinacy, and this criterion for it, are expressed more succinctly in the language of germs:

(i) two functions or maps f_1, f_2 defined in some neighbourhoods of $0 \in \mathbb{R}^n$ "have the same germ" at 0 if there is a neighbourhood of 0 on which both are defined and on which they coincide. This is clearly an equivalence relation, and a germ is an equivalence class.
(ii) The set of germs at $0 \in \mathbb{R}^n$ of smooth functions is a ring, under the obvious operations of pointwise addition and multiplication; it is denoted $\mathcal{E}_n$.
(iii) $\mathcal{E}_n$ has a unique maximal ideal, m_n, consisting of all germs whose value at 0 is 0. It is generated by the germs of the coordinate functions $x_1, \ldots, x_n$. The j-th power m_n^j of m_n consists of all germs of functions f such that f and all partials of order less than j vanish at 0.
(iv) The ideal in $\mathcal{E}_n$ generated by the first order partial derivatives of f is denoted J_f.
(v) The set of germs at 0 of diffeomorphisms of $\mathbb{R}^n$ mapping 0 to 0 is a group under composition, and is denoted (in this context) by $\mathcal{R}$. It acts on $\mathcal{E}_n$ by composition on the right [4] : $(f, \phi) \mapsto f \circ \phi$. If f_1 and f_2 are in the same orbit, we say they are *right equivalent*.

Then by definition, $f \in \mathcal{E}_n$ is k-determined if, for all $g \in m_n^{k+1}$, $f + g$ is right-equivalent to f. The criterion quoted above becomes

Theorem 4.12 (John Mather, 1968) *If $m_n^{k-1} \subset J_f$ then f is k-determined.* □

This theorem is proved by exactly the same argument we used for the Morse Lemma, which of course is just a special case.

The point of having a "normal form" for a non-degenerate critical point is that it allows us to describe the change in the topology of the level set $f^{-1}(a)$ and the sub-level set $M_{(-\infty,a]}$ as a passes through a critical value. First, we note

Proposition 4.13 *Suppose that M is a compact manifold without boundary and that $f : M \to \mathbb{R}$ is a smooth function. If the interval $[a_1, a_2]$ contains no critical value of f then the manifolds with boundary $M_{(-\infty,a_1]}$ and $M_{(-\infty,a_2]}$ are diffeomorphic.*

Proof The hypothesis implies that there is an open interval (b_1, b_2) containing $[a_1, a_2]$ and containing no critical value of f. For any set $X \subset \mathbb{R}$, denote $f^{-1}(X)$ by M_X. By the Ehresmann fibration theorem, $f_| : M_{(b_1,b_2)} \to (b_1, b_2)$ is a locally trivial fibre bundle. Indeed, it is trivial: in the argument we gave in the proof of the fibration theorem, we can take U to be all of (b_1, b_2). Thus $M_{[b_1,a_1]}$ is diffeomorphic to $M_{[b_1,a_2]}$ (simply stretch the interval). This diffeomorphism gives rise to a diffeomorphism $M_{(-\infty,a_1]} = M_{(-\infty,b_1]} \cup M_{[b_1,a_1]}$ to $M_{(-\infty,a_2]} = M_{(-\infty,b_1]} \cup M_{[b_1,a_2]}$. We have to be a little careful, though: in order to piece together the identity diffeomorphism on $M_{(-\infty,b_1]}$ and the stretching diffeomorphism $M_{[b_1,a_1]} \to M_{[b_1,a_2]}$, we have to start the stretching very slowly in the vicinity of a_1. In other words, we replace the linear stretch sending $b_1 + t$ in $[b_1, a_1]$ to $b_1 + \alpha t$ in $[b_1, a_2]$ (where $\alpha = (a_2 - b_1)/(a_1 - b_1)$) by a slow-starting stretch $b_1 + t \mapsto b_1 + \rho(t)t$, where ρ is a smooth function equal to 1 on a neighbourhood of 0 and equal to α in a neighbourhood of $a_1 - b_1$. □

The value of 4.10 becomes clear with the following result:

[4] Strictly speaking, in order to comply with the definition of group action, this should be $(f, \phi) \mapsto f \circ \phi^{-1}$; but we're only interested in the orbits, so we ignore this difference

Lemma 4.14 *Let $f : M \to \mathbb{R}$ be a Morse function, with M compact, and suppose that a is a critical value of f, with the (unique) critical point $p \in f^{-1}(a)$ having index k. Then if $\varepsilon > 0$ is so small that a is the only critical point in $[a-\varepsilon, a+\varepsilon]$, there is a closed neighbourhood $\bar{U}$ of p such that*

1. *$\bar{U}$ is homeomorphic to $D^k \times D^{n-k}$;*
2. *$\bar{U} \cap M_{(-\infty,a-\varepsilon]}$ is contained in the level set $M_a = f^{-1}(a)$ and is homeomorphic to $S^{k-1} \times D^{n-k}$,*
3. *$M_{(-\infty,a+\varepsilon]}$ is homeomorphic to $M_{(-\infty,a-\varepsilon]} \cup \bar{U}$.* □

This lemma is often stated as
"$M_{(-\infty,a+\varepsilon]}$ is homeomorphic to the space obtained from $M_{(-\infty,a-\varepsilon]}$ by gluing $D^k \times D^{n-k}$ to M_a along $(\partial D^k) \times D^{n-k}$".

Example 4.15 1. If $k = 0$ (i.e. if p is a local minimum of f) then $U \cap M_{(-\infty,a-\varepsilon]} = \emptyset$ (since $S^{k-1} = \emptyset$): $M_{(-\infty,a+\varepsilon]}$ is diffeomorphic to the disjoint union of $M_{(-\infty,a-\varepsilon]}$ and an n-ball. Compare the picture of the torus on page 32.

2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be $f(x, y, z) = x^2 + y^2 - z^2$. Here the index is 1. The following figure shows $\mathbb{R}^3_{(-\infty,-1]}$, $\mathbb{R}^3_{(-\infty,0]}$ and $\mathbb{R}^3_{(-\infty,1]}$.

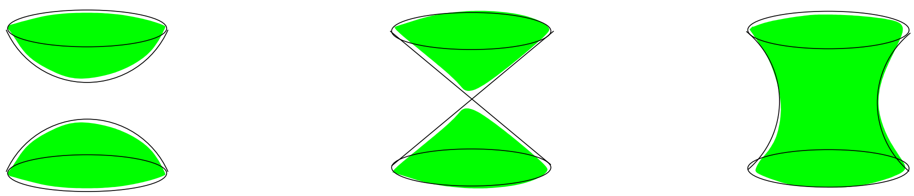


Figure 14

It is clear that $\mathbb{R}^3_{(-\infty,1]}$ is homeomorphic to $\mathbb{R}^3_{(-\infty,-1]}$ with a cylinder $D^1 \times D^2$ glued in along its top and bottom — i.e., along $(\partial D^1) \times D^2$:

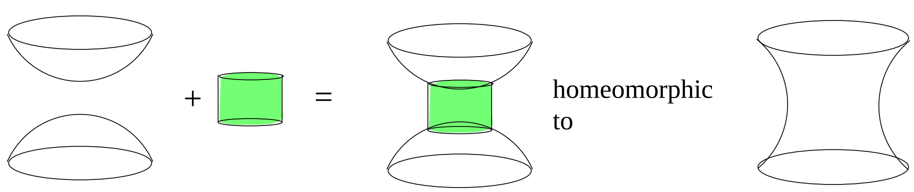


Figure 15

3. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be $f(x, y, z) = -x^2 - y^2 + z^2$. Here the index is 2. The following figure shows $\mathbb{R}^3_{(-\infty,-1]}$, $\mathbb{R}^3_{(-\infty,0]}$ and $\mathbb{R}^3_{(-\infty,1]}$ (in each case the complement of the solid shown).

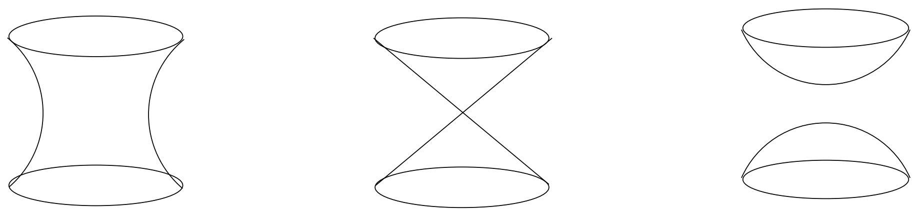


Figure 16

It is clear that $\mathbb{R}^3_{(-\infty,1]}$ is diffeomorphic to $\mathbb{R}^3_{(-\infty,-1]}$ with a cylinder $D^2 \times D^1$ glued in, but this time along $(\partial D^2) \times D^1$:

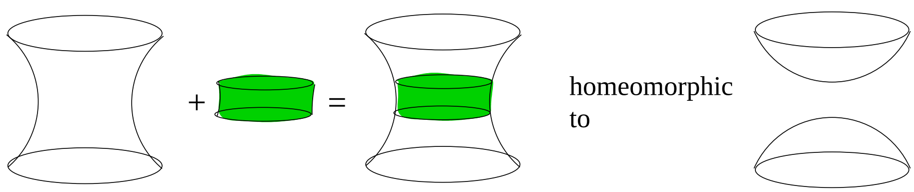


Figure 17

4. **Exercise** In the picture of the Morse function on the torus on page 32,
(i) Find the index of each of the critical points by choosing suitable local coordinates;
(ii) For each critical point c_i, make a drawing showing that $T^2_{(-\infty,c_i+\varepsilon]}$ is homeomorphic to $T^2_{(-\infty,c_i-\varepsilon]}$ with $D^{k_i} \times D^{2-k_i}$ glued in along $(\partial D^{k_i}) \times D^{2-k_i}$, where k_i is the index of the critical point lying over c_i.

Up to now our information about how $M_{(-\infty,a]}$ changes as a passes through a critical point, has been in terms of homeomorphisms. This can be improved to statement in terms of diffeomorphisms, at a slight cost in terms of precision:

Lemma 4.16 *Let $f : M \to \mathbb{R}$ be a Morse function, with M compact, and suppose that a is a critical value of f, with the (unique) critical point $p \in f^{-1}(a)$ having index k. Then if $\varepsilon > 0$ is so small that a is the only critical point in $[a - \varepsilon, a + \varepsilon]$, there is an open neighbourhood U of p such that*

1. *U is diffeomorphic to a contractible open set in $\mathbb{R}^n$;*

2. *$U \cap M_{(-\infty,a-\varepsilon)}$ is diffeomorphic to $S^{k-1} \times B^{n-k+1}$ (where B^j is the open unit ball in $\mathbb{R}^j$);*

3. *$M_{(-\infty,a+\varepsilon)}$ is diffeomorphic to $M_{(-\infty,a-\varepsilon)} \cup U$.* □

Note the difference between (ii) here and in 4.14: here we consider the $M_{(-\infty,a\pm\varepsilon)}$, which are open subsets of M, in place of the manifolds with boundary $M_{(-\infty,a\pm\varepsilon]}$; the intersection of U and $M_{(-\infty,a-\varepsilon)}$ is an open set, and thus of dimension n; it is diffeomorphic to $S^{k-1} \times B^{n-k+1}$, which one should think of as a the product of a thickened sphere $S^{k-1} \times (-1, 1)$ with an open ball B^{n-k}. It is worth trying to understand how the pictures in Example 4.15 change in this version of the lemma: the intersection $\bar{U} \cap M_{(-\infty,a-\varepsilon]}$, which lay in the level set $f^{-1}(a)$ in 4.14, is now thickened to an open set in $M_{(-\infty,a-\varepsilon]}$.

I will not prove either 4.14 or 4.16; a careful proof of the latter can be found in Appendix C of Madsen and Tornehave. It will not figure in the exam!

We go on to consider the consequences of 4.16 for cohomology. The first concerns the Euler characteristic. Recall that if M is a manifold then the Euler characteristic $\chi(M)$ is, by definition, equal to

$$\sum_q (-1)^q \dim\, H^q(M).$$

Exercise (Subadditivity of the Euler characteristic) Suppose that U_1 and U_2 are open subsets of the manifold M. Show, using Mayer Vietoris, that

$$\chi(U_1 \cup U_2) = \chi(U_1) + \chi(U_2) - \chi(U_1 \cap U_2).$$

Proposition 4.17 *In the situation of 4.16, suppose that $M_{(-\infty,a-\varepsilon)}$ has finite dimensional cohomology. Then so does $M_{(-\infty,a+\varepsilon)}$, and*

$$\chi(M_{(-\infty,a+\varepsilon)}) = \chi(M_{(-\infty,a-\varepsilon)}) + (-1)^k.$$

Proof The open set U of 4.16 is contractible, so $H^0(U) = \mathbb{R}$ and $H^q(U) = 0$ for $q > 0$. Hence $\chi(U) = 1$. Now apply subadditivity of the Euler characteristic:

$$\chi(M_{(-\infty,a+\varepsilon)}) = \chi(M_{(-\infty,a-\varepsilon)}) + \chi(U) - \chi(M_{(-\infty,a-\varepsilon)} \cap U).$$

Since $M_{(-\infty,a+\varepsilon)} \cap U$ is diffeomorphic to $S^{k-1} \times B^{n-k+1}$it is homotopy-equivalent to S^{k-1} and its Euler characteristic is $1 + (-1)^{k-1}$. The proposition follows. □

Corollary 4.18 *Suppose that $f : M \to \mathbb{R}$ is a Morse function, with M compact. Let c_k be the number of critical points of f with index k. Then*

$$\chi(M) = \sum_k (-1)^k c_k.$$

Proof **Exercise** (use 4.17, and work your way through the critical points of f, ordered by the size of their critical values). □

Exercise (The Morse alternative) Suppose that M is compact, the function $f : M \to \mathbb{R}$ has a single critical value in the interval $[a - \varepsilon, a + \varepsilon]$, and the corresponding critical point is non-degenerate and has index k. Show that for $q \neq k-1, k$ the inclusion $j : M_{(-\infty,a-\varepsilon)} \to M_{(-\infty,a+\varepsilon)}$ induces an isomorphism on q'th cohomology groups, and if $k \geq 2$ then either

1.
$$j^* : H^{k-1}(M_{(-\infty,a+\varepsilon)}) \to H^{k-1}(M_{(-\infty,a+\varepsilon)})$$

is an isomorphism and

$$j^* : H^{k}(M_{(-\infty,a+\varepsilon)}) \to H^{k}(M_{(-\infty,a+\varepsilon)})$$

has 1-dimensional kernel, or

2.
$$j^* : H^{k-1}(M_{(-\infty,a+\varepsilon)}) \to H^{k-1}(M_{(-\infty,a+\varepsilon)})$$

has 1-dimensional cokernel and

$$j^* : H^{k}(M_{(-\infty,a+\varepsilon)}) \to H^{k}(M_{(-\infty,a+\varepsilon)})$$

is an isomorphism.

What happens if $k = 0$ or 1?

Exercise 4.19 Recall that $SO(n)$ is the group of orientation-preserving linear isometries of $\mathbb{R}^n$.

1. Show that $A \in \mathrm{Gl}(n, \mathbb{R})$ is in $SO(n)$ if and only if $A^t A = I_n$, where I_n is the identity matrix, and $\det A = 1$.

2. Let $M_n(\mathbb{R})$ be the space of all $n \times n$ real matrices, and let $\mathrm{Sym}_n(\mathbb{R})$ be the subspace consisting of symmetric matrices. Show that the map $M_n(\mathbb{R}) \to \mathrm{Sym}_n(\mathbb{R})$ sending A to $A^t A$ has I_n as a regular value. (By (1), this shows that $SO(n)$ is a manifold of dimension $n(n-1)/2$).

3. Use (2) to show that the Lie algebra $T_{I_n} SO(n)$ is equal to the space of all skew-symmetric $n \times n$ real matrices, and go on find an expression for $T_A SO(n)$.

4. Consider the trace function $tr : M_n(\mathbb{R}) \to \mathbb{R}$, $tr(A) =$ the sum of the diagonal elements of A. Show that I_n is a critical point of tr, and go on to show that every diagonal matrix in $SO(n)$ is a critical point of tr.

5. Are the diagonal matrices non-degenerate critical points of tr?

The Poincaré-Hopf Theorem

From Corollary 4.18 one obtains an easy proof of the Poincaré- Hopf theorem on the sum of the indices of the isolated zeros of a vector field. We now sketch this, leaving out many details due to lack of time.

Let X be a vector field on the manifold M, and let $p \in M$ be an isolated zero of X. We define the *index* of X at p, denoted $\iota_p(X)$, as follows:
choose a chart $\phi : U \to V \subset \mathbb{R}^n$ around p, and define a vector field $\phi_*(X)$ (the "push-forward of X by ϕ") on V by

$$\phi_*(X)(y) = d_x\phi(X(x))$$

where $x = \phi^{-1}(y)$. Evidently $\phi_*(X)$ has an isolated zero at $\phi(p)$. For convenience let us now assume $\phi(p) = 0$. Choose a closed ball $\bar{B}_\varepsilon$ with centre 0, so small it contains no other zero of $\phi_*(X)$, and denote its boundary by S_ε. Then

$$\iota_p(X) = \deg\left\{\frac{\phi_*(X)}{\|\phi_*(X)\|} : S_\varepsilon \to S^{n-1}\right\}.$$

Lemma 4.20 *This is well-defined: it does not depend on the choice of ε, nor on the choice of chart ϕ.*

Proof The first assertion is easy: for any ε, there is an orientation-preserving diffeomorphism $g_\varepsilon : S^{n-1} \to S_\varepsilon$, which we may choose to depend smoothly on ε. Compose this with the map used to define $\iota_p(X)$ to get a map $S^{n-1} \to S^{n-1}$. As the degree of g_ε is 1, this new map has degree equal to the degree of the map $S_\varepsilon \to S^{n-1}$ used to define $\iota_p(v)$. As ε varies, we get a smooth homotopy, so the degree does not change.
The second assertion is more difficult; see Madsen and Tornehave, Lemma 11.18 page 107. □

We say the zero p of X is *non-degenerate* if the derivative at $\phi(p)$ of ϕ_*X (which we think of as a smooth map $V \to \mathbb{R}^n$) is non-singular.

Lemma 4.21 *p is a non-degenerate zero of X if and only if X is transverse to the zero-section $M \times \{0\}$ of TM at p. In this case*

$$\iota_p(X) = (X(M) \cdot M \times \{0\})_{(p,0)}$$

(Here $(X(M) \cdot M \times \{0\})_{(p,0}$ is the (oriented) intersection number at p of the image of X in TM and the zero section M. It is defined to be $+1$ if a positive basis for $T_{(p,0)}(X(M))$, followed by a positive basis for $T_{(p,0)}M$, gives a positive basis for $T_{(p,0)}(TM)$, and -1 if they give a negative basis. Although defined using the orientation of M, it is independent of the choice of orientation, because the orientation of TM is independent of that of M, and when we reverse the orientation of M then we also reverse the orientation of the zero section, so the changes cancel each other out.)

Proof Madsen and Tornehave, in Lemma 11.20 page 109, show that if p is a non-degenerate zero of X then

$$\iota_p(\phi_*(X)) = \begin{cases} 1 & \text{if } d_{\phi(p)}\phi_*(X) \text{ has positive determinant} \\ -1 & \text{if } d_{\phi(p)}\phi_*(X) \text{ has negative determinant} \end{cases}$$

The result follows. □

The key to the proof of Poincaré-Hopf is the fact that a Morse function $f : M \to \mathbb{R}$ gives rise to a vector field on M with only non-degenerate zeros. The idea is to construct a "gradient" vector field. Recall that the gradient $\nabla f(x)$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\nabla f(x) = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}).$$

Since on a manifold M there is in general no canonical system of coordinates it is not immediately clear how to define a gradient vector of a function on M. To make such a definition, we single out a crucial property of the gradient of a function on $\mathbb{R}^n$: for any vector v,

$$< \nabla f(x), v >= d_x f(v).$$

(the expression on the left is the inner product of $\nabla f(x)$ and v). To transfer this definition to a manifold M, we need a Riemann metric on M. Then *with respect to this metric* we define $\nabla f(x)$ to be the unique vector in T_xM such that for all $v \in T_xM$,

$$< \nabla f(x), v >= d_x f(v).$$

Another way of viewing this is to say that the Riemann metric gives rise to a bundle isomorphism

$$g : TM \to T^*M, \quad \text{defined by } (x, v) \mapsto (x, < v, \cdot >_x)$$

where $<,>_x$ is the scalar product in T_xM, so that for $v \in T_xM$ $< v, \cdot >: T_xM \to \mathbb{R}$ is a linear map. Then

$$\nabla f = g^{-1} \circ df.$$

Lemma 4.22 *g is an orientation-preserving diffeomorphism.*

Proof **Exercise** You have to supply the (natural) orientations on TM and T^*M. □

Lemma 4.23 *If $f : M \to \mathbb{R}$ has a non-degenerate critical point of index k at p, then*

$$(df(M), M \times \{0\})_{(p,0)} = (-1)^k.$$

Proof **Exercise** □

Corollary 4.24 *If M is compact and $f : M \to \mathbb{R}$ is a Morse function then*

$$\sum_{p \in M} \iota_p(\nabla f) = \chi(M).$$

Proof **Exercise** (using 3.20 and 3.21). □

Lemma 4.25 *If M is compact and X_1 and X_2 are any two vector fields with only non-degenerate zeros, then*

$$\sum_{p \in M} \iota_p(X_1) = \sum_{p \in M} \iota_p(X_2).$$

Proof Both sums are equal to the self-intersection number of the diagonal in $M \times M$. To understand this, recall that if Z is any compact oriented manifold of even dimension and W_1, W_2 are compact oriented submanifold with dim $W_i = (1/2)$dim Z, then $W_1 \cdot W_2$ is defined by moving one of them in a homotopy, say W_1, until it becomes transverse to W_2, and then counting intersection points with their signs. The usual argument about homotopy-invariance shows that the intersection number does not depend on the choice of perturbation of W_1: any two are homotopic to one another. This applies in particular if $W_1 = W_2$. Thus, if Δ is the diagonal in $M \times M$, then $\Delta \cdot \Delta$ is well-defined. It does not depend on the choice of orientation of M (indeed, it doesn't really even require M to be orientable at all — for $M \times M$ is always orientable).

To show that $\sum_{p \in M} \iota_p(X_1) = \Delta \cdot \Delta$, we show that there are neighbourhoods U of Δ in $M \times M$ and V of $M \times \{0\}$ in TM, and a diffeomorphism $\Psi : U \to V$ such that $\Psi(x, x) = (x, 0)$ and such that the diagram

$$\begin{array}{ccc} M \times M & & TM \\ \uparrow & & \uparrow \\ U & \stackrel{\Psi}{\longrightarrow} & V \\ \text{proj} \searrow & & \swarrow \pi \\ & M & \end{array}$$

(in which the vertical arrows are inclusions and proj is projection to the first factor) commutes.

This is easy: embed M in some $\mathbb{R}^N$ (so that $M \times M$ and TM are embedded in $\mathbb{R}^{2N}$), and define a map $\Psi : M \times M \to TM$ by

$$(x, y) \mapsto (x, \pi_x(y - x)),$$

where for each $x \in M$, $\pi_x : \mathbb{R}^n \to T_xM$ is orthogonal projection.

We have

(1) For all $x \in M$, $\Psi(x, x) = (x, 0)$, as required, and

(2) At each point $(x, x) \in \Delta$, $d_{(x,x)}\Psi : T_{(x,x)}M \times M \to T_{(x,0)}TM$ is an isomorphism (you should check this). Thus, at each point of Δ, Ψ is a local diffeomorphism.

It follows that

(3) Ψ is a local diffeomorphism at each point of some neighbourhood of Δ.

As Ψ is injective on Δ, it also follows that

(4) it is also injective on a neighbourhood of Δ (this uses (3) also - see Guillemin and Pollack Section 3 Exercise 10 page 19).

Thus,

(5) Ψ maps some neighbourhood U of Δ in $M \times M$ diffeomomorphically to some neighbourhhod $V = \Psi(U)$ of $M \times \{0\}$.

This diffeomorphism turns "a small perturbations of Δ" into (the image of) a vector field. That is, if Δ is shifted to Δ', so that $\Delta' \subset U$ and $\Delta' \pitchfork \Delta$, then (provided $\pi : \Psi(\Delta') \to M$ is a diffeomorphism), in an obvious way we can define a vector field X on M so that $\Psi(\Delta')$ is the image of X. As $\pi : \Psi(\Delta) \to M$ is a diffeomorphism, then provided Δ' is close enough to Δ, $\pi : \Psi(\Delta') \to M$ will indeed be a diffeomorphism. By Lemma 4.21 X has only non-degenerate zeros. In fact Ψ preserves intersection numbers (it respects orientations) and thus

$$\sum_{(x,x)\in\Delta'\cap\Delta} (\Delta'\cdot\Delta)_{(x,x)} = \sum_{x\in M} (X(M)\cdot M\times\{0\})_{(x,0)} = \sum_{x\in M} \iota_x(X).$$

The proof is complete. □

Before completing the proof of Poincaré Hopf, we need

Lemma 4.26 *Suppose that the vector field X has an isolated zero of index k at $p\in M$. If U is a neighbourhood of p in which X has no other zero, then*
(i) there exists a vector field Y which coincides with X outside U and has only non-degenerate zeros in U, and
(ii) $\sum_{q\in U}\iota_q(Y) = k$.

Proof (i) Choose a chart $\phi: U_1 \to V$ around p, with $U_1\subset U$, and let Z denote $\phi_*(X)$. We can think of Z simply as a map $V\to\mathbb{R}^n$. By Sard's theorem, almost all $v\in\mathbb{R}^n$ are regular values of Z, and so it follows that for almost all $v\in\mathbb{R}^n$, the map $Z+v$ defined by $(Z+v)(x) = Z(x)+v$ is transverse to 0 — and thus the vector field $Z+v$ has only non-degenerate zeros in V.

The basic idea of the proof now is to replace Z by $Z+v$, for suitable v, in some neighbourhood of $\phi(p)$.

Choose radii δ_1,δ_2 such that $0<\delta_1<\delta_2$ and $\overline{B(\phi(p),\delta_2)}\subset V$. As Z has no zero in $V\setminus\{p\}$, there exists $\varepsilon>0$ such that for all $x\in B(\phi(p),\delta_2)\setminus B(\phi(p),\delta_1)$, $\|Z(x)\|>\varepsilon$. Choose a regular value v of Z with $\|v\|<\varepsilon$, and choose a non-negative smooth function ρ on V such that ρ is identically zero outside $B(\phi(p),\delta_2)$ and ρ is identically 1 in $B(\phi(p),\delta_1)$. Then if $Z'(x) = Z(x)-\rho(x)v$, we have

(a) $Z' = Z$ outside $B(\phi(p),\delta_2)$

(b) Z' has only non-degenerate zeros in V. (You should check this.)

To construct Y, we simply replace X inside U_1 by $\phi_*^{-1}(Z')$.

(ii) It's clearly enough to show that $\sum_{q\in B(\phi(p),\delta_2)}\iota_q(Z') = k$. This follows from the second exercise after corollary 2.13: inside $B(\phi(p),\delta_2)$ we choose disjoint balls B_i around the zeros of Z', and a ball B_0 around $\phi(p)$. Apply the exercise first with $W = \overline{B(\phi(p),\delta_2)}\setminus B_0$ to deduce that

$$\iota_{\phi(p)}(Z) = \deg\left\{\frac{Z}{\|Z\|} : \partial\overline{B(\phi(p),\delta_2)} \to S^{n-1}\right\},$$

and then apply it with $W = \overline{B(\phi(p),\delta_2)}\setminus\cup_i B_i$ to deduce that

$$\sum_{q\in B(\phi(p),\delta_2)}\iota_q(Z') = \deg\left\{\frac{Z'}{\|Z'\|} : \partial\overline{B(\phi(p),\delta_2)} \to S^{n-1}\right\}.$$

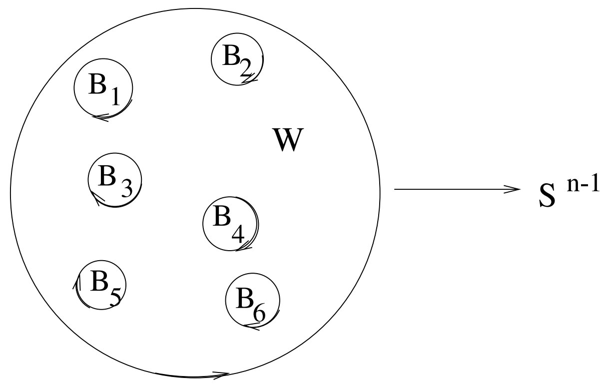


Figure 18: The orientation of ∂B_i is reversed when we consider it as part of ∂W

The right hand sides of these two equalities coincide, since $Z = Z'$ on $\partial \overline{B(\phi(p), \delta_2)}$. □

Corollary 4.27 *If X is any vector field with only isolated zeros on the manifold M, there is a vector field Y on M having only non-degenerate zeros and such that*

$$\sum_{p \in M} \iota_p(X) = \sum_{p \in M} \iota_p(Y).$$

□

The Poincaré-Hopf Theorem now follows:

Theorem 4.28 *Let M be a compact oriented manifold without boundary, and let X be a vector field on M with only isolated zeros. Then*

$$\sum_{p \in M} \iota_p(X) = \chi(M).$$

Proof Immediate from 4.24,4.25 and 4.27. □

Note that 4.25 and 4.28 give another interpretation of the Euler characteristic, as the self-intersection number of the diagonal in $M \times M$.

Exercise In the proof of Lemma 4.25 we (more or less) proved the following statement: suppose that $f : M \to N$ is a smooth map of smooth manifolds of the same dimension, that at every point of some compact subset K of M, $d_x f$ is an isomorphism, and that f is 1-1 on K. Then there is a neighbourhood U of K and a neighbourhood V of $f(K)$ in N, such that $f_| : U \to V$ is a diffeomorphism. Use this result to prove that if M is a compact submanifold of $\mathbb{R}^N$, then there is a diffeomorphism Ψ from some neighbourhood of the zero section in the normal bundle [5] NM to some neighbourhood V of M in $\mathbb{R}^N$. Using $r : \pi\Psi^{-1} : V \to M$, where $\pi : NM \to M$ is the bundle projection, we get a left inverse to the inclusion $i : M \to V$. If we choose U sensibly, we can ensure that $i : M \to V$ is a homotopy-equivalence, with homotopy inverse r.

[5] The normal bundle is by definition the set $NM = \{(x, v) \in M \times \mathbb{R}^N : v \in (T_x M)^\perp\}$. It is a smooth manifold, and indeed a smooth vector bundle over M.

5 The complex projective space $\mathbb{CP}^n$

$\mathbb{CP}^n$ is the space of all complex lines through 0 in $\mathbb{C}^{n+1}$. Thus, it is equal, as a set, to the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the equivalence relation which identifies points $(z_0, \dots, z_n)$ and $(\lambda z_0, \dots, \lambda z_n)$ (where λ is any non-zero complex number). We use square brackets to denote points in $\mathbb{CP}^n$ by the coordinates of any of their preimages in $\mathbb{C}^{n=1}$: thus $[z_0, \dots, z_n] = [\lambda z_0, \dots, \lambda z_n]$. This description also enables us to endow it with the natural quotient topology: $U \subset \mathbb{CP}^n$ is open if its preimage in $\mathbb{C}^{n+1}$ is open (in the usual Euclidean topology). Since $\mathbb{CP}^n$ is also the image under the quotient map $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ of the unit sphere $S^{2n+1} = \{(z_0, \dots, z_n) : \sum_i |z_i|^2 = 1\}$, it is compact.

$\mathbb{CP}^n$ is also a complex manifold of complex dimension n. We take

$$U_i = \{[z_0, \dots, z_n] \in \mathbb{CP}^n : z_i \neq 0\}$$

and define $\phi_i : U_i \to \mathbb{C}^n$ by

$$\phi_i[z_1, \dots, z_n] = (z_0/z_i, \dots, z_n/z_i)$$

(leaving out the z_i/z_i, of course).

Exercise U_i is open and ϕ_i is a homeomorphism.

Note that $\mathbb{CP}^n \setminus U_i$ is diffeomorphic to $\mathbb{CP}^{n-1}$.

Each change of coordinates $\phi_i \circ \phi_j^{-1}$ is a rational map, and thus holomorphic. This shows that $\mathbb{CP}^n$ is a complex manifold. Any holomorphic map on $\mathbb{C}^n$ is also a smooth map of the underlying $\mathbb{R}^{2n}$, so that $\mathbb{CP}^n$ is also a smooth $2n$-dimensional manifold.

Exercise $\mathbb{CP}^1$ is diffeomorphic to the 2-sphere S^2.

The Hopf fibration

We have observed that the quotient map $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ restricts to a surjection $S^{2n+1} \to \mathbb{CP}^n$. This map is smooth, and a submersion (**Exercise**). As it is evidently proper, by the Ehresmann fibration theorem it is a locally trivial fibre bundle, the *Hopf fibration*. Its fibre over $\ell \in \mathbb{CP}^n$ is the unit circle in the complex line ℓ. When $n = 1$, the Hopf fibration is a map $S^3 \to \mathbb{CP}^1 \simeq S^2$. H. Hopf showed that it is not homotopic to a constant map, and thus represents a non-trivial element of $\pi_3(S^2)$. This was the one of the first examples known of non-vanishing of a higher homotopy group (i.e. where $\pi_k(M) \neq 0$ for some $k > \dim M$).

Theorem 5.1

$$H^q(\mathbb{CP}^n) = \begin{cases} \mathbb{R} & \textit{if } q \textit{ is even} \\ 0 & \textit{if } q \textit{ is odd} \end{cases}$$

Proof The proof is essentially an induction, based on the fact that the open set $U_i \subset \mathbb{CP}^n$ is diffeomorphic to $\mathbb{C}^n = \mathbb{R}^{2n}$ and has complement in $\mathbb{CP}^n$ diffeomorphic to $\mathbb{CP}^{n-1}$. To make use of this decomposition, we need a general cohomological result:

Proposition 5.2 *Suppose that N is a smooth compact manifold and M is a smooth compact submanifold. Let $U = N \setminus M$, and let $j : M \to N$ and $i : U \to N$ denote the inclusions. Then there is a long exact sequence*

$$\cdots \to H^{q-1}(M) \xrightarrow{\delta} H^q_c(U) \xrightarrow{i_*} H^q(N) \xrightarrow{j^*} H^q(M) \to \cdots .$$

Proof Here i_* is the push-forward defined on page 21. We prove first

(1) $j^* : \Omega^q(N) \to \Omega^q(M)$ is an epimorphism.

(2) If $\omega \in \Omega^q(M)$ is closed, then there exists $\tau \in \Omega^q(N)$ such that $j^*(\tau) = \omega$ and such that $d\tau$ is identically zero on some open set in N containing M.

(3) If $\tau \in \Omega^q(N)$ and the support of $d\tau$ does not meet M, and if $j^*(\tau)$ is exact, then there exists $\sigma \in \Omega^{q-1}(N)$ such that $\tau - d\sigma$ is identically zero on some open neighbourhood of M in N.

We use a tubular neighbourhood of M in N. For the moment, this means just a neighbourhood V of M in N together with a retraction $r : V \to M$ such that $r \circ j = \mathrm{id}_M$, and such that $j \circ r$ is homotopic to the identity (so that r and j are "homotopy-inverses" to one another). Choose a smooth function ρ on N such that $\mathrm{supp}(\rho) \subset V$ and $\rho \equiv 1$ on a (smaller) tubular neighbourhood V_1 of M.

To prove (1), simply observe that for $\omega \in \Omega^q(M)$, the form $\rho \cdot r^*(\omega)$ extends to a form defined on all of N, and restricts to ω on M.

Exactly the same argument proves (2) also.

For (3), we can shrink V so that $V \cap \mathrm{supp}(d\tau) = \emptyset$. Thus, $\tau_{|\,V}$ is closed, and hence represents a cohomology class in $H^q(V)$. As $j : M \to V$ is a homotopy-equivalence, $j^* : H^q(V) \to H^q(M)$ is an isomorphism. It follows that $(\tau_{|\,V})$ is exact. Let $\sigma \in \Omega^{q-1}(V)$ satisfy $d\sigma = \tau_{|\,V}$. Extend σ to a form defined on all of N by multiplying by the bump function ρ described above. Then on the neighbourhood V_1 of M, $\tau - d\sigma \equiv 0$.

Now we proceed with the proof of the proposition. Let $\Omega^q(N, M)$ denote the subset of $\Omega^q(N)$ consisting of forms ω such that $j^*(\omega) = 0$. Then by (1) above, we have a short exact sequence of complexes

$$0 \to \Omega^\bullet(N, M) \hookrightarrow \Omega^\bullet(N) \xrightarrow{j^*} \Omega^\bullet(M) \to 0.$$

As usual, this gives a long exact sequence of cohomology,

$$\cdots \to H^{q-1}(M) \to H^q(N, M) \to H^q(N) \to H^q(M) \to \cdots .$$

To complete the proof, we show that the chain map $i_* : \Omega^\bullet_c(U) \to \Omega^\bullet(N, M)$ gives rise to an isomorphism on cohomology. We can then substitute $H^q_c(U)$ for $H^q(N, M)$ in the above long exact sequence.

To prove that i_* gives an isomorphism on cohomology, we use (2) and (3) above. First we show that it is surjective. Let $\omega \in \Omega^q(N, M)$ represent a cohomology class. Trivially, $j^*(\omega)$ is exact on M, and moreover $d\omega = 0$ on *all* of N. Thus by (3) above, there exists $\sigma \in \Omega^q(N)$ such that $\omega - d\sigma \equiv 0$ on some neighbourhood of M. The form $\omega - d\sigma$ is in $i_*(\Omega_c^q(U))$; it represents the same cohomology class as ω in $H^q(N)$, but unfortunately not in $H^q(N, M)$ since σ is not necessarily in $\Omega^{q-1}(N, M)$. To remedy this, we have to show that we can indeed modify σ to make it lie in $\Omega^{q-1}(N, M)$. For this we use (2): we know that $j^*(\sigma)$ is closed, since $dj^*(\sigma) = j^*(\omega) = 0$. Thus by (2) there exists $\tau \in \Omega^{q-1}(N)$ such that $j^*(\tau) = j^*(\sigma)$ and such that $d\tau \equiv 0$ on some neighbourhood of M. So we replace $\omega - d\sigma$ by $\omega - d(\sigma - \tau)$. For now $\sigma - \tau \in \Omega^{q-1}(N, M)$, so that $[\omega] = [\omega - d(\sigma - \tau)]$ in $H^q(N, M)$; moreover since $\omega - d\sigma \equiv 0$ on a neighbourhood of M and $d\tau \equiv 0$ on a neighbourhood of M, it follows that $\omega - d(\sigma - \tau) \equiv 0$ on a neighbourhood of M, and thus lies in the image of $i_* : \Omega_c^q(U) \to \Omega^q(N, M)$. This shows that i_* is surjective on cohomology.

To see that it is injective, suppose that $\omega \in \Omega_c^q(U)$ represents a cohomology class in $H_c^q(U)$, and suppose that $i_*(\omega)$ is exact on N. That is,

$$i_*(\omega) = d\tau$$

for some $\tau \in \Omega^{q-1}(M, N)$. We have to replace τ in this equation by some $\tau_1 \in \Omega_c^{q-1}(U)$, in order to conclude that $[\omega] = 0$ in $H_c^q(U)$. Now $j^*(\tau) = 0$, and $\mathrm{supp}(d\tau) \cap M = \emptyset$; it follows by (3) above that there exists $\sigma \in \Omega^{q-1}(N)$ such that $\tau - d\sigma \equiv 0$ on some neighbourhood of M in N. This means that $\tau - d\sigma \in \Omega_c^{q-1}(U)$; and of course $d(\tau - d\sigma) = d\tau = \omega$, so that $[\omega] = 0$ in $H_c^q(U)$, as required. □

Now we continue with the proof of 5.1. We apply the proposition with $N = \mathbb{CP}^n, M = \mathbb{CP}^{n-1}$. We assume the result for $n = 1$, since we know $\mathbb{CP}^1 \simeq S^2$. The long exact sequence from the lemma gives us

$$\cdots \to H_c^q(U_i) \xrightarrow{i_*} H^q(\mathbb{CP}^n) \xrightarrow{j^*} H^q(\mathbb{CP}^{n-1}) \to H_c^{q+1}(U_i) \to \cdots;$$

as $U_i \simeq \mathbb{C}^n = \mathbb{R}^{2n}$, we know $H_c^q(U_i) = 0$ if $q \neq 2n$ and $H_c^{2n}(U_i) = \mathbb{R}$. Thus if $q < 2n - 1$, $j^* : H^q(\mathbb{CP}^n) \to H^q(\mathbb{CP}^{n-1})$ is an isomorphism; moreover, $H^q(\mathbb{CP}^{n-1}) = 0$ if $q > 2n - 2$, and so the exact sequence gives us $H^{2n-1}(\mathbb{CP}^n) = 0, H^{2n}(\mathbb{CP}^n) = \mathbb{R}$, as stated. □

Theorem 5.3 *The cohomology algebra $H^*(\mathbb{CP}^n)$ is a truncated polynomial algebra*

$$H^*(\mathbb{CP}^n) = \mathbb{R}[c]/(c^{n+1})$$

where c is any non-zero cohomology class in $H^2(\mathbb{CP}^n)$ and (c^{n+1}) is the ideal generated by c^{n+1}.

Proof It's true for $n = 1$, so by induction assume it true for $n - 1$. Let $c_1 \neq 0$ in $H^2(\mathbb{CP}^{n-1})$, and let $c = j^*(c_1)$ in $H^2(\mathbb{CP}^n)$. By induction hypothesis, $c_1^q \neq 0$ in $H^{2q}(\mathbb{CP}^{n-1})$ for $q = 1, \ldots, n - 1$.

The inclusion $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$ induces a map

$$j^* : H^q(\mathbb{CP}^n) \to H^q(\mathbb{CP}^{n-1}),$$

which is an isomorphism for $q \leq 2n-2$. Hence $c^q \neq 0$ in $H^2q(\mathbb{CP}^n)$ for $q = 1, \ldots, n-1$, and thus c^q generates $H^{2q}(\mathbb{CP}^n)$ for $q = 1, \ldots, n-1$. It remains only to show that $c^n \neq 0$. But by Poincaré duality, the wedge pairing

$$H^{2n-2}(\mathbb{CP}^n) \times H^2(\mathbb{CP}^n) \to \mathbb{R}$$

is non-degenerate; since $H^{2n-2}(\mathbb{CP}^n) = \mathbb{R} \cdot c^{n-1}$ and $H^2(\mathbb{CP}^n) = \mathbb{R} \cdot c$, and the pairing takes $(\alpha c^{n-1}, \beta c)$ to

$$\int_{\mathbb{CP}^n} \alpha c^{n-1} \wedge \beta c = \alpha\beta \int_{\mathbb{CP}^n} c^n,$$

we must have $c^n \neq 0$. □

Like any complex manifold, $\mathbb{CP}^n$ acquires a canonical orientation coming from the complex structure on its tangent spaces. This orientation is defined as follows. Suppose that V is a finite dimensional complex vector space. Then V is also a real vector space. In what follows it will be convenient to denote V, thought of as a real vector space, by rV, though of a course as a set it is the same as V. If $e_1, \ldots, e_n$ is any basis for V, then $e_1, ie_1, \ldots, e_n, ie_n$ is a (real) basis for rV. Any complex-linear map $T : V \to V$ can also be thought of as a real linear map $rV \to rV$, in which case it will be denoted rT.

Lemma 5.4 $\det(rT) = |\det(T)|^2$.

Proof Induction on $n = \dim_{\mathbb{C}} V$. If $n = 1$, then T is just multiplication by a complex number $z = x + iy$. With respect to a basis e, ie of rV, rT has matrix

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

and thus $\det(rT) = x^2 + y^2 = |z|^2$, as claimed.

Now assume the result true for all spaces of dimension $n-1$, and suppose V has dimension n. Choose a complex line $\ell \subset V$ such that $T(\ell) \subset \ell$ (ℓ can be any line generated by an eigenvector). Then T induces $\mathbb{C}$-linear maps $T_0 : \ell \to \ell$ and $T_1 : V/\ell \to V/\ell$. By induction hypothesis, we can assume $\det(rT_0) = |\det(T_0)|^2$ and $\det(rT_1) = |\det(T_1)|^2$. Finally, we have

$$\det(T) = \det(T_0)\det(T_1) \quad \text{and} \quad \det(rT) = \det(rT_0)\det(rT_1),$$

and thus $\det(rT) = |\det(T)|^2$. □

From the lemma it follows that if $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases for V, then the two bases $u_1, iu_1, \ldots, u_n, iu_n$ and $v_1, iv_1, \ldots, v_n, iv_n$ for rV are equivalent, i.e., the change of basis matrix is positive. For this matrix is the same as the matrix of the map $rT : rV \to rV$ arising from the complex linear map $T : V \to V$ sending a_i to b_i for $i = 1, \ldots, n$. By the lemma, $\det(rT) > 0$.

Corollary 5.5 *(1) If V is a complex vector space of dimension n then rV has a well-defined orientation in which every basis $v_1, iv_1, \ldots, v_n, iv_n$ is positive.*
(2) If $T : V_1 \to V_2$ is a complex-linear isomorphism then $rT : rV_1 \to rV_2$ preserves this orientation.

Proof (1) is proved in the paragraph preceding the lemma. For (2), just observe that tT sends a basis of rV_1 of the form $v_1, iv_1, \ldots, v_n, iv_n$ to a basis of rV_2 of the same form. $\square$

Now suppose that M is a complex manifold. That is, it is a 2n-dimensional manifold with an atlas $\{U_\alpha, \phi_\alpha\}_{\alpha \in A}$ with the property that if we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ then each of the crossover maps $\phi_\alpha \circ \phi_\beta^- 1$ is complex analytic. Write $\phi_\alpha \circ \phi_\beta^{-1}$ simply as h. Like any smooth map, h has a derivative $d_x h$ at each point; and in fact $d_x h$ is not only real linear but complex linear. It follows that each T_pM has a natural complex vector space structure.

It follows from the corollary that the derivative of h preserves the orientation of $\mathbb{R}^{2n}$ coming from $\mathbb{C}^n$; as all the crossover maps preserve this orientation, it thus defines an orientation on M.

Despite this general fact, we consider the complex structure on $\mathbb{CP}^n$ from a slightly different point of view, which enables us also to define a Hermitian metric on each tangent space $T_p\mathbb{CP}^n$.

Let $< \, , \, >$ denote the usual hermitian inner product on $\mathbb{C}^{n+1}$,

$$< (w_1, \ldots, w_{n+1}), (z_1, \ldots, z_{n+1}) >= \sum_j w_j \bar{z}_j.$$

Then Re $< \, , \, >$ is the usual real inner product on the underlying $\mathbb{R}^{2n+2}$ (check this!). For $v \in \mathbb{C}^{n+1}$ let $(\mathbb{C}v)^\perp$ denote the orthogonal complement of $\mathbb{C}v$ *with respect to* $< \, , \, >$. It is an n-dimensional complex subspace of $\mathbb{C}^{n+1}$. Similarly, let $(\mathbb{R}v)^\perp$ be the orthogonal complement of $\mathbb{R}v$ with respect to the real inner product Re $< \, , \, >$. Evidently, $(\mathbb{C}v)^\perp \subset (\mathbb{R}v)^\perp$.

Lemma 5.6 *(1) Let $p \in \mathbb{CP}^n$ and $v \in \pi^{-1}(p) \subset S^{2n+1}$. Then there is an open neighbourhood U of p in $\mathbb{CP}^n$ and a smooth map $s : U \to S^{2n+1}$ such that $s(p) = v$ and $\pi \circ s = id_U$.*

(2) Let $v \in S^{2n+1}$ and $p = \pi(v)$. The differential $d_v\pi$ induces an $\mathbb{R}$-linear isomorphism $(\mathbb{C}v)^\perp \to T_p(\mathbb{CP}^n)$.

(3) $T_p\mathbb{CP}^n$ has a well-defined structure as complex vector space with a Hermitian inner product, with respect to which the isomorphism of (2) is a $\mathbb{C}$-linear isometry.

Proof Choose $U + U_j$ such that $p \in U_j$. Define $s_j : U_j \to S^{2n+1}$ by

$$s_j([z_0, \ldots, z_n]) = \left(\sum_{k=0}^{n} |z_k|^2 \right)^{-1/2} (z_0, \ldots, z_k)$$

where homogeneous coordinates are chosen so that $z_j = 1$. In other words, $s_j(q)$ is just $\phi_j(q)$, translated by 1 in the e_j direction, and then divided by its norm, to pull it onto the unit sphere.

(**Exercise** Show that (thinking of $q \in \mathbb{CP}^n$ as a line through 0 in $\mathbb{C}^{n+1}$) $s_j(q)$ is the unique point of $q \cap \mathbb{C}^j \times \{1\} \times \mathbb{C}^{n-j}$, divided by its norm.)

Note that $\pi \circ s_j = \mathrm{id}_U$. We then define s by multiplying s_j by the unique $\lambda \in S^1$ such that $\lambda s_j(p) = v$. Clearly $\pi \circ s = \mathrm{id}_U$. This proves (1). It follows by the chain rule that

$$d_v\pi : T_vS^{2n+1} \to T_p\mathbb{CP}^n$$

is surjective. Its kernel contains iv, for iv spans the tangent space at v to the unit circle $\{e^{it}v : 0 \le t \le 2\pi\}$ though v in S^{2n+1}, all of which is mapped to p by π. As $\ker d_v\pi$ is 1-dimensional, it is spanned by iv, and thus $d_v\pi$ is an isomorphism on any complement in T_vS^{2n+1} to the linear span of iv. Now $(\mathbb{C}v)^\perp \subset T_vS^{2n+1}$ (since $T_vS^{2n+1} = (\mathbb{R}v)^\perp$), and moreover $(\mathbb{C}v)^\perp = (\mathbb{C}iv)^\perp$, so $(\mathbb{C}v)^\perp$ is a complement to iv in T_vS^{2n+1}. This proves (2).

We use the isomorphism $d_v\pi_| : (\mathbb{C}v)^\perp \simeq T_p\mathbb{CP}^n$ to give $T_p\mathbb{CP}^n$ a complex structure in the obvious way : for $\lambda \in \mathbb{C}$ and $\hat{p} \in T_p\mathbb{CP}^n$, $\lambda\hat{p} = d\_v\pi_|(\lambda(d_v\pi_|)^{-1}(\hat{p}))$. All this uses is that $d_v\pi_|$ is a bijection. However, because $d_v\pi$ is a real linear isomorphism, this complex structure is compatible with its pre-existing structure as real vector space (remember that $\mathbb{R} \subset \mathbb{C}$).

By an analagous procedure we transfer to $T_p\mathbb{CP}^n$ the Hermitian metric from $(\mathbb{C}v)^\perp$.

Only one thing remains to check: that the complex structure and Hermitian metric we have just constructed on $T_p\mathbb{CP}^n$ *do not depend on the choice of v in $\pi^{-1}(p)$*. In fact, if v' is any other point in S^{2n+1} mapping to p, then there exists $\lambda \in \mathbb{C}$ of unit modulus such that $v' = \lambda v$. The map on $\mathbb{C}^{n+1}$ defined by multiplying by λ restricts to a map $L : S^{2n+1} \to S^{2n+1}$, and we have a commutative diagram

$$\begin{array}{ccc} T_vS^{2n+1} & \xrightarrow{d_vL} & T_{v'}S^{2n+1} \\ d_v\pi \searrow & & \swarrow d_{v'}\pi \\ & T_p\mathbb{CP}^n & \end{array}$$

- $d_vL = L$ restricts to a complex linear isomorphism $(\mathbb{C}v)^\perp \to (\mathbb{C}v')^\perp$, and it follows that the complex structure on $T_p\mathbb{CP}^n$ is independent of choice of v, and thus well defined;
- L is a Hermitian isometry, and it follows that the Hermitian metric on $T_p\mathbb{CP}^n$ is well defined.

□

Proposition 5.7 *Let V be an n-dimensional complex vector space with Hermitian inner product $<\ ,\ >$. Then*

(1) $g(v_1, v_2) = Re < v_1, v_2 >$ defines an inner product on rV.

(2) $\omega(v_1, v_2) = g(iv_1, v_2) = -Im < v_1, v_2 >$ defines an element of $Alt^2(rV)$.

(3) If $vol \in Alt^{2n}(rV)$ denotes the volume form with respect to the metric g and the orientation defined in 5.5, then $\omega^n = n!vol$.

Proof (1) and (2) I leave as an **Exercise**. For (3), suppose that $v_1, \ldots, v_n$ is an orthonormal basis for V with respect to $<\ ,\ >$. Then $v_1, iv_1, \ldots, v_n, iv_n$ is an orthonormal basis for rV with respect to g (check this!). Let $\epsilon_1, \tau_1, \ldots, \epsilon_n, \tau_n$ denote the dual basis for $(rV)^* = \mathrm{Alt}^1(rV)$. We have $\omega(v_j, iv_j) = -\omega(iv_j, v_j) = 1$, and ω vanishes on all other pairs of bais vectors. Thus,

$$\omega = \sum_{j=1}^{n} \epsilon_j \wedge \tau_j.$$

Moreover, vol $= \epsilon_1 \wedge \tau_1 \wedge \cdots \wedge \epsilon_n \wedge \tau_n$ (both sides are 1 on $(v_1, iv_1, \ldots, v_n, iv_n)$). The equality $\omega^n = n!\text{vol}$ can now easily be proved by induction on n. □

Now recall that on each tangent space $T_p\mathbb{CP}^n$ we have a complex structure and a Hermitian inner product. We define

- a 2-form ω on $\mathbb{CP}^n$ by taking the alternating 2-tensor defined in 5.7 on each tangent space $T_p\mathbb{CP}^n$, and
- a Riemannian metric (the *Fubini-Study metric*) on $\mathbb{CP}^n$ by taking the inner product g defined in 5.7 on each tangent space $T_p\mathbb{CP}^n$.

Theorem 5.8 *ω and g are, respectively, a smooth 2-form and a Riemannian metric. Moreover, ω is closed, and*

$$\omega^n = n!vol_{\mathbb{CP}^n}$$

where $vol_{\mathbb{CP}^n}$ is the volume form determined by g and the orientation coming from the complex structure.

Proof Smoothness of both ω and g is clear: both come from the Hermitian metric on the spaces $T_p\mathbb{CP}^n$ which is induced by the $\mathbb{R}$-linear ismorphism $d_v\pi : (\mathbb{C}v)^\perp \to T_p\mathbb{CP}^n$, and thus "depends smoothly on p". To see that ω is closed, we show that in fact it is the pull-back, via the smooth section s of π that we defined in 5.6, of a closed 2-form on S^{2n+1}. The 2-form in question is the restriction to S^{2n+1} of the canonical 2-form $\omega_{\mathbb{C}^{n+1}}$ defined by $\omega_{\mathbb{C}^{n+1}}(\hat{x_1}, \hat{z_2}) = -\text{Im} < \hat{z_1}, \hat{z_2} >$.

Exercise Show that $\omega_{\mathbb{C}^{n+1}} = \sum_{j=0}^n dx_j \wedge dy_j$ on $\mathbb{C}^{n+1}$, where x_j and y_j are the real and imaginary parts of the j-th complex coordinate function z_j. (This form is obviously closed on $\mathbb{C}^{n+1}$ and therefore on S^{2n+1}.)

Suppose that $\hat{v_1}, \hat{v_2} \in (\mathbb{C}v)^\perp$. Via $d_v\pi$ they map to $\hat{p_1}, \hat{p_2} \in T_p\mathbb{CP}^n$. We have

$$\omega(\hat{p_1}, \hat{p_2}) = -\text{Im} < \hat{p_1}, \hat{p_2} >= -\text{Im} < \hat{v_1}, \hat{v_2} >= \omega_{\mathbb{C}^{n+1}}(\hat{v_1}, \hat{v_2})$$

This almost shows what we want; the only problem is that although $\hat{p_k} = d_v\pi(\hat{v_k})$ for $k = 1, 2$, we don't know that $\hat{v}_k = d_ps(\hat{p_k})$, which is apparently what we need to conclude that $\omega = s^*(\omega_{\mathbb{C}^{n+1}})$. What we do know is that $d_ps(\hat{p_1}) = \hat{v}_1 + \alpha_1 iv, d_ps(\hat{p_2}) = \hat{v}_2 + \alpha_2 iv$ for some real scalars α_1, a_2. But now since $\hat{v_k} \in (\mathbb{C}iv)^\perp$, we have

$$-\text{Im} < \hat{v_1} + \alpha_1 iv, \hat{v_2} + \alpha_2 iv >= -\text{Im}\,(< \hat{v_1}, \hat{v_2} > + < \alpha_1 iv, \alpha_2 iv >) = -\text{Im} < \hat{v_1}, \hat{v_2} >$$

and we have won.

The fact that $\omega^n = n!\text{vol}$ follows directly from the last part of 5.7. □

We conclude this section by remarking that the closed 2-form ω we have defined on $\mathbb{CP}^n$ generates its cohomology ring. This is now clear: $\omega^n = n!\text{vol}$ and therefore $\int_{\mathbb{CP}^n} \omega^n \neq 0$, so $[\omega^n] \neq 0$ in $H^{2n}(\mathbb{CP}^n)$. It follows that $[\omega] \neq 0$ in $H^2(\mathbb{CP}^n)$.

Exercise Show that if n is even, there can be no orientation-reversing diffeomorphism $\mathbb{CP}^n \to \mathbb{CP}^n$; show also that if n is odd then the map induced by complex conjugation,

$$[z_0, \ldots, z_n] \mapsto [\bar{z_0}, \ldots, \bar{z_n}],$$

reverses orientation.

Exercise Show that any smooth map $f : \mathbb{CP}^m \to \mathbb{CP}^n$ induces the zero map on cohomology if $m > n$.(Hint: use the structure of the cohomology algebra described in 5.3).

Note (for the purposes of the next exercise) that any continuous map between manifolds can be approximated by a smooth map homotopic to it, and that any two smooth approximations both homotopic to f are therefore homotopic to one another, and hence induce the same morphism on cohomology. This morphism is therefore determined by f alone. In this way we can associate to a merely continuous map between manifolds a well-defined morphism on cohomology. The result of the last exercise still holds, of course, when $f : \mathbb{CP}^m \to \mathbb{CP}^n$ is a continuous map.

Exercise* Show that the Hopf map $S^{2n+1} \to \mathbb{CP}^n$ is not homotopic to a constant map. Here is a sketch of how to do it (taken from Madsen and Tornehave): suppose that $F : S^{2n+1} \times [0, 1] \to \mathbb{CP}^n$ is a homotopy from a constant map F_0 to the Hopf map π. Then it is possible to extend π to a continuous map $g : D^{2n+2} \to \mathbb{CP}^n$, defined by $g(x) = F(x/\|x\|, \|x\|)$. Now define $h : D^{2n+2} \to \mathbb{CP}^{n+1}$ by

$$h(z_0, \ldots, z_n) = [z_0, \ldots, z_n, \left(1 - \sum_{j=0}^{n} |z_j|^2\right)^{1/2}].$$

Observe that h maps the interior B^{2n+2} of D^{2n+2} bijectively onto $U_{n+1} = \{[w_0, \ldots, w_{n+1}] \in \mathbb{CP}^{n+1} : \ w_n \neq 0\} = \mathbb{CP}^{n+1} \setminus \mathbb{CP}^n$. Also, $h_{| \, S^{2n+1}}$ is the composite of the Hopf map $\pi : S^{2n+1} \to \mathbb{CP}^n$ with the inclusion $j : \mathbb{CP}^n \to \mathbb{CP}^{n+1}$. Now find $f : \mathbb{CP}^{n+1} \to \mathbb{CP}^n$ such that $f \circ j = \mathrm{id}_{\mathbb{CP}^n}$, and pass to de Rham cohomology to obtain a contradiction.

Exercise Show that $H^q(S^2 \times S^4) \simeq H^q(\mathbb{CP}^3)$ for every q, but that the cohomology algebras are not isomorphic.

Exercise (i) Show that any element $T \in \mathrm{Gl}(n + 1, \mathbb{C})$ passes to the quotient to give a map $\tilde{T} : \mathbb{CP}^n \to \mathbb{CP}^n$.
(ii) Which elements of $\mathrm{Gl}(n + 1, \mathbb{C})$ give rise in this way to isometries of $\mathbb{CP}^n$ (with respect to the Fubini-Study metric)?
(iii)* If $T \in \mathrm{Gl}(n + 1, \mathbb{C})$ and ω is the 2-form of 5.8, what is $\tilde{T}^*(\omega)$?

Exercise (i) Use the long exact sequence constructed in 5.2 to give another calculation of $H^*(S^n)$.

(ii) Ditto for the cohomology of the 2-torus.

Exercise Suppose that M^m is a compact submanifold of S^n, with $0 < m < n$, and let $U = S^n \setminus M$. Construct isomorphisms

$$H^q(U) \simeq H^{n-q-1}(M)^*$$

for $1 \leq q \leq n-2$. Show that $H^n(U) = 0$ and find short exact sequences

$$0 \to H^{n-1}(U) \to H^0(M)^* \to \mathbb{R} \to 0$$

and

$$0 \to \mathbb{R} \to H^0(U) \to H^{n-1}(M)^* \to 0.$$

Exercise A *symplectic manifold* is a manifold M equipped with a closed 2-form ω such that on each tangent space T_pM, ω induces a non-degenerate pairing. Show that

1. A symplectic manifold must be even-dimensional (Hint: After choosing a basis for T_pM, the the pairing on T_pM can be written in the form $\omega(v_1, v_2) = [v_1]^{\mathrm{t}} A[v_2]$, where A is a skew-symmetric matrix and $[v_1], [v_2]$ are the expressions of v_1, v_2 in the chosen basis. Show that $\det A \neq 0$ to conclude that $\dim M$ must be even.)

2. Suppose M is a symplectic manifold of dimension $2n$. Show that ω^n must be a nowhere-vanishing $2n$-form. Deduce that if M is compact then $[\omega] \neq 0$ in $H^2(M)$, and that $H^*(M)$ contains a subalgebra isomorphic to $H^*(\mathbb{CP}^n)$.

Exercise On the manifold T^*M there is a canonical 1-form α, defined as follows: a point $x \in TM$ is a linear form on the vector space $T_{\pi(x)}M$, where $\pi : TM \to M$ is the bundle projection. If $v \in T_xT^*M$, then $d_x\pi(v) \in T_{\pi(x)}M$, so we can evaluate x on it. This is how we define the 1-form α:

$$\alpha(v) = x(d_x\pi(v)).$$

(i) Find coordinate expressions for α and for $d\alpha$ when $M = \mathbb{R}^n$.
(ii) Find an expression for α using local coordinates on a manifold M.
(iii) Show that $\omega := d\alpha$ is a symplectic form on M.
(iv) Show that if $f : M \to \mathbb{R}$ is a smooth function and $L = df(M) \subset T^*M$ then the restriction of ω to L is zero.

6 Vector Bundles

There are two equivalent definitions:
(1) $E \xrightarrow{\pi} M$ is a smooth real vector bundle of rank k if each fibre $E_p = \pi^{-1}(p)$ is a real vector space of dimension k, and if for all $p \in M$ there is a neighbourhood U of p in M and a diffeomorphism $\phi_U : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that $\mathrm{proj} \circ \phi_U = \pi$ and such that the restriction of ϕ to each fibre is a linear isomorphism.
(2) $E \xrightarrow{\pi} M$ is a smooth real vector bundle of rank k if there is an open cover $\{U_\alpha\}_{\alpha \in A}$ of M and diffeomorphisms $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ such that for all $\alpha, \beta \in A$, the composite

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^k \xrightarrow{\phi_\alpha^{-1}} \pi^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\phi_\beta} (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$

restricts to a linear isomorphism $\{p\} \times \mathbb{R}^k \to \{\partial\} \times \mathbb{R}^k$ for all $p \in U_a \cap U_\beta$.
In the second definition, the vector space structure on the fibre E_p is the one pulled back from $\{p\} \times \mathbb{R}^k = \mathbb{R}^k$ by ϕ_α. The second part of this definition is in order that the structure be independent of the choice of local trivialisation ϕ_α.

The definition for complex vector bundle is analogous.

Example 6.1 1. For any smooth n-manifold M, $\pi : TM \to M$ and its dual T^*M are real vector bundles of rank n.

2. If $M^m \subset Y^n$ and Y has a Riemannian metric then the normal bundle $N(M, Y) = \{(x, v) \in M \times TY : v \in (T_x M)^\perp\}$ is a smooth vector bundle over M of rank $n - m$, with the obvious projection to X. It seems somewhat unsatisfactory to have to use a Riemann metric on Y to define $N(M, Y)$. Later we will see how to do it without.

3. There is a canonical complex line bundle (i.e. complex bundle of rank 1) over $\mathbb{CP}^n$, denoted H_n and defined as follows: its total space is

$$\{(\ell, y) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : y \in \ell\}$$

and the bundle projection simply maps (ℓ, x) to ℓ. That is, the fibre over ℓ is the line ℓ. **Exercise** Show that H_n is a smooth complex line bundle.

A *section* of the bundle $\pi : E \to M$ is a map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$.

Example 6.2 1. If $M \subset \mathbb{R}^N$ and g is a smooth function vanishing on M and defined on some neighbourhood V of M in $\mathbb{R}^N$, then restriction to M of the vector field grad g is a section of the normal bundle $N(M, \mathbb{R}^N)$. If $M \subset \mathbb{R}^N$ is the preimage of a regular value of a map $g : V \to \mathcal{R}^k$ (where V is some neighbourhood of M), then the k sections grad $g_{i \mid M}$ of $N(M, \mathbb{R}^N)$ are everywhere linearly independent, and can be used to define a global trivialisation $N(M, \mathbb{R}^N) \to M \times \mathbb{R}^k$ of $N(M, \mathbb{R}^N)$.

2. Suppose that G is a Lie group with group identity element e, and for each $g \in G$ let $\ell_g : G \to G$ denote left-multiplication by g, $\ell_g(h) = gh$. For any vector $v \in T_e G$ we use the maps ℓ_g to propagate v to give a vector field χ_v on G:

$$\chi_v(g) = d_e \ell_g(v).$$

In this way we get n vector fields on G; together they define a global trivialisation of TG.

At this point these Lecture Notes end: the remainder of the course consists of
1. Morita, *Geometry of Differential Forms*, Chapter 5.
2. Madsen and Tornehave, *From Calculus to Cohomology*, Chapters 15, 16, 17 and 18.
Read Morita first: he gives a clearer and more motivated account, occasionally skipping technical details, which can be found in Madsen and Tornehave. If time runs short, make sure you understand Morita, and worry less about Madsen and Tornehave. I strongly advise you to do the relevant exercises in both books.