These memories of George Mackey will form part of a longer tribute to appear in the Notices of the AMS, 2007.

In my memory, George Mackey scarcely changed from the time I first approached him as a prospective graduate student in the mid-seventies, until the last time I saw him a couple of years before his death. He was a scholar in the truest sense, his entire life dedicated to mathematics. He lived a life of extraordinary self discipline and regularity, timing his walk to his office like clockwork and managing, how one cannot imagine, to avoid teaching in the mornings, this prime time being devoted to research. I was once given a privileged view of his study on the top floor of their elegant Cambridge house. There, surrounded on all sides by books and journals shelved from floor to ceiling, he had created a private library in which he immersed himself in a quiet haven of mathematical and intellectual scholarship.

I was lucky to arrive in Harvard at the time when he was working on what subsequently became his famous Oxford and Chigaco lecture notes [7, 8]. These masterly surveys convey the sweep of huge parts of mathematics from group representations up. I do not know any other writer with quite his gift of sifting out the essentials and exposing the bare bones of a subject. There is no doubt that his unique ability to cut through the technicalities and draw diverse strands together into one grand story has been a hugely wide and enduring influence.

Mackey believed strongly in letting students find their own thesis problem. Set loose reading his notes, I reported to him every couple of weeks and he was always ready to point me down some new yet relevant avenue. He never helped with technical problems, always saying "Think about it and come back next week if you are still stuck." Sometimes I envied the other students whom I, somewhat naïvely, assumed were being told exactly what to do, but in retrospect this was a most valuable training. Stemming from his interest in ergodic theory, I was given a partially guided tour of a wide swathe of dynamical systems. This loose but broad direction stood me in good stead later; I often think of it when under pressure to hand out precisely doable thesis problems when students have barely started their studies.

I finally settled for working on Mackey's wonderful invention *virtual groups*. The idea, touched on in C.C. Moore's article, is laid out in most detail in Mackey's paper [5]. His explanation to me was characteristically simple. He was interested in group actions on measure spaces, because a measure preserving action of a group induces a natural unitary representation on the associated L^2 space. As with any class of mathematical objects, he said, if you want to understand group actions, you should split them up into the simplest possible pieces. The simplest kind of group action is a transitive one, that is, one with a single orbit. Being a generalist, Mackey wanted to work in the category of Borel actions, so in particular the groups he cared about always had a standard Borel structure, that is, were either countable or Borel isomorphic to the unit interval. As he pointed out, an action being Borel has unexpectedly strong consequences; in particular the stabilisers of points are closed. A transitive action is clearly determined by the stabiliser of a single point, or more precisely, since the choice of point is arbitrary, by a conjugacy class of such. Thus a transitive action of a Borel group on a standard Borel space is equivalent to the specification of a conjugacy class of closed subgroups.

What is the next simplest type of action? Since we are in a category of measure spaces, an 'indecomposable' action means that the underlying space should not split into non-trivial and measurable 'subactions'. Assuming the group preserves a measure or at least a measure class, this is precisely what is meant by the action being ergodic: there are no 'non-trivial' group invariant measurable subsets, where 'non-trivial' means neither a null set nor the complement of a null set. Mackey's idea was that, since a transitive action is determined by a closed subgroup, then wouldn't it be nice if an ergodic action were similarly determined by a new kind of 'subobject' of the group which he named in advance a 'virtual group'. There is a touch of genius in his passage from this apparently simplistic idea to a formal mathematical structure yielding deep insights. The key point is that the sought after virtual group should be the *groupoid* $S \times G$, in which the base space is the underlying space of the action S and the arrows are pairs (s, g) where (s, g) has initial point s and final point $g \cdot s$ (or rather $s \cdot g$, since Mackey insisted on doing all his group actions from the right). Thus (s, g) could be composed with (s', g') if and only if $s' = s \cdot g$ and then $(s, g) \circ (s\cdot, g') = (s, gg')$.

The next step is perhaps the most interesting: a homorphism from $S \times G$ to a group H should be a cocycle: that is, a map $a : S \times G \to H$ such that $a(s, g)a(sg, g') = a(a, gg')$. This led Mackey to his construction of the 'range of the homomorphism'. Observe that, in deference to Mackey's order, the direct product $G \times H$ acts on $S \times H$ by $(s, h) \cdot (g, h') = (sg, h'^{-1}ha(s, g))$. The 'range' is essentially the action of H on the space of G orbits: if this space is not a standard Borel space, as for example if the G action is properly ergodic, one replaces it by the largest standard Borel quotient. Mackey saw this as the generalisation of the dynamical systems construction of a 'flow built under a function'. (A cocycle for a $\mathbb{Z}$-action can be defined additively given a single function from S to H.) This circle of ideas was seminal for much future work, in particular that of Mackey's former student Robert Zimmer. A special case is the Radon Nikodym derivative of a measure class preserving group action, which can be viewed as a groupoid homorphism to $\mathbb{R}$, of which more shortly.

A special case arises when all the stabilisers of points are trivial. Mackey's natural relation of similarity between groupoids leads to the classification of free ergodic actions up to 'orbit equivalence'. Two measure class preserving actions of groups G, G' on spaces S, S' are called *orbit equivalent* if there is a Borel measure class preserving map $\phi : S \to S'$ with the property that two points in the same G-orbit in S are mapped to points in the same G'-orbit in S'. (This is *much* weaker than the usual notion of conjugacy in which one insists that $G = G'$, and that $\phi(sg) = \phi(s)g$.) Just how much weaker is expressed in a remarkable theorem discovered by Mackey's student Peter Forrest: *any* two finite measure preserving actions of $\mathbb{Z}$ are orbit equivalent [3]. Subsequently, Mackey learnt that the theorem had previously been proved by H. Dye [2] and became a great publicist. He took pleasure in telling me it had also been proved by the Russian woman mathematician R. M. Belinskaya [1]. More generally, any equivalence relation orbit equivalent to a $\mathbb{Z}$-action is called *hyperfinite*. Dye's theorem extends to show that actions of a much wider class, including all abelian groups, are hyperfinite, culminating in Zimmer's result [9] that an action is hyperfinite if and only if the equivalence relation is amenable in a suitable sense.

The classification of non-measure preserving $\mathbb{Z}$-actions up to orbit equivalence is even more remarkable. Regarding the Radon Nikodym derivative as a cocycle to $\mathbb{R}$ as above, Mackey's 'range' is known to dynamicists as the Poincaré flow. At the time Mackey did not perhaps appreciate just how far reaching a construction this was. If the original $\mathbb{Z}$-action is measure preserving, it is called Type I, II_0, II_∞ depending on whether the range $\mathbb{R}$-action is transitive, or preserves a finite or infinite measure respectively. If the original action only preserves a measure class, it is called Type III_1, III_λ and III_0 depending on whether the range groupoid is $\{\text{id}\}, \lambda\mathbb{Z}$ for some $\lambda \in \mathbb{R}^+$, or properly ergodic. In a beautiful and remarkable piece of mathematics, pushed to its conclusion by W. Krieger [4], it turns out that the range completely classifies the original $\mathbb{Z}$-action up to orbit equivalence. The same construction gives rise to a rich fund of examples of von Neumann algebras, a fact widely exploited by A. Connes. My training under Mackey was an ideal foundation from which to appreciate all this work, which was developing rapidly in the late 70's just about the time I finished my thesis.

To return to Mackey as a person. Everyone who knew him will remember his uncompromising and sometimes uncomfortably forthright intellectual honesty. He took pleasure in following through a line of thought to its conclusion: political correctness was not for him. He wrote several articles about the invidious effects of federal research funding [6]. He may have lost the battle, but what he said was quite true.

I must say something about the widely held view that Mackey was against women mathematicians. All that I can say is that I never experienced the slightest prejudice from him and am proud to be what he referred to as 'his first mathematical daughter'. His straightforwardness was perhaps easy to misinterpret. For example, he was might say something like "There have been historically been almost no women mathematicians of stature. Therefore on a statistical basis it is unlikely that a particular woman will be one." Of course such a view ignores all the complicated historical and cultural factors which explain why this might be so; nevertheless, the fact was hard to dispute. But what remains in my memory is that Mackey was always open minded and unprejudiced, willing to take on board new insights or experiences and accept new people on their merits, exactly as they came along.

Mackey stood for the highest standards. He lived his life by precise rules but he enjoyed it to the full. Within his mathematics he found a fulfilment to which few can aspire. The spartan simplicity and mild disorder of his office contrasted sharply with the comfortable elegance created by Alice in their Cambridge home. Her wonderful old world dinner parties were memorable occasions at which it was a privilege to be a guest. Mackey was sometimes disarmingly open about his family life but through it all shone human warmth and love: their strictly scheduled but vastly important time reading aloud in the evenings, his pride in their daughter Ann.

Mackey had the habit of writing lengthy letters about his latest discoveries. Long after retirement, indeed right up to a couple of years before his death, he continued working on various projects which between them seemed to involve nothing less than unravelling the entire mathematical history of the twentieth century. Subjects expanded to include statistical mechanics, number theory, complex analysis, probability and more. He explained that group representations encompassed more or less everything, given that starting from quantum theory one obviously had to include chemistry and thus also biology. One might argue that things are a little more complicated, indeed I am sure with a twinkle in his eye he would agree. What is certain is, that his ability to strip things down to their essential mathematical structure put a hugely influential stamp on generations of mathematicians and physicists. He was a man whose unique qualities, insights and enthusiasms touched us all.

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