Lectures on Singularities of Mappings

David Mond

1 Introduction

These lecture notes are intended as a very brief introduction to the theory of singularities of mappings. The quantity of material is of course greater than I will cover in three hours of lectures. I hope in the lectures to follow a particular thread which runs through the notes: that of an example in which we can follow all of the theoretical developments. My own involvement with the subject began with calculations in examples, and it seems to me that one of the pleasures of the subject is the range of calculable examples one meets and works with. The notes begin with basic notions of stability and codimension, and progress towards the vanishing homology of images and discriminants, one of the unexpectedly close parallels between this subject and the classical theory of isolated hypersurface singularities. Unlike the classical theory, the theory of singularities of mappings is still incomplete in certain basic aspects, and the lectures end with a discussion of one open question.

Mostly I refer to complex analytic germs, but except for the material in Section 4, everything applies equally to real C^∞ map-germs. In the complex analytic context, "diffeomorphism" means bi-analytic isomorphism.

This is an exclusively analytic (and C^∞) account; I have not attempted to cover the recent developments in Whitney equisingularity theory which began with Terry Gaffney's work on polar varieties and David Massey's work on Lê varieties.

The notes are incomplete, especially at the end, where they become rather sketchy. I hope they will continue to evolve, on my homepage. Please help them to do so by pointing out errors and omissions.

I am grateful to the organisers of the Summer School for the opportunity to lecture on this material, and for the stimulus that this has given me to try to gather it together in written form.

2 Stability, $\mathscr{A}_e$-codimension, Classification, Finite Determinacy

2.1 Stability and Codimension

Mather and Thom, in their work in the 60's on smooth maps, thought in global terms: a C^∞ map $f : N \to P$ is *stable* if its orbit under the natural action of $\mathrm{Diff}(N) \times \mathrm{Diff}(P)$ is open in $C^\infty(N, P)$, with respect to a suitable topology. Here we are interested in local geometry, and so we give a local version of this definition: a map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is stable if every deformation is trivial: roughly speaking, if f_t is a deformation of f then there should exist deformations of the identity maps of $(\mathbb{C}^n, 0)$ and $(\mathbb{C}^p, 0)$, φ_t and ψ_t, such that

$$f_t = \psi_t \circ f \circ \varphi_t. \tag{2.1}$$

A substantial part of Mather's papers [16]-[21] is devoted to showing that if all the germs of a mapping f are stable in this local sense then f is stable in the global sense. We will not discuss global stability any further.

To take account of base-points we make a slightly more formal definition:

Definition 2.1. (1) An *unfolding* of f is a map-germ

$$F : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$$

of the form

$$F(x, u) = (\tilde{f}(x, u), u)$$

such that $\tilde{f}(x, 0) = f(x)$.

Retaining the parameters u in the second component of the map makes the following definition easier to write down:

(2) The unfolding F is *trivial* if there exist germs of diffeomorphisms

$$\Phi : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^n \times \mathbb{C}^d, 0)$$

and

$$\Psi : (\mathbb{C}^p \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$$

such that

1. $\Phi(x, u) = (\varphi(x, u), u)$ and $\varphi(x, 0) = x$
2. $\Psi(y, h) = (\psi(y, u), u)$ and $\psi(y, 0) = y$
3. $$F = \Psi \circ (f \times \mathrm{id}) \circ \Phi \tag{2.2}$$

   (where $f \times \mathrm{id}$ is the 'constant' unfolding $(x, u) \mapsto (f(x), u)$).

(3) The map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is *stable* if every unfolding of f is trivial.

By writing $\varphi(x, u) = \varphi_u(x)$ and $\psi(y, u) = \psi_u(y)$, from 2.2 we recover the heuristic definition 2.1. We do not insist that the mappings φ_u and ψ_u preserve the origin of $\mathbb{C}^n$ and $\mathbb{C}^p$ respectively. After all, if the interesting behaviour merely changes its location, we should not regard the unfolding as non-trivial.

Example 2.2. (1) Consider the map-germ $f(x) = x^2$, and its unfolding $F(x, u) = (x^2 + ux, u)$. This is trivialised by the families of diffeomorphisms $\Phi(x, u) = (x + u/2, u)$, $\Psi(y, u) = (y - u^2/4, u)$. Both Φ and Ψ are just families of translations.

Exercise Check that indeed $F = \Psi \circ (f \times \mathrm{id}) \circ \Phi$.

(2) A germ of submersions is stable. For any submersion is equivalent to

Fortunately, there exists a simple and computable criterion for stability. If f is stable, then the quotient

$$T^1(f) := \frac{\{\frac{d}{dt}f_t|_{t=0} : f_0 = f\}}{\{\frac{d}{dt}(\psi_t \circ f \circ \varphi_t)|_{t=0} : \varphi_0 = \mathrm{id}\}}, \tag{2.3}$$

is equal to 0. In general this quotient is a vector space whose dimension measures the failure of stability. Mather ([17]) proved

Theorem 2.3. Infinitesimal stability implies stability: *f is stable if and only if $T^1(f) = 0$.*

One of the aims of this lecture is to develop techniques for calculating $T^1(f)$, and apply them in some examples.

Before continuing, we note that the denominator in (2.3) is very close to being the tangent space to the orbit of f under the group $\mathscr{A} = \mathrm{Diff}(\mathbb{C}^n, 0) \times \mathrm{Diff}(\mathbb{C}^p, 0)$. It is not quite equal to it, because we are allowing ϕ_t and ψ_t to move the origin (so they are not "paths in $\mathrm{Diff}(\mathbb{C}^n, 0)$ and $\mathrm{Diff}(\mathbb{C}^p, 0)$"). For this reason we write it as $T\mathscr{A}_e f$ and call it the 'extended' tangent space. The tangent space to the $\mathscr{A}$-orbit of f is denoted $T\mathscr{A}f$. We have

$$tf(m_{\mathbb{C}^n,0}\theta_{\mathbb{C}^n,0}) + \omega f(m_{\mathbb{C}^p,0}\theta_{\mathbb{C}^p,0}); \tag{2.4}$$

the maximal ideal $m_{\mathbb{C}^n,0}$ appears here because since $\varphi_t(0) = 0$ for all t, $d\varphi_t/dt|_{t=0}$ vanishes at 0 and thus belongs to $m_{\mathbb{C}^n,0}\theta_{\mathbb{C}^n,0}$; similarly for $m_{\mathbb{C}^p,0}$.

2.2 Notation and then calculation

By the chain rule,

$$\frac{d}{dt}\Big(\psi_t \circ f \circ \varphi_t\Big)|_{t=0} = df(\frac{d\phi_t}{dt}|_{t=0}) + (\frac{d\psi_t}{dt}|_{t=0}) \circ f.$$

Both $(d\varphi_t/dt)|_{t=0}$ and $(d\psi_t/dt)|_{t=0}$ are germs of vector fields, on $(\mathbb{C}^n, 0)$ and $(\mathbb{C}^p, 0)$ respectively: quite simply, $(d\varphi_t(x)/dt)|_{t=0}$ is the tangent vector at x to the trajectory $\varphi_t(x)$. In the same way, the elements of the numerator of 2.3 should be thought of as 'vector fields along f'; $(df_t/dt)|_{t=0}$ is the tangent vector at $f(x)$ to the trajectory $x \mapsto f_t(x)$. By associating to $(df_t/dt)|_{t=0}$ the map

$$\hat{f} : x \mapsto (x, (d/dt)f_t|_{t=0}) \in T\,\mathbb{C}^p,$$

we obtain a commutative diagram:

$$\begin{tikzcd} T\,\mathbb{C}^n \arrow[r, "df"] \arrow[d] & T\,\mathbb{C}^p \arrow[d] \\ \mathbb{C}^n \arrow[r, "f"] \arrow[ur, dashed, "\hat{f}"] & \mathbb{C}^p \end{tikzcd} \tag{2.5}$$

in which the vertical maps are the bundle projections. Elements of $\theta_{\mathbb{C}^n,0}$ can be written in various ways: as n-tuples,

$$\xi(x) = (\xi_1(x), \ldots, \xi_n(x))$$

(sometimes as columns rather than rows), or as sums:

$$\xi(x) = \sum_{j=1}^{n} \xi_j(x)\partial/\partial x_j.$$

The second notation emphasizes the role of the coordinate system on $\mathbb{C}^n, 0$. Similarly, elements of $\theta(f)$ can be written as row vectors or column vectors, or as sums:

$$\hat{f}(x) = \sum_{j=1}^{p} \hat{f}_j(x)\partial/\partial y_j.$$

We denote by

$\theta(f)$	the numerator of (2.3)
$\theta_{\mathbb{C}^n,0}$	the space of germs at 0 of vector fields on $\mathbb{C}^n$
$\theta_{\mathbb{C}^p,0}$	the space of germs at 0 of vector fields on $\mathbb{C}^p$
$tf : \theta_{\mathbb{C}^n,0} \to \theta(f)$	the map $\xi \mapsto df \circ \xi$
$\omega f : \theta_{\mathbb{C}^p,0} \to \theta(f)$	the map $\eta \mapsto \eta \circ f$

The notation "tf" is slightly fussy. We use it instead of *df* here because we think of *df* as the bundle map between tangent bundles induced by f, as in the diagram (2.5), whereas tf is the map "left composition with *df*" from $\theta_{\mathbb{C}^n,0}$ to $\theta(f)$. Some authors use "*df*" for both. In any case,

$$T^1(f) = \theta(f)/tf(\theta_{\mathbb{C}^n,0}) + \omega f(\theta_{\mathbb{C}^p,0}) =: \theta(f)/T\mathscr{A}_e f.$$

These spaces are not just vector spaces:

$\theta_{\mathbb{C}^n,0}$	is an $\mathcal{O}_{\mathbb{C}^n,0}$-module
$\theta(f)$	is an $\mathcal{O}_{\mathbb{C}^n,0}$-module
$tf : \theta_{\mathbb{C}^n,0} \to \theta(f)$	is $\mathcal{O}_{\mathbb{C}^n,0}$-linear, so
$\theta(f)/tf(\theta_{\mathbb{C}^n,0})$	is an $\mathcal{O}_{\mathbb{C}^n,0}$-module

But $T^1(f)$ is not an $\mathcal{O}_{\mathbb{C}^n,0}$-module, because $\mathcal{O}_{\mathbb{C}^p,0}$ is not. It is, however, an $\mathcal{O}_{\mathbb{C}^p,0}$-module; for via composition with f, $\mathcal{O}_{\mathbb{C}^n,0}$ becomes an $\mathcal{O}_{\mathbb{C}^p,0}$- module: we can 'multiply' $g \in \mathcal{O}_{\mathbb{C}^n,0}$ by $h \in \mathcal{O}_{\mathbb{C}^p,0}$ using composition with f to transport $h \in \mathcal{O}_{\mathbb{C}^p,0}$ to $h \circ f \in \mathcal{O}_{\mathbb{C}^n,0}$:

$$h \cdot g := (h \circ f)g.$$

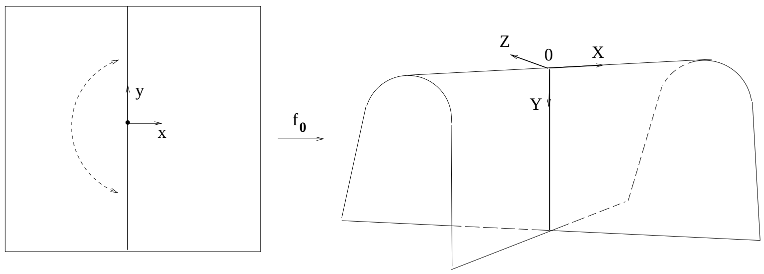
By this 'extension of scalars', every $\mathcal{O}_{\mathbb{C}^n,0}$-module becomes an $\mathcal{O}_{\mathbb{C}^p,0}$-module. This is where commutative algebra enters the picture. But we will not open the door to it in any serious way just yet. We simply note that

$\theta_{\mathbb{C}^p,0}$	is an $\mathcal{O}_{\mathbb{C}^p,0}$-module
$\omega f : \theta_{\mathbb{C}^p,0} \to \theta(f)$	is $\mathcal{O}_{\mathbb{C}^p,0}$-linear, so
$T^1(f)$	is an $\mathcal{O}_{\mathbb{C}^p,0}$-module

Example 2.4. (1) The map-germ

$$f(x, y) = (x, y^2, xy)$$

parametrising the cross-cap (pinch point, Whitney umbrella) is stable. We use coordinates (x, y) on the source and (X, Y, Z) on the target. We now calculate that $T^1(f) = 0$. For this purpose we divide $\mathcal{O}_{\mathbb{C}^2,0}$ into *even* and *odd* parts with respect to the y variable, and denote them by $\mathcal{O}^e$ and



$\mathscr{O}^o$. Every element of $\mathscr{O}^e$ can be written in the form $a(x, y^2)$, and every element of $\mathscr{O}^o$ in the form $ya(x, y^2)$. Then (we hope the notation is self-explanatory)

$$\theta(f) = \begin{pmatrix} \mathscr{O}^e \oplus \mathscr{O}^o \\ \mathscr{O}^e \oplus \mathscr{O}^o \\ \mathscr{O}^e \oplus \mathscr{O}^o \end{pmatrix}$$

and since

$$\omega f \begin{pmatrix} a(X,Y) \\ b(X,Y) \\ c(Y,Y) \end{pmatrix} = \begin{pmatrix} a(x,y^2) \\ b(x,y^2) \\ c(x,y^2) \end{pmatrix} \tag{2.6}$$

we see that the even part of $\theta(f)$ is indeed contained in $T^1(f)$, and we need worry only about the odd part. Since

$$tf\big(a(x,y^2)\partial/\partial x\big) = \begin{pmatrix} 1 & 0 \\ 0 & 2y \\ y & x \end{pmatrix} \begin{pmatrix} a(x,y^2) \\ 0 \end{pmatrix} = \begin{pmatrix} a(x,y^2) \\ 0 \\ ya(x,y^2) \end{pmatrix} \tag{2.7}$$

we get all of the odd part of the third row. Since

$$tf\big(a(x,y^2)\partial/\partial y\big) = \begin{pmatrix} 1 & 0 \\ 0 & 2y \\ y & x \end{pmatrix} \begin{pmatrix} 0 \\ a(x,y^2) \end{pmatrix} = \begin{pmatrix} 0 \\ 2ya(x,y^2) \\ xa(x,y^2) \end{pmatrix} \tag{2.8}$$

we get all of the odd part of the second row. Since

$$tf\big(ya(x,y^2)\partial/\partial x\big) = \begin{pmatrix} 1 & 0 \\ 0 & 2y \\ y & x \end{pmatrix} \begin{pmatrix} ya(x,y^2) \\ 0 \end{pmatrix} = \begin{pmatrix} ya(x,y^2) \\ 0 \\ y^2a(x,y^2) \end{pmatrix} \tag{2.9}$$

we get all of the odd part of the first row. So $T\mathscr{A}_e f = \theta(f)$, $T^1(f) = 0$ and f is stable.

(2) The map-germ $f(x,y) = (x, y^2, y^3 + x^2y)$ is not stable. The calculation of (2.6), (2.8) and (2.9) still apply, with insignificant modifications. The only change from (1) is that (2.7) now shows that

$$T\mathscr{A}_e f \supset \big(x\,\mathscr{O}^o\big)\partial/\partial Z \tag{2.10}$$

and we need an extra calculation

$$tf\big(ya(x,y^2)\partial/\partial y\big) = \begin{pmatrix} 1 & 0 \\ 0 & 2y \\ 2xy & x^2+3y^2 \end{pmatrix} \begin{pmatrix} 0 \\ ya(x,y^2) \end{pmatrix} = \begin{pmatrix} 0 \\ 2y^2a(x,y^2) \\ x^2ya(x,y^2) + 3y^3a(x,y^2) \end{pmatrix} \tag{2.11}$$

In view of (2.10) and what we know about the even terms, this completes the proof that

$$T^1(f) = \begin{pmatrix} \mathcal{O}^e + \mathcal{O}^o \\ \mathcal{O}^e + \mathcal{O}^o \\ \mathcal{O}^e + x\,\mathcal{O}^o + y^2\,\mathcal{O}^o \end{pmatrix} \tag{2.12}$$

It follows that $T^1(f)$ is generated, as a vector space over $\mathbb{C}$, by $y\partial/\partial Z$.

Definition 2.5. The $\mathscr{A}_e$-codimension of $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is the dimension, as a $\mathbb{C}$-vector space, of $T^1(f)$.

Exercise 2.6. Calculate the $\mathscr{A}_e$-codimension, and a $\mathbb{C}$-basis for $T^1(f)$, when

1. $f(x, y) = (x, y^2, y^3 + x^{k+1}y)$
2. $f(x, y) = (x, y^2, x^2y + y^5)$
3. $f(x, y) = (x, y^2, x^2y + y^{2k+1})$.

Remark 2.7. If $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is not an immersion then the ideal $f^*m_{\mathbb{C}^3,0}$ generated in $\mathcal{O}_{\mathbb{C}^2,0}$ by the three component functions of f is strictly contained in $m_{\mathbb{C}^2,0} = (x, y)$. It follows that $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0}/f^*m_{\mathbb{C}^3,0} \geq 2$. It can be shown (cf [23]) that every germ for which this dimension is exactly 2 (as in all the examples above) is $\mathscr{A}$-equivalent to one of the form $f(x, y) = (x, y^2, yp(x, y^2))$. Alternative characterisation: these are the map-germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ of Boardman type $\sum^{1,0}$.

Question to ponder for later: what is the significance here of the involution $(x, y) \mapsto (x, -y)$?

2.3 Lighter notation

Since we are nearly always referring to germs at 0, write

$$\begin{array}{rll} \mathcal{O}_n & \text{in place of} & \mathcal{O}_{\mathbb{C}^n,0} \\ \theta_n & \text{in place of} & \theta_{\mathbb{C}^n,0} \\ m_n & \text{in place of} & m_{\mathbb{C}^n,0} \end{array}$$

2.4 More sophisticated calculations

These examples are somewhat atypical. Calculating $T\mathscr{A}_ef$ is generally rather complicated. Checking that a given map-germ is it stable, however, is made much easier by a theorem of John Mather, which makes use of an auxiliary module known as the *contact tangent space*, and denoted $T\mathscr{K}_ef$, defined by

$$T\mathscr{K}_ef = tf(\theta_{\mathbb{C}^n,0}) + f^*m_{\mathbb{C}^p,0}\theta(f).$$

Here $f^*m_{\mathbb{C}^p,0}$ is the ideal in $\mathcal{O}_{\mathbb{C}^n,0}$ generated by the component functions of f. When $p = 1$, $T\mathscr{K}_ef$ is just the ideal $(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ of $\mathcal{O}_{\mathbb{C}^n,0}$. In any case it is always an $\mathcal{O}_{\mathbb{C}^n,0}$-module, which makes calculating with it very much easier than calculating $T\mathscr{A}_ef$. Like $T\mathscr{A}_ef$, $T\mathscr{K}_ef$ is the 'extended' tangent space to the orbit of f under a group action, which we will not say anything about. The true tangent space here is $T\mathscr{K}f = tf(m_n\theta_n) + f^*(m_p)\theta(f)$.

Mather's theorem is

Theorem 2.8. *If $T\mathscr{K}_e f + Sp_{\mathbb{C}}\{\partial/\partial y_1, \ldots, \partial/\partial y_p\} = \theta(f)$ then $T^1(f) = 0$ (so f is stable).*

Example 2.9. (1) We apply this theorem to the map-germ f of Example 2.4(1). We have

$$\begin{aligned} T\mathscr{K}_e f &= tf(\theta_{\mathbb{C}^2,0}) + f^* m_{\mathbb{C}^3,0}\theta(f) \\ &= \mathscr{O}_{\mathbb{C}^2,0} \cdot \{\partial f/\partial x, \partial f/\partial y\} + (x, y^2)\theta(f) \\ &= \mathscr{O}_{\mathbb{C}^2,0} \cdot \left\{ \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ 2y \\ x \end{pmatrix} \right\} + \begin{pmatrix} (x, y^2) \\ (x, y^2) \\ (x, y^2) \end{pmatrix} \end{aligned}$$

You can easily show that the condition of the theorem holds; in particular, since (x, y^2) contains the square of the maximal ideal of $\mathscr{O}_{\mathbb{C}^2,0}$, it's necessary only to check for terms of degree 0 and 1.
(2) The same theorem can be used to show that the map-germs

1. $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ defined by
$$f(x_1, x_2, x_3) = (x_1, x_2, x_3^4 + x_1 x_3^2 + x_2 x_3)$$
2. $f : (\mathbb{C}^4, 0) \to (\mathbb{C}^5, 0)$ defined by
$$f(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4^3 + x_1 x_4, x_2 x_4^2 + x_3 x_4)$$
3. $f : (\mathbb{C}^5, 0) \to (\mathbb{C}^6, 0)$ defined by
$$f(x, y, a, b, c, d) = (x^2 + ay, xy + bx + cy, y^2 + dx, a, b, c, d)$$

are stable. These are left as **Exercises**.

Remark 2.10. The reader will note that each of the germs listed in Example 2.9(2) is itself an unfolding of a germ of rank 0 (i.e. whose derivative at 0 vanishes). Of course, by means of the inverse function theorem *any* germ can be put in this form, in suitable coordinates. But in fact there is a general procedure for finding *all* stable map-germs as unfoldings of lower-dimensional germs of rank zero, based on Mather's theorem quoted here. The procedure is the following:

1. Given $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ of rank 0, calculate $T\mathscr{K}_e f$, and find a basis for the quotient $\theta(f)/T\mathscr{K}_e f$.
2. If $g_1, \ldots, g_d \in \theta(f)$ project to a basis for the quotient $\theta(f)/T\mathscr{K}_e f$ then the unfolding $F : (\mathbb{C}^n \times \mathbb{C}^d, (0,0)) \to (\mathbb{C}^p \times \mathbb{C}^d, (0,0))$ defined by
$$F(x, u_1, \ldots, u_d) = (f(x) + \sum_j u_j g_j(x), u_1, \ldots, u_d)$$
is stable.

Exercise 2.11. *Apply this procedure starting with $f(x, y) = (x^2, y^2)$.*

An ingenious result, due to Terry Gaffney, and extending Mather's, allows one to transform a guess for $T\mathscr{A}_e f$, (based perhaps on a calculation modulo some power of the maximal ideal (i.e. ignoring all terms of degree higher than some fixed k)) into a rigorous calculation.

Theorem 2.12. *Suppose that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is a map-germ such that*

$$T\mathscr{K}_e f \supset \mathfrak{m}_{\mathbb{C}^n,0}^{\ell}\, \theta(f)$$

and $C \subset \theta(f)$ is an $\mathcal{O}_{\mathbb{C}^p,0}$-submodule such that

$$C \supset \mathfrak{m}_{\mathbb{C}^n,0}^{k}\, \theta(f)$$

(where $k > 0$). Then

$$C = T\mathscr{A}_e f \qquad \Leftrightarrow \qquad C = T\mathscr{A}_e f + f^*\, \mathfrak{m}_{\mathbb{C}^p,0}\, C + \mathfrak{m}_{\mathbb{C}^n,0}^{k+\ell}\, \theta(f).$$

The proof I know of is in [23, 3:2]

Exercise 2.13. Find the smallest integer ℓ such that $T\mathscr{K}_e f \supset \mathfrak{m}_2^{\ell}\, \theta(f)$ when f is the map germ of Example 2.4(2).

2.5 Consequences of Finite Codimension

Let $f : \mathbb{C}^n \to \mathbb{C}^p$ (or $\mathbb{R}^n \to \mathbb{R}^p$) be an analytic (or C^∞) map. Its k-jet at a point x is the p-tuple consisting of the Taylor polynomials of degree k of its component functions. The k-jet of f at x is denoted by $j^k f(x)$. We say that a map-germ $f : (\mathbb{C}^n, x) \to (\mathbb{C}^p, y)$ is *k-determined for $\mathscr{A}$-equivalence* if any other map-germ having the same k-jet at x is $\mathscr{A}$-equivalent to f, and *finitely determined for $\mathscr{A}$-equivalence* if this holds for some finite value of k.

Theorem 2.14. (J.Mather [18]) *f is finitely determined if and only if $\dim_{\mathbb{C}} T^1(f) < \infty$.*

The smallest value of k for which this holds is the *determinacy degree* of f. Finding good estimates for the determinacy degree of f in terms of easily calculable data was once a major endeavour. Mather's original estimates (in [18]) were impractically large. They were greatly improved by Terry Gaffney and Andrew du Plessis ([5], [28]). In particular the following estimate due to Gaffney is useful:

Theorem 2.15. ([5]) *If $T\mathscr{A}_e f \supset \mathfrak{m}_{\mathbb{C}^n,0}^{k}\, \theta(f)$ and $T\mathscr{K}_e f \supset \mathfrak{m}_{\mathbb{C}^n,0}^{\ell}\, \theta(f)$ then f is $k + \ell$-determined.*

Since we are reaching conclusions about the $\mathscr{A}$-orbit of f, it is slightly curious that our hypotheses are framed in terms of $T\mathscr{A}_e f$ and not $T\mathscr{A} f$. Indeed it is (almost) obvious that if f is k-determined then

$$T\mathscr{A} f \supset \mathfrak{m}_n^{k+1}\, \theta(f) \tag{2.13}$$

To make this clear, we introduce the jet spaces $J^k(n, p)$.

Definition 2.16.

1. $\mathfrak{m}(n, p)$ is the vector space of all germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. It can be identified with $\mathfrak{m}_n\, \theta(f)$ for any $f \in \mathcal{O}(n, p)$.
2. $J^k(n, p)$ is the set of k-jets of germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$.
3. $j^k : \mathcal{O}(n, p) \to J^k(n, p)$ is the operation "take the k-jet". The map $j^k : \mathcal{O}(n, p) \to J^k(n, p)$ is surjective. Its kernel is $\mathfrak{m}_n^k\, \mathfrak{m}(n, p)$, so we can view $J^k(n, p)$ as $\mathfrak{m}(n, p)/\, \mathfrak{m}_n^k\, \mathfrak{m}(n, p)$.
4. For $k \leq \ell$, $\pi_k^\ell : J^\ell(n, p) \to J^k(n, p)$ is the projection ("truncate at degree k")

5. $\mathscr{A}^k = j^k(\mathscr{A}) \subset J^k(n,n) \times J^k(p,p)$ is the quotient of $\mathscr{A}$ acting naturally on $J^k(n,p)$.

The diagram (in which the rows are group actions)

$$\begin{array}{ccc} \mathscr{A} \times \mathfrak{m}_n\, \mathfrak{m}(n,p) & \longrightarrow & \mathfrak{m}_n\, \mathcal{O}(n,p) \\ \Big\downarrow{\scriptstyle j^k \times j^k} & & \Big\downarrow{\scriptstyle j^k} \\ \mathscr{A}^{(k)} \times J^k(n,p) & \longrightarrow & J^k(n,p) \end{array} \tag{2.14}$$

is commutative. The lower row is a finite-dimensional model of the upper row. In the lower row we really do have an algebraic group acting algebraically on an algebraic variety - indeed, on a finite dimensional complex vector space. This model provides motivation for many assertions, such as the statement that if f is k-determined then $T\mathscr{A}f \supset \mathfrak{m}_n^{k+1}\,\theta(f)$. What is clear is that if f is k-determined then

$$\mathscr{A}^{(\ell)} j^\ell f(0) = (\pi^\ell_k)^{-1}\big(\mathscr{A}^{(k)} j^k f(0)\big).$$

Now π^ℓ_k is linear, and its kernel is $j^\ell\big(\mathfrak{m}^{k+1}\,\theta(f)\big)$. So if f is k-determined,

$$T\mathscr{A}^{(\ell)} j^\ell f(0) \supset j^\ell\big(m^{k+1}\theta(f)\big)$$

Since

$$J^\ell(n,p) = \mathfrak{m}_n\,\theta(f)/\,\mathfrak{m}_n^{\ell+1}\,\theta(f),$$

this can be rewritten

$$T\mathscr{A}f + \mathfrak{m}_n^{\ell+1}\,\theta(f) \supset \mathfrak{m}_n^{k+1}\,\theta(f), \tag{2.15}$$

almost the statement (2.13) described as obvious above. If we knew that $\mathfrak{m}_n^{k+1}\,\theta(f)$ were a finitely generated module over $\mathcal{O}_{\mathbb{C}^p,0}$ then an application of Nakayama's Lemma would prove (2.13). But we don't know it, and in fact if $n > p$ it can't be true. Neverthless, it is possible to deduce (2.13) from (2.15) using some algebraic/analytic geometry:

1. $T\mathscr{K}_e f \supset T\mathscr{A}f$, so (2.15) implies

$$T\mathscr{K}_e f + \mathfrak{m}_n^{\ell+1}\,\theta(f) \supset \mathfrak{m}_n^{k+1}\,\theta(f). \tag{2.16}$$

2. Because (2.16) involves only $\mathcal{O}_{\mathbb{C}^n,0}$-modules, by Nakayama's Lemma we deduce that $T\mathscr{K}_e f \supset \mathfrak{m}_n^{k+1}\,\theta(f)$. This implies that $\dim_{\mathbb{C}}\big(\theta(f)/T\mathscr{K}_e f\big) < \infty$ (f is "$\mathscr{K}$-finite", or has "finite singularity type".)

3. Let J_f be the ideal in $\mathcal{O}_{\mathbb{C}^n,0}$ generated by the $p \times p$ minors of the matrix of df. Its locus of zeros is the critical set $\sum_f$, the set of points where f is not a submersion. By taking the determinants of p-tuples of elements of $\theta(f)$, from the fact that f is $\mathscr{K}$ finite we deduce that $\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0}\,/J_f + f^*\,\mathfrak{m}_p\,\mathcal{O}_{\mathbb{C}^n,0}) < \infty$. This condition has a clear geometrical significance (over the complex numbers!):

$$V(J_f + f^*\,\mathfrak{m}_p\,\mathcal{O}_{\mathbb{C}^n,0}) = \sum\nolimits_f \cap f^{-1}(0),$$

so f *is finite-to-one on its critical locus.*

4. From this it follows that every coherent sheaf of $\mathcal{O}_{\mathbb{C}^n,0}$ modules supported on $\sum_f$ is finite over $\mathcal{O}_{\mathbb{C}^p,0}$. In particular
$$\big(\mathfrak{m}^{\ell+1}\,\theta(f) + tf(\theta_n)\big)/tf(\theta_n)$$
is a finite $\mathcal{O}_{\mathbb{C}^p,0}$-module! So now we can apply Nakayama's Lemma to deduce (2.13) from (2.15): simply take the quotient on both sides by $tf(\theta_n)$.

It took some quite non-elementary steps to get to the "obvious" statement (2.13) from the truly obvious statement (2.15)!

Exercise 2.17. Use the techniques just introduced to prove Theorem 2.8. Note that the hypothesis of 2.8 is equivalent to
$$\theta(f) = T\mathscr{A}_e f + T\mathscr{K}_e f = T\mathscr{A}_e f + f^*\,\mathfrak{m}_p\,\theta(f).$$

In view of the fact that (2.13) *is* true, one might hope that its converse, which also seems reasonable, should also be true. But things are not so simple. They become simpler if we replace the group $\mathscr{A}$ by its subgroup $\mathscr{A}_1$ consisting of pairs of germs of diffeomorphisms whose derivative at 0 is the identity. This observation by Bill Bruce led to what was probably the final paper on finite determinacy, [1], in which *unipotent groups* $\mathscr{G}$ are identified as those for which the determinacy degree is equal to one less than the smallest power k such that $m_n^k\theta(f) \subseteq T\mathscr{G}_e f$.

2.6 Multi-germs

We have spoken only of 'mono'-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. But many of the interesting phenomena associated with deformations of mono-germs require description in terms of multi-germs, so they cannot sensibly be avoided. For example, a parametrised plane curve singularity splits into a certain number of nodes on deformation; each of these is stable; their number is an important invariant of the singularity.

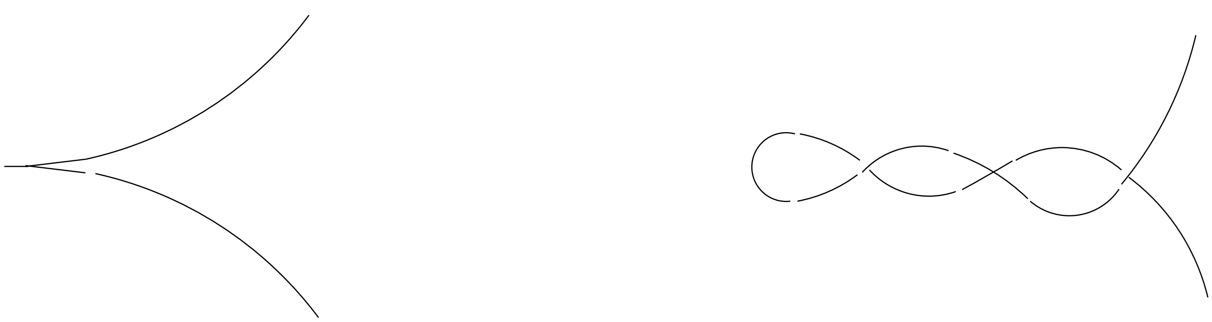
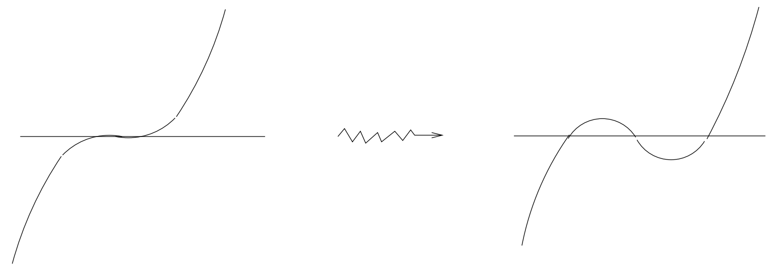


Figure 1: $t \mapsto (t^2, t^7)$ $\qquad$ $t \mapsto (t^2, t(t^2-4u)(t^2-9u)(t^2-16u))$

Example 2.18. The bi-germ consisting of two germs of immersion from $\mathbb{C}$ to $\mathbb{C}^2$ which meet tangentially is not stable. In suitable coordinates such a germ can be written
$$\begin{cases} f^{(1)} : s \mapsto (s, 0) \\ f^{(2)} : t \mapsto (t, h(t)) \end{cases} \tag{2.17}$$



We use independent coordinate systems s, t centred on each of the base-points. It will be useful to label the base-points $0^{(1)}$ and $0^{(2)}$. The two branches meet tangentially if $h \in (t^2)$. Let us calculate $T^1(f)$. We have

$$\theta(f) = \theta\big(f^{(1)}\big) \oplus \theta\big(f^{(2)}\big)$$

$$tf : \theta_{\mathbb{C},\{0^{(1)},0^{(2)}\}} \to \theta(f) \qquad \text{is equal to} \qquad tf^{(1)} \oplus tf^{(2)}$$

$$\theta_{\mathbb{C}^2,0} \to \theta(f) \qquad \text{is given by} \qquad \eta \mapsto (\eta \circ f^{(1)}, \eta \circ f^{(2)})$$

We represent elements of $\theta(f)$ as 2×2-matrices, in which the first column is in $\theta\big(f^{(1)}\big)$ and the second in $\theta\big(f^{(2)}\big)$. Elements of $\theta_{\mathbb{C},\{0^{(1)},0^{(2)}\}}$ are written as pairs $(a(s)\partial/\partial s, b(t)\partial/\partial t)$. Then

$$tf\big(a(s)\partial/\partial s, 0\big) = \begin{bmatrix} a(s) & 0 \\ 0 & 0 \end{bmatrix} \tag{2.18}$$

so in $T\mathscr{A}_e f$ we have everything in the top left corner; also

$$tf\big(0, b(t)\partial/\partial t\big) = \begin{bmatrix} 0 & b(t) \\ 0 & h'(t)b(t) \end{bmatrix} \tag{2.19}$$

$$\omega f\left(\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}\right) = \begin{bmatrix} \eta_1(s,0) & \eta_1(t,h(t)) \\ \eta_2(s,0) & \eta_2(t,h(t)) \end{bmatrix}. \tag{2.20}$$

Using (2.20) with $\eta_2 = 0$, in view of (2.18) we get everything in the top right corner. Now using (2.19), in the bottom right hand corner we get everything in the Jaocbian ideal J_h, and using (2.20) with $\eta_1 = 0$ and $\eta_2(X,Y) = p(X)$ we get everything of the form

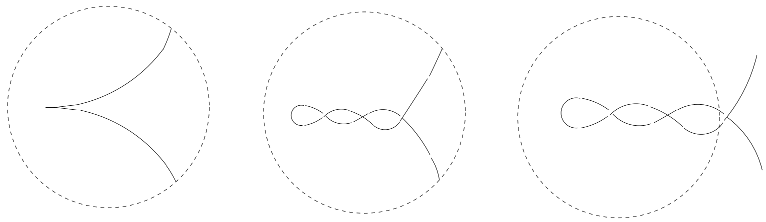
$$\begin{bmatrix} 0 & 0 \\ p(s) & p(t) \end{bmatrix}.$$

We have essentially shown

Proposition 2.19.

$$\theta(f)/T\mathscr{A}_e f \simeq \mathcal{O}_{\mathbb{C},0^{(2)}}/J_h$$

Notice that f can be perturbed to a bi-germ with ν nodes, where ν is the order of h. So the number of nodes is one more than the codimension. The relation between the $\mathcal{A}_e$-codimension of a map-germ and the geometry and topology of a stable perturbation is one of the most interesting aspects of the subject, and will be explored further below.



Milnor ball for undeformed singularity   Sufficiently small deformation   Too large deformation!

2.7 Finite codimension equals isolated instability

The next theorem is stated int two parts; the first is a special case of the second, but is easier to make sense of.

Theorem 2.20. (Terry Gaffney)*(1)* $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ *(*$n < p$*) has finite* $\mathscr{A}_e$*-codimension if and only if for every representative* $f : U \to V$ *of* f *there is a neighbourhood* V_0 *of* $0 \in V$ *such that for every* $y \in V_0 \smallsetminus \{0\}$ *the multi-germ* $f : (\mathbb{C}^n, f^{-1}(y)) \to (\mathbb{C}^p, y)$ *is stable.*
(2) $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ *(*$n \geq p$*) has finite* $\mathscr{A}_e$*-codimension if and only if for every representative* $f : U \to V$ *of* f *there is a neighbourhood* V_0 *of* $0 \in V$ *such that for every* $y \in V_0 \smallsetminus \{0\}$ *the multi-germ* $f : (\mathbb{C}^n, f^{-1}(y) \cap \sum_f) \to (\mathbb{C}^p, y)$ *is stable.*

This theorem is an easy application of the theory of coherent analytic sheaves; there is a a proof in [31]. As a consequence of 2.20, when a germ of finite codimension is deformed, the only qualitative changes occur in the vicinity of the unique unstable point. Near the boundary of the domain of any representative of the germ, nothing changes, in a sufficiently small deformation.

3 Versal Unfoldings and Stable Perturbations

An unfolding of a map-germ f_0 is *versal* if it contains, up to parametrised equivalence, every possible unfolding of the germ. In this section we make precise sense of this idea, and study some examples.

Definition 3.1. (1) Let $F, G : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$ be unfoldings of the same map germ $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. They are *equivalent* if there exist germs of diffeomorphisms

$$\Phi : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^n \times \mathbb{C}^d, 0)$$

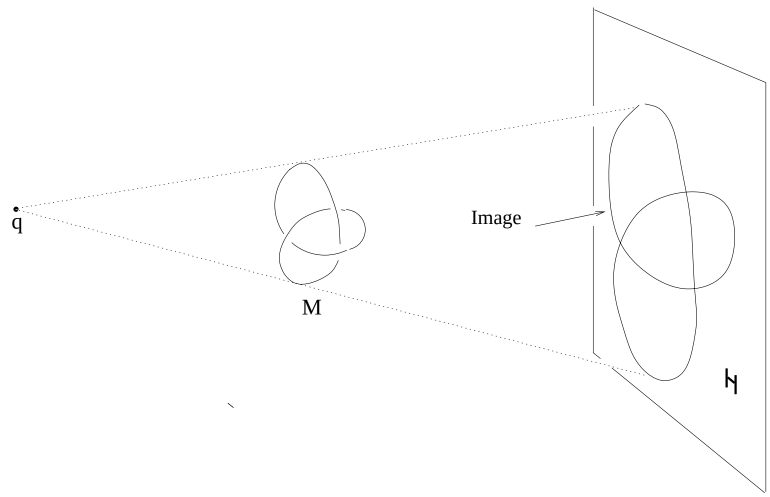
and

$$\Psi : (\mathbb{C}^p \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$$

such that

1. $\Phi(x, u) = (\varphi(x, u), u)$ and $\varphi(x, 0) = x$
2. $\Psi(y, h) = (\psi(y, u), u)$ and $\psi(y, 0) = y$
3. $F = \Psi \circ G \circ \Phi$

Note that an unfolding is trivial (Definition 2.1) if it is equivalent to the constant unfolding.



(2) With $F(x,u) = (f(x,u),u)$ as in (1), let $h : (\mathbb{C}^e,0) \to (\mathbb{C}^d,0)$ be a map germ. The unfolding $(\mathbb{C}^n \times \mathbb{C}^e,0) \to (\mathbb{C}^p \times \mathbb{C}^e,0)$ defined by

$$(x,v) \mapsto (f(x,h(v)),v)$$

is called the *pull-back* of F by h, and denoted by h^*F. The map-germ h in this context is often called the 'base-change' map, and we say that h^*F is the unfolding *induced from* F by h.

(3) The unfolding F of f_0 is *versal* if for every other unfolding $G : (\mathbb{C}^n \times \mathbb{C}^e,0) \to (\mathbb{C}^p \times \mathbb{C}^e,0)$ of f_0, there is a base-change map $h : (\mathbb{C}^e,0) \to (\mathbb{C}^d,0)$ such that G is equivalent (in the sense of (1)) to the unfolding h^*F (as defined in (2)).

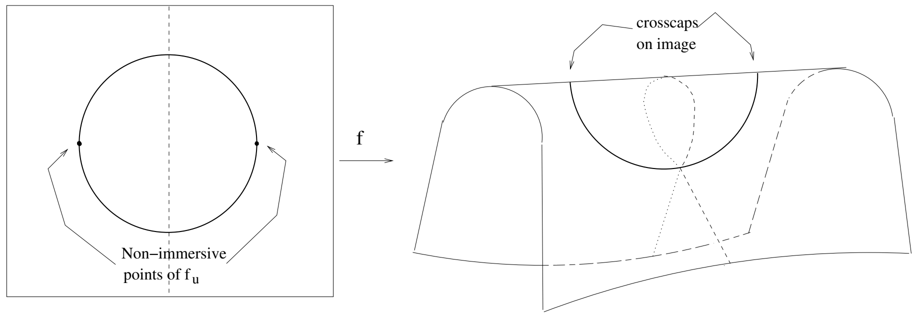
The term 'versal' if the intersection of the words 'universal' and 'transversal'. Versal unfoldings were once upon a time called universal, but later it was decided that they did not deserve this term, because the base-change map h of part (3) of the definition is not in general unique. Uniqueness is an important ingredient in the "universal properties" which characterise many mathematical objects, and so universal unfoldings were stripped of their title. However the intersection with the word 'transversal' is serendipitous, as we will see.

Example 3.2. Some light relief Consider a manifold $M \subset \mathbb{C}^N$. Radial projection from a point q into a hyperplane H is defined by the following picture: It defines a map $P_q : M \to H$. If the hyperplane H is replaced by another hyperplane H', then the corresponding projection $P'_q : M \to H'$ is left-equivalent to P_q; composing P'_q with the restriction of P_q to H', we get P_q. On the other hand, if we vary the point q then we may well deform the projection P_q non-trivially. So we consider the unfolding

$$P : M \times \mathbb{C}^N \to H \times \mathbb{C}^N .$$

It's instructive to look at this over $\mathbb{R}$ with the help of a piece of bent wire and an overhead projector. Are the unstable map-germs one sees versally unfolded in the family of all projections? This is discussed in [30] and again in [25].

Like stability, versality can be checked by means of an infinitesimal criterion. Let $F(x,u) = (f(x,u),u)$ be an unfolding of f_0. Write $\partial f/\partial u_j|_{u=0}$ as $\dot{F}_j$.



Theorem 3.3. (Infinitesimal versality is equivalent to versality) *The unfolding F of f_0 is versal if and only if*

$$T\mathscr{A}_e f_0 + Sp_{\mathbb{C}}\{\dot{F}_1, \ldots, \dot{F}_d\} = \theta(f_0)$$

– in other words, if the images of $\dot{F}_1, \ldots, \dot{F}_d$ in $T^1(f_0)$ generate it as (complex) vector space.

Proof See Chapter X of Martinet's book [14]. □

Exercise 3.4. Prove 'only if' in Theorem 3.3. It follows in a straightforward way from the definitions: let g be an arbitrary element of $\theta(f_0)$ and take, as G, the 1-parameter unfolding $G(x,t) = (f(x) + tg(x), t)$. Show that if G is equivalent to an unfolding induced from F then $g \in T\mathscr{A}_e f_0 + \mathrm{Sp}_{\mathbb{C}}\{\dot{F}_1, \ldots, \dot{F}_d\}$

Example 3.5. Consider the map-germ $f_0(x,y) = (x, y^2, y^3 + x^2 y)$ of Example 2.4. We saw that $y\partial/\partial Z$ projects to a basis for $T^1(f_0)$. So

$$F(x,y,u) = (x, y^2, y^3 + x^2 y + uy, u)$$

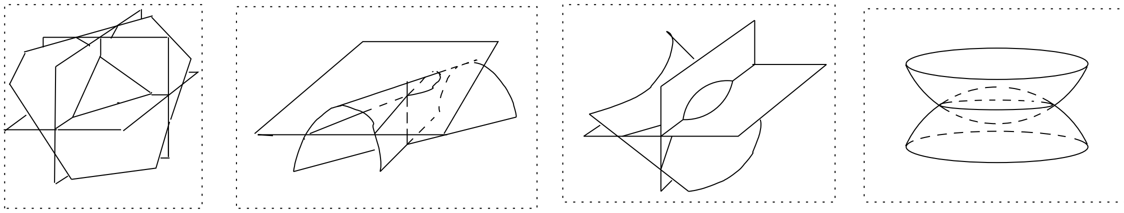
is a versal deformation. What is the geometry here? Think of F as a family of mappings,

$$f_u(x,y) = (x, y^2, y^3 + x^2 y + uy).$$

The *ramification ideal* $\mathscr{R}_{f_u} \subset \mathcal{O}_{\mathbb{C}^2}$ generated by the 2×2 minors of the matrix $[df_u]$ defines the set of points where f_u fails to be an immersion. Here $\mathscr{R}_{f_u} = (y, x^2 + u)$. So for $u \neq 0$, f_u has two non-immersive points. They are only visible over $\mathbb{R}$ when $u < 0$. How does f_u behave in the neighbourhood of each of these points? At each, $\mathscr{R}_{f_u}$ is equal to the maximal ideal; it follows that df_u is transverse to the submanifold $\sum^1 \subset L(\mathbb{C}^2, \mathbb{C}^3)$ consisting of linear maps of rank 1. In fact this transversality *characterises* the map-germ f of 2.4(1) up to $\mathscr{A}$-equivalence, though here we are not yet able to show that. Using this characterisation, we see that in a neighbourhood of the image of each of the two points $(\pm\sqrt{-u}, 0)$, the image of f_u looks like the drawing in Example 2.4. The key to assembling the image of f_u from its constituent parts is the curve of self-intersection. The only points mapped 2-1 by f_u are the points of the curve
$\{x^2 + y^2 + u = 0\}$; for $u < 0$ this is a circle when viewed over $\mathbb{R}$. Here points $(x, \pm y)$ share the same image. The two non-immersive points of f_u are the fixed points of the involution $(x,y) \mapsto (x, -y)$ which interchanges pairs of points sharing the same image.

The image contains a chamber; indeed it is homotopy-equivalent to a 2-sphere. This is no coincidence. The next figure shows images of stable perturbations of each of the remaining codimension 1 singularities of maps from surfaces into 3-space. Each is homotopy-equivalent to a 2-sphere.

Figure 2: Images of stable perturbations of codimension 1 germs of maps from the plane to 3-space



Some choices have been made regarding the real form: sometimes a change of sign which makes no difference over $\mathbb{C}$ does make a difference over $\mathbb{R}$. Nevertheless in all of these cases it is possible to choose a suitable real form whose perturbation is a homotopy 2-sphere.

3.1 Stable perturbations

We have looked at examples of mappings from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ for $n = 1, 2$. By inspection, we can see that the perturbations of the unstable maps we considered were at least locally stable: every (mono- and multi-) germ they contain is stable. In the dimension range we have looked at, every germ of finite codimension can be perturbed so that it becomes stable. These are "nice dimensions", to use a term due to John Mather. These dimension-pairs may be characterised by the following property: in the base of a versal deformation, the set of parameter-values u such that f_u has an unstable multi-germ is a proper analytic subvariety. It is known as the *bifurcation set*.

Mather carried out long calculations to determine the nice dimensions, published in [21]. Curiously, the nice dimensions are also characterised by the fact that every stable germ in these dimensions is weighted homogeneous, in appropriate coordinates.

When the bifurcation set B is a proper analytic subvariety of a smooth space, it does not separate it topologically (remember we're working in $\mathbb{C}^d$). That is, any two points u_1 and u_2 in its complement can be joined by a path $\gamma(t)$ which does not meet B. Because f_{u_1} and f_{u_2} are locally stable, each germ of the unfolding

$$(x, t) \mapsto (f_{\gamma(t)}(x), t)$$

is trivial; so f_{u_1} and f_{u_2} are locally isomorphic and globally C^∞-equivalent. Thus, to each complex germ of finite codimension we can associate a *stable perturbation* (any one of the mappings f_u for $u \notin B$) which is independent of the choice of u, at least up to diffeomorphism. Some care must be taken to define the domain of f_u; it is more than a germ, but not a global mapping $\mathbb{C}^n \to \mathbb{C}^p$. The situation is analogous to the construction of the Milnor fibre, in which several choices of neighbourhoods must be made, but in which the final result is nevertheless independent of the choices. Details may be found in [15].

4 Topology of the Disentanglement: Stable Images and Discriminants

In the theory of isolated hypersurface singularities a key role is played by the Milnor fibre. Here is a very brief description.

1. Let f be a complex analytic function defined on some neighbourhood of 0 in $\mathbb{C}^{n+1}$, and suppose it has isolated singularity at 0. Then by the curve selection lemma, there exists $\varepsilon > 0$

such that for ε' with $0 < \varepsilon' \leq \varepsilon$, the sphere of radius ε' centred at 0 is transverse to $f^{-1}(0)$. Let B_ε be the closed ball centred at 0 and with radius ε. Then from the transversality it follows that $f^{-1}(0) \cap B_\varepsilon(0)$ is homeomorphic (indeed, diffeomorphic except at 0) to the cone on its boundary $f^{-1}(0) \cap S_\varepsilon$. The ball $B_\varepsilon(0)$ is a *Milnor ball* for the singularity.

2. By an argument involving properness, one can show that for suitably small $\eta > 0$, all fibres $f^{-1}(t)$ with $|t| < \eta$ are transverse to S_ε. Let D_η be the closed ball in $\mathbb{C}$ with radius η and centre 0, and let $D^*_\eta = D_\eta \smallsetminus \{0\}$.

3. By the Ehresmann fibration theorem,

$$f| : B_\varepsilon \cap f^{-1}(D^*_\eta) \to D^*_\eta$$

   is a C^∞-locally trivial fibration. It is known as the Milnor fibration. Up to fibre-homeomorphism, it is independent of the choice of ε.

4. Its fibre is called the *Milnor fibre* of f. It has the homotopy type of a wedge of n-spheres, whose number μ, the *Milnor number of* f, is equal to the dimension of the Jacobian algebra of f,

$$\mathcal{O}_{\mathbb{C}^{n+1},0}/J_f.$$

The argument for the last statement is based on two facts:

1. if $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{n+1},0}/J_f = 1$ (in which case f is said to have a 'non-degenerate" critical point), then by the holomorphic Morse lemma, f is right-equivalent to $x \mapsto x_1^2 + \cdots + x_{n+1}^2$. An explicit calculation now shows that the Milnor fibre is diffeomorphic to the unit ball sub-bundle of the tangent bundle of S^n. This has S^n as a deformation-retract.

2. f can be perturbed so that the critical point at 0 splits into non- degenerate critical points. There are exactly μ of them, and each contributes one sphere to the wedge.

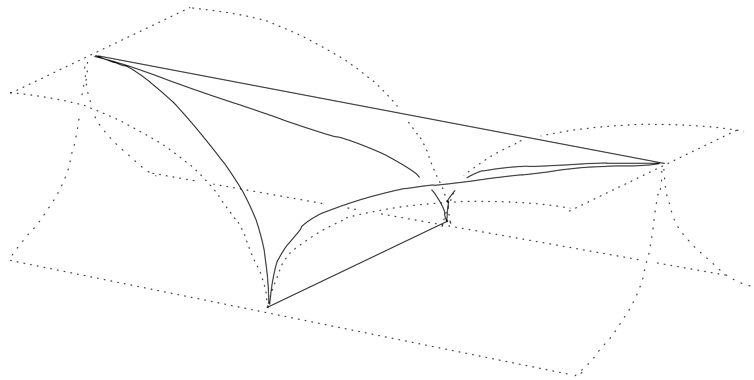
The dimension of the Jacobian algebra plays a second, completely different, role in the theory. If we consider only right-equivalence (composition of f with diffeomorphisms of the source) rather than right-left equivalence, then the quotient (2.3) which we used to measure instability, becomes the self-same Jacobian algebra, and indeed the Jacobian ideal itself is the extended tangent space for right-equivalence. The analogue of Theorem 3.3 shows that one can construct a versal deformation of f (versal for right-equivalence, that is) by taking $g_1, \ldots, g_\mu \in \mathcal{O}_{\mathbb{C}^{n+1},0}$ whose images in the Jacobian algebra span it as vector space, and defining

$$F(x, u_1, \ldots, u_\mu) = f(x) + \sum_j u_j g_j.$$

The Milnor fibration extends to a fibration over the complement of the discriminant Δ in the base-space $S = \mathbb{C}^\mu$; taking its associated cohomology bundle we obtain a holomorphic vector bundle of rank μ over the μ-dimensional space S. It is equipped with a canonical flat connection, the *Gauss-Manin connection.*

The objective now is to show that many of these same ingredients can be found in the theory of singularities of mappings.

We have already seen, in Example 3.5, that the *real* image of each codimension 1 germ f of mappings from surfaces to 3-space grows a 2-dimensional homotopy-sphere when f is suitably perturbed.



Proposition 4.1. *(1) Suppose that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is a map-germ of finite codimension. Then the image of a stable perturbation of f has the homotopy type of a wedge of n-spheres.*

(2 Suppose that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is a map-germ of finite codimension, with $n \geq p$. Then the discriminant (= set of critical values) of a stable pertubation of f has the homotopy-type of a wedge of $(p-1)$-spheres.

Terminology The number of spheres in the wedge is called the *image Milnor number*, μ_I, in case (1), and the *discriminant Milnor number*, μ_Δ, in case (2).

Proof of 4.1 Both statements are consequences of a fibration theorem of Lê Dung Trang ([22]), that says, in effect, that if (X, x_0) is a p-dimensional complete intersection singularity and $\pi : (X, x_0) \to (\mathbb{C}, 0)$ is a function with isolated singularity, in a suitable sense, then the analogue of the Milnor fibre of π (i.e. the intersection of a non-zero level set with a Milnor ball around x_0) has the homotopy-type of a wedge of spheres of dimension $p-1$. To apply this theorem here, we take, as X, the germ of the image in case (1), or discriminant, in case (2), of a 1-parameter stabilisation of f: that is, an unfolding $F : (\mathbb{C}^n \times \mathbb{C}, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}, 0)$ with $F(x, u) = (\tilde{f}(x, u), u) = (f_u(x), u)$ such that f_u is stable for $u \neq 0$. Then $(X, 0)$ is a hypersurface singularity, and thus a complete intersection. We take, as π, the projection to the parameter space. Thus $\pi^{-1}(u)$ is the image (or discriminant) of f_u. The fact that π has isolated singularity is a consequence of the fact that f_u is stable for $u \neq 0$. For this implies that the unfolding is trivial away from $u = 0$, so that the vector field $\partial/\partial u$ in the target of π lifts to a vector field tangent to X. □

Discriminant of stable perturbation of the bi-germ

$$\begin{cases} (u, v, w) \mapsto (u, v, w^3 - uw) \\ (x, y, z) \mapsto (x, y^3 + xy, z) \end{cases}$$

Siersma proves in [29] that the number of spheres in the wedge is counted by the sum of the Milnor numbers of the isolated critical points of the defining equation g of the image/discriminant which move off the image/discriminant as f (and with it g) is deformed. The proof can be understood as follows. Let $g_u : B_\varepsilon \to \mathbb{C}$ be a reduced defining equation for the image/discriminant of f_u, varying analytically with u for $u \in (\mathbb{C}, 0)$. We apply Morse theory. Up to homotopy, the space B_ε is obtained from $g_u^{-1}(0)$ by progressively thickening it: considering

$$|g_u|^{-1}([0, \eta])$$

and increasing η. Away from critical points of $|g_u|$, this thickening does not change the homotopy type. Changes in homotopy-type occur only when η passes through a critical value of $|g_u|$. The critical points of $|g_u|$ off $g_u^{-1}(0)$ are the same as those of g_u, and each has index equal to the ambient dimension, because of the complex structure. Thus, the contractible space B_ε is obtained from $g_u^{-1}(0)$ by gluing in cells of dimension p. It follows by a standard Mayer-Vietoris type argument that $g^{-1}(0)$ is homotopy-equivalent to the wedge of the boundaries of these cells. We can assume that g_u has only non-degenerate critical points off $g_u^{-1}(0)$; so the number of cells is the sum of their Milnor numbers.
This counting procedure is esential for the proofs of the following theorems.

Theorem 4.2. ([4]) *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a map-germ of finite codimension, with $n \geq p$ and (n, p) nice dimensions. Then*

$$\mu_\Delta(f) \geq \mathscr{A}_e - codim(f)$$

with equality if f is weighted homogeneous.

Theorem 4.3. *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ ($n = 1$ or 2) have finite codimension. Then*

$$(1)\mu_I(f) \geq \mathscr{A}_e - codim(f)(2)\text{Equality holds if } f \text{ is weighted homogeneous.} \tag{4.1}$$

Theorem 4.3 was proved for $n = 2$ by de Jong and van Straten in [12]; another proof, also inspired by de Jong and van Straten, was given in [24], and an analogous proof for the case $n = 1$ was given in [25].

A number of examples ([2],[8],[10],[27]) of map-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ for $n \geq 3$ support the conjecture that (4.1) should hold for all n for which $(n, n+1)$ are nice dimensions, but it remains unproven. Part of the difficulty in proving the conjecture lies in the fact that we do not have an effective method for computing image Milnor numbers. The best we can do here involves the image-computing spectral sequence (see [6], [7], [11]), and this only yields an answer when f has corank 1.

In contrast, we do have a method for computing discriminant Milnor numbers.

To explain it we begin by simplifying our initial description of $T^1(f)$, using an idea of Jim Damon's. Given a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ \uparrow & & \uparrow i \\ X \times_Y Z & \xrightarrow{f} & Z \end{array} \tag{4.2}$$

in which we suppose X, Y and Z smooth spaces and $i \pitchfork F$, we say that f is the *transverse pull-back of F by i*. Every map-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ of finite singularity type can be obtained by transverse pull-back from a stable map-germ: simply construct a stable unfolding $F : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$ (along the lines described in Remark 2.10), and then recover f from F by the map $i : (\mathbb{C}^p, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$ given by $i(y) = (y, 0)$.

Definition 4.4. If $F : (X, x_0) \to (Y, y_0)$ is any map-germ, the *iso-singular locus* of F is the set-germ

$$\mathscr{I}_F := \{y \in (Y, y_0) : f : (X, F^{-1}(y) \cap \sum\nolimits_F \to (Y, y) \text{ is } \mathscr{A}\text{-equivalent to } F.\}$$

Just as the domain $X \times_Y Z$ of f is smooth if and only if $i \pitchfork F$, the map f is stable if and only if $i \pitchfork \mathscr{I}_F$. This suggests that the instability of f should be reflected in the failure of i to be transverse to $\mathscr{I}_F$. A theorem of Damon (4.9 below) makes this precise. We need

Definition 4.5. (1) If $D \subset Y$ is an analytic subvariety, $\mathrm{Der}(-\log D)$ is the $\mathcal{O}_Y$-module (sheaf) of germs of vector fields on Y tangent to D at its smooth points.
(2) If D is a divisor (hypersurface) in Y, we say D is a *free divisor* if $Der(-\log D)$ is a locally free $\mathcal{O}_Y$-module.

It is easy to show that if D is the variety of zeros of an ideal I then

$$\mathrm{Der}(-\log D) = \{\chi \in \theta_Y : \chi \cdot g \in I \text{ for all } h \in I\},$$

and in particular if D is a hypersurface with equation h then

$$\mathrm{Der}(-\log D) = \{\chi \in \theta_Y : \chi \cdot h = \alpha h \text{ for some } \alpha \in \mathcal{O}_Y\}.$$

Let F be a map-germ of finite $\mathscr{A}_e$-codimension, and let $\Delta(F)$ be its discriminant.

Proposition 4.6. $T_{y_0}\mathscr{I}_F = \{\chi(y_0) : \chi \in Der(-\log \Delta(F))_{y_0}$. □

The vector space on the right is known as the *logarithmic tangent space to* $\Delta(F)$ *at* y_0; we denote it by $T^{\log}_{y_0}\Delta(F)$.

Proposition 4.7. *If* $F : (\mathbb{C}^n, S) \to (\mathbb{C}^p, Y)$ *(*$n \geq p$*) is stable then* $\Delta(F)$ *is a free divisor.*

Proof Looijenga's book [13, 6.13]. □

Given a diagram

$$\begin{tikzcd} X \arrow[r, "F"] & Y \\ & Z \arrow[u, "i"'] \end{tikzcd}$$

we measure the failure of transversality of i to $\mathscr{I}_F$ by the module

$$\theta(i)/ti(\theta_Y) + i^*\big((\mathrm{Der}(-\log \Delta(F))\big),$$

which is denoted by $T^1_{\mathscr{K}_{\Delta(F)}} i$. We say i is "logarithmically transverse to $\Delta(F)$" at z if

$$d_z i(T_z Z) + T_{i(z)}\Delta(F) = T_{iz}Y$$

Proposition 4.8. *Let* $D \subset Y$ *be a hypersurface and* $i : Z \to Y$ *a map. Then* i *is logarithmically transverse to* D *at* z *if and only if* $T^1_{\mathscr{K}_D} i = 0$.

Proof Nakayama's Lemma □

Theorem 4.9. (J.N.Damon,[3]) *If* f *is obtained form the stable map* F *by transverse pull back by* i*, as in the diagram (4.2), then*

$$T^1(f) \simeq T^1_{\mathscr{K}_{\Delta(F)}} i.$$

A simpler proof than Damon's original one can be found in [26, Section 8].

Let h be the equation of $\Delta(F)$, and define $\mathrm{Der}(-\log h)$ to be the $\mathcal{O}_Y$-module of germs of vector fields which annihilate h; that is, which are tangent not only to $\Delta(F) = h^{-1}(0)$, but to all level sets of h. Clearly $\mathrm{Der}(-\log h)$ is a submodule of $\mathrm{Der}(-\log D)$.

Theorem 4.10. ([4]) *If $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, with $n \geq p$ and (n,p) nice dimensions, and f is obtained from the stable map-germ F by transverse pull back by i, then*

$$\mu_I(f) = dim_{\mathbb{C}} \theta(i)/ti\big(\theta_{\mathbb{C}^p,0}\big) + i^*\big(Der(-\log h)\big).$$

The proof of this result depends in an essential way on the fact that $\Delta(F)$ is a free divisor. The inequality in Theorem 4.2 follows immediately from 4.10 and 4.9.

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