LIMITS OF QUASI-FUCHSIAN GROUPS WITH SMALL BENDING

CAROLINE SERIES

Abstract
We study limits of quasi-Fuchsian groups for which the bending measures on the convex hull boundary tend to zero, giving necessary and sufficient conditions for the limit group to exist and be Fuchsian. As an application, we complete the proof of a conjecture made in [24, Conjecture 6.5], that the closures of pleating varieties for quasi-Fuchsian groups meet Fuchsian space exactly in Kerckhoff's lines of minima of length functions. Doubling our examples gives rise to a large class of cone manifolds which degenerate to hyperbolic surfaces as the cone angles approach 2π.

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1. Introduction

Considerable recent interest has focussed on the two components of the convex hull boundary of a quasi-Fuchsian group G. The object of this paper is to study what happens when these components flatten out, the obvious expectation being that, under suitable conditions, a limit group should exist and be Fuchsian.

The hyperbolic 3-manifold $\mathbb{H}^3/G$ associated to the group G is homeomorphic to $S \times (0, 1)$ for some topological surface S. The convex hull boundary, that is, the boundary of the convex hull of all closed geodesics in $\mathbb{H}^3/G$, has two connected components, each itself homeomorphic to S. Each component is bent along some geodesic lamination on S, the amount of bending being measured by the *bending*

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measures $\mathrm{pl}^{\pm} = \mathrm{pl}^{\pm}(G)$. Given two nonzero measured laminations μ and ν, the *pleating variety* $\mathscr{P}_{\mu,\nu}$ consists of all groups $G \in \mathscr{QF}(S)$ for which the bending measures are nonzero and $\mathrm{pl}^{+}(G)$ is projectively equivalent to μ and $\mathrm{pl}^{-}(G)$ to ν.

Recall that measured laminations μ and ν are said to fill up S if $i(\mu,\xi)+i(\nu,\xi) > 0$ for any measured lamination ξ. It is not hard to show that $\mathscr{P}_{\mu,\nu}$ is empty unless μ and ν fill up S. The converse is a special case of central recent results of Bonahon and Otal [2].

THEOREM 1.1
Let S be a hyperbolic surface and let μ, ν be measured geodesic laminations that fill up S. Suppose also that each closed leaf of μ and of ν has weight strictly less than π. Then there is a quasi-Fuchsian group $G(\mu,\nu)$ for which $\mathrm{pl}^{+}(G) = \mu$ *and* $\mathrm{pl}^{-}(G) = \nu$. *If μ, ν are rational, then $G(\mu,\nu)$ is unique.*

One could well conjecture that the final uniqueness statement is true in general. Thus from now on, we use $G(\mu,\nu)$ to denote any quasi-Fuchsian group for which $\mathrm{pl}^{+}(G) = \mu$ and $\mathrm{pl}^{-}(G) = \nu$. We prove the following.

THEOREM 1.2
Let μ, ν be two measured laminations that together fill up S. Then for any sequence $\theta_n \to 0$, up to extracting a subsequence, $G(\theta_n\mu, \theta_n\nu)$ converges to a Fuchsian group.

The Bonahon-Otal result for irrational laminations involves a delicate limit process that, however, says nothing about what happens when the bending measures tend to zero.

We also show that the limit Fuchsian group in Theorem 1.2 is unique by identifying it precisely. Using Thurston's earthquake theorem, Kerckhoff [14] proved the following result about length functions on Teichmüller space.

THEOREM 1.3
Let S be a hyperbolic surface, and let μ, ν be measured geodesic laminations that fill up S. Then the length function $l_\mu + l_\nu$ has a unique minimum $M(\mu,\nu)$ on the Teichmüller space $\mathscr{T}(S)$.

This enables us to state our main result, special cases of which were already proved in [11] and [24].

THEOREM 1.4
Let μ, ν be two measured laminations that together fill up S. Then as $\theta_n \to 0$, the sequence $G(\theta_n \mu, \theta_n \nu)$ of Theorem 1.2 converges to $M(\mu, \nu)$.

It is essential for the convergence in this result that the bending measures pl^+ and pl^- stay in bounded proportion. For example, one might consider the case in which μ, ν are unit measures δ_α and δ_β supported on fixed geodesics α, β and study the groups $G(\theta\delta_\alpha, \phi\delta_\beta)$ with $\phi/\theta \to 0$. In the case of a once-punctured torus with α and β a pair of generators, one can check by direct calculation (see Section 4) that if $\phi/\theta \to 0$, then this sequence has no (algebraic) limit as $\theta \to 0$. We show that a similar phenomenon holds in general.

THEOREM 1.5
Let μ, ν be two measured laminations that together fill up S. Then any sequence of groups $G(\theta_n \mu, \phi_n \nu)$ with $\theta_n, \phi_n \to 0$ diverges (i.e., no subsequence has an algebraic limit) unless θ_n/ϕ_n is uniformly bounded away from zero and ∞.

One also has to be careful if one wishes to allow μ, ν to vary. It is easy to see by example that it is important that the limit laminations themselves fill up S. We prove the following theorem.

THEOREM 1.6
Let μ, ν be two measured laminations that together fill up S, and suppose that $\mu_n \to \mu$, $\nu_n \to \nu$ and $\theta_n \to 0$. Then the sequence of groups $G(\theta_n \mu_n, \theta_n \nu_n)$ converges to $M(\mu, \nu)$ as $n \to \infty$.

The set of minima $M(\mu, t\nu)$ for $t \in (0, \infty)$ is a line $\mathscr{L}_{\mu,\nu} \subset \mathscr{F}(S)$ called the *Kerckhoff line of minima* of μ and ν. Combining the above results, we obtain a complete proof of [24, Conjecture 6.5].

THEOREM 1.7
Let $\mu, \nu \in \mathscr{ML}$ be laminations that fill up S. Then the closure of $\mathscr{P}_{\mu,\nu}$ meets $\mathscr{F}$ precisely in $\mathscr{L}_{\mu,\nu}$.

Notice that to prove the conjecture we need to invoke Theorem 1.1 to construct the sequence $G(\theta_n \mu, \theta_n \nu)$. The Bonahon-Otal theorem for rational laminations is based on the Kerckhoff-Hodgson theory of deformations of cone manifolds, so our proof rests ultimately on the same thing. In [24], we were able to avoid this in some cases by directly proving the existence of a sequence in $\mathscr{P}_{\mu,\nu}$ approximating a point $p \in \mathscr{L}_{\mu,\nu}$.

However, we required that the supports of μ and ν be pants decompositions and that a certain condition on the partial derivatives of the lengths of these pants curves be satisfied at p. It would be nice to have a more general direct proof.

If the pleating loci $\mathrm{pl}^{\pm}$ are both rational, then, after removing the pleating locus, one can double the convex core of the 3-manifold $\mathbb{H}^3/G$ to obtain a cone manifold whose singular locus is the removed bending lines. If the bending angle along an axis is ϕ, then the corresponding cone angle is $2(\pi - \phi)$. Thus one can regard the above results as describing a special class of degeneration of cone manifolds to 2-dimensional hyperbolic structures as all the cone angles approach 2π in a controlled way. Theorems 1.7 and 1.5 give necessary and sufficient conditions for such degeneration to occur.

Of the above theorems, the new results are Theorems 1.2, 1.4, 1.5, 1.6, and 1.7. The heart of the paper is Sections 5 and 6, in which we establish the following variant of Theorem 1.2.

PROPOSITION 1.8

Let S be a hyperbolic surface of finite type, and suppose that $\mu, \nu \in \mathscr{ML}(S)$ fill up S. Then there exists $\epsilon > 0$, depending only on μ, ν and the topology of S, such that the groups $G(\theta\mu, \theta\nu)$ for $\theta < \epsilon$ lie in a relatively compact set in $\mathscr{QF}$ and any accumulation point as $\theta \to 0$ is Fuchsian. Moreover, for any finite set Γ of simple curves on S, there exists $c > 0$ such that $|l_\gamma(p^+(G_\theta)) - l_\gamma(p^-(G_\theta))| \leq c\theta^2$ for all $\gamma \in \Gamma$ and all sufficiently small θ.

The paper is organised as follows. After briefly summarising the background in Section 2, in Section 3 we show that Theorems 1.4 and 1.5 follow from Proposition 1.8. One easily deduces Theorem 1.2 from Proposition 1.8 and Theorem 1.4. In Section 4 we discuss the example of the once-punctured torus referred to Theorem 1.4. In Section 5 we prove Proposition 1.8 for rational laminations. In Section 6 we prove it in the general case. Finally, in Section 7 we discuss diagonal limits, showing that there is sufficient uniformity in the estimates needed to prove Proposition 1.8 to deduce Theorem 1.6.

2. Preliminaries

The background we need is mostly well known and explained at length elsewhere. Here we give only a brief summary and refer to [24] and elsewhere for more details.

Throughout the paper, we frequently replace consideration of convergence of a sequence depending on θ_n with $\theta_n \to 0$ as $n \to \infty$, with the behaviour of a function depending on a real variable θ as $\theta \to 0$. We write $g(\theta) = O(\theta)$ to mean that $g(\theta) \leq c\theta$ for some fixed $c > 0$ as $\theta \to 0$. We also write $g(\theta) > O(\theta)$ to mean that there exists $c > 0$ such that $g(\theta) > c\theta$ as $\theta \to 0$.

In general, we shall be careful to specify the dependence of our constants. Symbols c, k, and so on may denote different constants in different places, but we label by a subscript if we need to refer back to some particular earlier choice.

2.1. Quasi-Fuchsian groups

Let S be an oriented surface of negative Euler characteristic, homeomorphic to a closed surface with at most a finite number of points removed. A *quasi-Fuchsian group* G is the image of a discrete faithful representation $\rho : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ such that the limit set of G is a topological circle. If S has punctures, we insist that the images of loops around boundary components are parabolic. The limit set separates the regular set into two simply connected G-invariant components $\Omega^{\pm}$, and each quotient $\Omega^{\pm}/G$ is homeomorphic to S.

Two quasi-Fuchsian groups are *equivalent* if the corresponding representations are conjugate in $\mathrm{PSL}(2, \mathbb{C})$. *Quasi-Fuchsian space* $\mathscr{QF}(S)$ is the space of equivalence classes. It has a holomorphic structure induced from the natural holomorphic structure of $\mathrm{SL}(2, \mathbb{C})$. A quasi-Fuchsian group is *Fuchsian* if the limit set is a round circle. *Fuchsian space* $\mathscr{F} = \mathscr{F}(S)$ is the subset of $\mathscr{QF}(S)$ corresponding to Fuchsian groups.

As proved by Kerckhoff [13, p. 24], the induced real-analytic structure on $\mathscr{F}$ is determined by the lengths of a finite number of geodesics in $\pi_1(S)$. These curves can always be taken to be simple. The arguments extend to show the same is true for the complex analytic structure on $\mathscr{QF}(S)$ if we replace length with *complex length* $\lambda(g)$ given by the formula $\mathrm{Tr}\,\rho(g) = 2\ \cosh \lambda(g)/2$ for $g \in \pi_1(S)$.

For further details on these definitions, good references are [17] and [21].

2.2. Geodesic laminations

Let S be a surface as above. Given a hyperbolic structure on S, a *geodesic lamination* on S is a closed union of pairwise disjoint simple close geodesics which are called its *leaves*. A *measured geodesic lamination* μ consists of a geodesic lamination together with a transverse invariant measure on the leaves. We denote the underlying lamination by $|\mu|$. For reasons that will be clear in the next section, we consider only laminations with no leaves which end in a puncture.

We denote the set of such measured laminations by $\mathscr{ML}(S)$. This space is topologised with the topology of weak convergence: $\mu_n \to \mu$ if $\mu_n(T) \to \mu(T)$ for any transversal T. It is well known that $\mathscr{ML}(S)$ is independent of the hyperbolic structure on S. For $\mu \in \mathscr{ML}$, the length l_μ (relative to a given hyperbolic structure on S) is the total mass of the measure which is the product of hyperbolic distance along the leaves of $|\mu|$ with the transverse measure μ.

We call a measured geodesic lamination μ *rational* if its support $|\mu|$ consists entirely of closed leaves; otherwise, we call it *irrational*. (Note: this is quite a different meaning from the term *"arational"* used, e.g., in [21].) Let $\mathscr{S} = \mathscr{S}(S)$ denote the set of all homotopy classes of simple closed nonboundary parallel curves on S. (By definition, the homotopy classes in $\mathscr{S}$ are assumed primitive; i.e., they are not powers of other elements in $\pi_1(S)$.) If α_i are a set of disjoint curves in $\mathscr{S}$, then by $\sum_i a_i\alpha_i$, $a_i \in \mathbb{R}^+$, we mean the measured lamination with support $\cup_i \alpha_i$ which gives mass a_i to each intersection with α_i. Note that the maximum number of curves α_i in such a sum is $3g-3+b$, where g is the genus of S and b is the number of punctures. We denote the set of all rational measured laminations by $\mathscr{ML}_Q(S)$; the set $\mathscr{ML}_Q$ is dense in $\mathscr{ML}$.

The length of the rational lamination $\sum_i a_i\alpha_i$ is just $\sum_i a_i l_{\alpha_i}$, where l_{α_i} is the hyperbolic length of the geodesic α_i. Kerckhoff [12], [13] has shown that if $\mu_n \in \mathscr{ML}_Q$ converges to μ in $\mathscr{ML}$, then l_{μ_n} converges to l_μ uniformly on compact subsets of $\mathscr{F}$ and, hence, is a real-analytic function on $\mathscr{F}$. In a similar way, the geometric intersection number $i(\alpha, \alpha')$ of two closed geodesics α, α' extends by linearity and continuity to a continuous function $i(\mu, \nu)$ on $\mathscr{ML}$ (see, e.g., [12], [21]).

Laminations $\mu, \nu \in \mathscr{ML}$ are said to *fill up* S if $i(\mu, \eta) + i(\nu, \eta) > 0$ for all $\eta \in \mathscr{ML}$. An equivalent condition is that every component of $S - |\mu| \cup |\nu|$ contains at most one puncture whose closure, after filling in the puncture (if needed), is compact and simply connected.

There is an obvious action of $\mathbb{R}^+$ on $\mathscr{ML}$ given by scalar multiplication $\mu \to t\mu$ for any $t > 0$. A *projective measured lamination* is an equivalence class under this action. We write $[\mu]$ for the projective class of μ and denote the set of all nonzero projective measured laminations by $\mathrm{P}\mathscr{ML}$. Thurston showed that $\mathrm{P}\mathscr{ML}(S)$ can be viewed as the boundary of $\mathscr{F}(S)$: a sequence of structures $p_n \in \mathscr{F}$ converges to $\xi \in \mathrm{P}\mathscr{ML}$ if the lengths $\{l_\gamma(p_n)\}_{\gamma\in\mathscr{S}}$ converge projectively to the intersection numbers $\{i(\gamma, \xi)\}_{\gamma\in\mathscr{S}}$. It is not hard to deduce that if the laminations μ, ν fill up S and if $p_n \in \mathscr{F}$ diverges, then at least one of $l_\mu(p_n)$ or $l_\nu(p_n)$ tends to ∞.

For more details on this material, see, for example, [4] or [21].

2.3. The convex hull boundary and bending measures

For any Kleinian group G, let $\mathscr{C} = \mathscr{C}(G)$ be the hyperbolic convex hull of the limit set of G in hyperbolic 3-space $\mathbb{H}^3$. If G is quasi-Fuchsian, then $\partial\mathscr{C}$ has exactly two components $\partial\mathscr{C}^\pm$ that "face" the components $\Omega^\pm$ of Ω. The quotients $\partial\mathscr{C}^\pm/G$ are homeomorphic to $\Omega^\pm/G$ and, hence, to S. (In the special case in which G is Fuchsian, $\mathscr{C}$ is contained in a single flat plane. We regard this as a degenerate case in which $\partial\mathscr{C}$ is two-sided, each side facing one component of $\Omega(G)$.)

The structure of $\partial\mathscr{C}$ is studied in detail in [7]. Note that by convexity, $\partial\mathscr{C}$ must be embedded in $\mathbb{H}^3$. The ambient hyperbolic metric induces a metric on $\partial\mathscr{C}$ which endows each component with its own hyperbolic metric; for a quasi-Fuchsian group G, we shall denote the corresponding hyperbolic structures in $\mathscr{F}(S)$ by $p^{\pm}(G)$. Each component of $\partial\mathscr{C}$ is the closure of a set of infinite-sided ideal polygons, each contained in a hyperbolic plane in $\mathbb{H}^3$. These polygons, called the flat pieces of $\partial\mathscr{C}$, are geodesic not only in $\mathbb{H}^3$ but also in the induced metrics on $\partial\mathscr{C}^{\pm}(G)$. (For a nice picture of this, see [20, Figure 12.6].) In each component of $\partial\mathscr{C}/G$, the closure of the complement of the flat pieces is a geodesic lamination on S, the *bending lamination*, which carries a transverse measure, the *bending measure*, denoted $\mathrm{pl}^{\pm}(G)$.

We note that no leaves of the bending lamination can limit on cusps of S. Consider a horocycle of length ϵ around the cusp. The lift of the horocycle to $\mathbb{H}^3$ bends by a definite amount δ (fixed and independent of ϵ) in every interval of length ϵ. By making ϵ sufficiently small, a comparison with the Euclidean situation shows that it is impossible for $\partial\mathscr{C}$ to be embedded. This explains our assumption that laminations in $\mathscr{ML}$ contain no leaves that end in a puncture.

Each cusp on a hyperbolic surface is surrounded by a horocycle of fixed area with the property that any simple geodesic that penetrates it ends in the cusp. If S_0 is a hyperbolic structure on the surface S, let S_0^C denote the surface S_0 minus these fixed-area horocyclic neighbourhoods of the punctures. For $x \in S$, the injectivity radius $\mathrm{inj}(x)$ at x is the radius of the largest embedded closed disk at x. The *noncuspidal injectivity radius* $\mathrm{inj}(S_0^C)$ of S_0 is the infimum over $x \in S_0^C$ of $\mathrm{inj}(x)$. By the above observation, a lower bound on $\mathrm{inj}(S_0^C)$ is equivalent to a lower bound on the lengths of all essential simple curves on S_0.

For a curve $\gamma \in \pi_1(S)$, we always use γ^* and $\gamma^{\pm}$ to denote the geodesic γ in $\mathbb{H}^3/G$ and its geodesic representatives on $\partial\mathscr{C}^{\pm}/G$, respectively, and we denote by l_{γ^*} and $l_{\gamma^{\pm}}$ the corresponding geodesic lengths. Thus $l_{\gamma^*} \le l_{\gamma^{\pm}}$ and $l_{\gamma^*} = l_{\gamma^+}$ if γ is contained in the support of pl^+. Notice that if $\mu \in \mathscr{ML}$ and $\mathrm{pl}^+(G) = \theta\mu$, then the total bending measure along γ^+ is $i(\gamma, \mu)\theta$.

We shall need the main result of [9].

PROPOSITION 2.1
The maps $\mathscr{QF} \to \mathscr{F}$, $q \mapsto p^{\pm}(q)$ *and* $\mathscr{QF} \to \mathscr{ML}$, $q \mapsto \mathrm{pl}^{\pm}(q)$ *are continuous, where by definition* $\mathrm{pl}^{\pm}(q) = 0$ *if* $q \in \mathscr{F}$.

We remark that the results of [9] are stated for holomorphic families depending on one complex variable only. However, the theory of holomorphic motions extends to several variables (see [18]), and identical methods apply.

The following straightforward result is [24, Proposition 3.2] (see also [2]).

PROPOSITION 2.2
Let G be quasi-Fuchsian, $G \in \mathscr{QF}(S)$. Then the bending measures $\mathrm{pl}^{\pm}(G)$ *fill up S.*

We remark that the proof of [24, Proposition 3.2] is not quite complete in the case in which $\mathrm{pl}^{\pm}$ are irrational because we omitted the possibilities that a complementary region of the union of the two laminations is simply connected but noncompact or that it is a once-punctured disk for which the complement of a neighbourhood of the puncture is noncompact. In this case, however, one obtains a semi-infinite geodesic α in the complement of both $|\,\mathrm{pl}^{\pm}\,|$. The accumulation points of α form a geodesic lamination contained in both $\partial\mathscr{C}^{+}$ and $\partial\mathscr{C}^{-}$. Following the same idea as in [24], this is easily seen to be impossible. For more details, see [14, Lemma 4.4].

Rational bending laminations
The structure of a component of $\partial\mathscr{C}$ is particularly simple when its bending measure is rational, say, $\mathrm{pl}^{+} = \sum_i \theta a_i \alpha_i$ for some $\theta > 0$. In this case, $\partial\mathscr{C}^{+}$ consists of pieces of hyperbolic planes which meet exactly along the lifts of axes of the curves α_i in such a way that the exterior angle of intersection (i.e., the angle *outside* $\mathscr{C}$) along an axis which projects to α_i is θa_i. Since $\partial\mathscr{C}$ is convex, all the angles θa_i have the same sign. They are measured so that $\theta = 0$ exactly when the oriented planes containing the adjacent flat pieces coincide. In particular, G is Fuchsian if and only if $\mathrm{pl}^{+} = 0$ (so that $\mathrm{pl}^{-} = 0$ follows automatically).

We remark that if the bending lamination is rational, then each flat piece of $\partial\mathscr{C}$ faces a disk in the regular set which contains a Cantor set of limit points in its boundary. Such "ghost circles" are a highly visible feature of many limit set pictures (see, e.g., [20]).

2.4. Earthquakes and quakebends
The time t left earthquake along a lamination $\mu \in \mathscr{ML}$ is a real-analytic map $\mathscr{E}_\mu(t) : \mathscr{F} \to \mathscr{F}$ which generalises the classical Fenchel-Nielsen twist. Let T be a transversal to $|\mu|$ with endpoints in distinct complementary components of $|\mu|$. The earthquake shifts the component on the right a distance $t\mu(T)$ relative to the one on the left, inducing a new hyperbolic metric $\mathscr{E}_\mu(t)(p)$ on an initial hyperbolic structure $p \in \mathscr{F}$. In particular, if $\mu = \sum_i a_i \alpha_i$, then for each i, the earthquake $\mathscr{E}_\mu(t)$ twists by hyperbolic distance ta_i around the closed geodesic α_i.

The map $(p, t) \mapsto \mathscr{E}_\mu(t)(p)$ is a flow on $\mathscr{F}$ which induces a tangent vector field $\partial/\partial t_u$. In [12], Kerckhoff showed that if $\nu \in \mathscr{ML}$, then the length l_ν is a real-analytic function of t along the flow; it is strictly convex if $i(\mu, \nu) > 0$ and constant otherwise. Wolpert [28] proved the famous antisymmetry relations $\partial l_\nu/\partial t_\mu = -\partial l_\mu/\partial t_\nu$.

Complexifying the parameters corresponds to passing from Fuchsian to quasi-

Fuchsian groups. In this context, the earthquake $\mathscr{E}_\mu(t)$ has a natural extension to a *left quakebend* $\mathscr{E}_\mu(\tau)$, $\tau \in \mathbb{C}$. We only need the construction relative to a hyperbolic structure p_0 corresponding to an initial Fuchsian group G_0, in which form it is explained in detail in [7]. In addition to shifting complementary components of $|\mu|$ on p_0 through a relative distance $\Re\tau\,\mu(T)$, the map $\mathscr{E}_\mu(\tau)$ bends the right-hand component through the angle $\Im\tau\,\mu(T)$ relative to the left-hand one. This deforms the group G_0, given by a representation $\rho_0 : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$, into a group $\mathscr{E}_\mu(\tau)(G_0)$, given by a representation $\rho(\tau) : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$. A quakebend with the purely imaginary parameter $\tau \in i\mathbb{R}$ is called a *pure bend*.

The following result is [10, Theorem 8.8]. In [10], it is explained in the context of a punctured torus, but the proof clearly extends to a general surface. Note that the first statement follows from the fact that quasi-Fuchsian groups are structurally stable.

PROPOSITION 2.3
Let $G_0 \in \mathscr{F}$ and $\tau \in \mathbb{C}$. Then, provided $|\Im\tau|$ is sufficiently small (with bounds depending on $\Re\tau$), $\mathscr{E}_\mu(\tau)(G_0)$ is quasi-Fuchsian. If $\mathscr{E}_\mu(\tau)(G_0)$ is quasi-Fuchsian, then $p^+(\mathscr{E}_\mu(\tau)(G_0)) = \mathscr{E}_\mu(\Re\tau)(p^+(G_0))$ and $\mathrm{pl}^+(\mathscr{E}_\mu(\tau)(G_0)) = (\Im\tau)\mu$.

By [7, Lemma 3.8.1], the groups $G_{\tau\mu}$ depend holomorphically on τ. One can therefore define the complex derivatives $\partial l_\nu/\partial\tau_\mu$; complex versions of the antisymmetry formulae have been proved by Kourouniotis [15].

3. Identification of the limit

In this section we show that Theorems 1.4 and 1.5 follow from Proposition 1.8. The proof is based on Kerckhoff's result, which we stated as Theorem 1.3. In fact, he proved a rather stronger statement: if μ and ν fill up S, then $l_\mu + l_\nu$ has a unique *critical point* on $\mathscr{F}$.

As usual, let $G(\theta) = G(\theta\mu, \theta\nu)$ denote a quasi-Fuchsian group for which $\mathrm{pl}^+ = \theta\mu$ and $\mathrm{pl}^- = \theta\nu$, and let $p^\pm(\theta)$ denote the Fuchsian structures on $\partial\mathscr{C}^\pm/G(\theta)$.

PROPOSITION 3.1
Let μ, ν be two measured laminations that together fill up S. Suppose that as $\theta_n \to 0$, the groups $G(\theta_n\mu, \theta_n\nu)$ converge to $p \in \mathscr{F}$ and, in addition, that for any finite set $\Gamma \subset \mathscr{S}$, there exists $c > 0$, depending on Γ, such that $|l_\gamma(p^+(\theta) - l_\gamma(p^-(\theta)| \le c\theta^2$ for all $\gamma \in \Gamma$ and all sufficiently small θ. Then $p = M(\mu, \nu)$.

(We remark that the bound $|l_\gamma(p^+(\theta)) - l_\gamma(p^-(\theta))| \le c\theta$ would be sufficient but our work leads naturally to θ^2.)

Proof
By the extension to Theorem 1.3, it is sufficient to show that p is a critical point of the length function $l_\mu + l_\nu$ on $\mathscr{F}$. Since by [14, Theorem 3.5] the tangent vectors $\partial/\partial t_\xi, \xi \in \mathscr{ML}$ span the tangent space $T_p(\mathscr{F})$ to $\mathscr{F}$ at p, it is enough to show that

$$\frac{\partial l_\mu}{\partial t_\xi}(p) + \frac{\partial l_\nu}{\partial t_\xi}(p) = 0, \quad \forall \xi \in \mathscr{ML},$$

or, equivalently, by the antisymmetry of the derivatives, that

$$\frac{\partial l_\xi}{\partial t_\mu}(p) + \frac{\partial l_\xi}{\partial t_\nu}(p) = 0, \quad \forall \xi \in \mathscr{ML}.$$

The proof of [13, Lemma 2.4] shows that l_ξ is a real-analytic function on $\mathscr{F}(S)$. By a straightforward extension of the arguments (see also [11, Theorem 6.3]), one sees that l_ξ extends to a complex analytic function λ_ξ on $\mathscr{QF}$. As noted above, for fixed $p_0 \in \mathscr{F}$ and $\mu \in \mathscr{ML}$, the group $\mathscr{E}_\mu(\tau)(p_0)$, obtained by quakebending by τ along μ, depends holomorphically on τ; in particular, $\lambda_\xi(\mathscr{E}_\mu(\tau)(p_0))$ is an analytic function of τ.

Now a quasi-Fuchsian group is completely determined by the Fuchsian structure p^+ on $\partial\mathscr{C}^+$ and by the bending measure pl^+. This leads to the key observation that we can reach $G(\theta\mu, \theta\nu)$ either by starting at $p^+(\theta)$ and making the pure bend $\mathscr{E}_\mu(i\theta)$ through the angle θ along μ, or by starting at $p^-(\theta)$ and making a pure bend through $-\theta$ along ν. The idea is to use this to make Taylor series expansions of the length of an arbitrary geodesic in $G(\theta\mu, \theta\nu)$ in two ways and compare the results.

Write $q(\theta)$ for $G(\theta\mu, \theta\nu)$, and let $\sigma^+ : [0, 1] \to \mathscr{QF}$ be the pure bend path between $p^+(\theta) = \sigma^+(0)$ and $q(\theta) = \sigma^+(1)$ so that $\sigma^+(t) = \mathscr{E}_\mu(it\theta)$ is the quasi-Fuchsian group obtained by bending $p^+(\theta)$ through the angle $t\theta$ along μ. Expanding from $p^+(\theta)$, we obtain

$$\lambda_\xi\big(q(\theta)\big) = \lambda_\xi\big(p^+(\theta)\big) + i\theta\frac{\partial \lambda_\xi}{\partial t_\mu}\big(p^+(\theta)\big) - \theta^2\frac{\partial^2 \lambda_\xi}{\partial t_\mu^2}\big(r^+(\theta)\big), \tag{1}$$

where $r^+(\theta) \in \sigma^+$. (Notice that since λ_ξ is real valued on $\mathscr{F}$, we have $\lambda_\xi(p^+(\theta)) = l_\xi(p^+(\theta))$ and $(\partial\lambda_\xi/\partial t_\mu)(p^+(\theta)) = (\partial l_\xi/\partial t_\mu)(p^+(\theta))$.) With a similar definition of σ^-, expanding from $p^-(\theta)$, we get

$$\lambda_\xi\big(q(\theta)\big) = \lambda_\xi\big(p^-(\theta)\big) - i\theta\frac{\partial \lambda_\xi}{\partial t_\nu}\big(p^-(\theta)\big) - \theta^2\frac{\partial^2 \lambda_\xi}{\partial t_\nu^2}\big(r^-(\theta)\big), \tag{2}$$

where $r^-(\theta) \in \sigma^-$.

Now suppose given $\theta_n \to 0$ such that $G(\theta_n\mu, t_n\nu) \to p \in \mathscr{F}$ (as in the statement of Proposition 3.1). Then by Proposition 2.1, $\lim_{n\to\infty} p^\pm(\theta_n) = p$, so the points

$r^{\pm}(\theta_n)$ lie in some compact neighbourhood of p in $\mathscr{QF}$. It follows that the second derivatives in equations (1) and (2) are uniformly bounded as $n \to \infty$. Equating imaginary parts, we find that

$$\frac{\partial l_\xi}{\partial t_\mu}\big(p^+(\theta_n)\big) + \frac{\partial l_\xi}{\partial t_\nu}\big(p^-(\theta_n)\big) = O(\theta_n) \tag{3}$$

with constants depending on ξ.

Now choose the set Γ to be a finite set of curves which determine the analytic structure on $\mathscr{F}$. This means that $p \mapsto \{\lambda_\gamma(p)\}$ is a real-analytic embedding of $\mathscr{F}$ into $\mathbb{R}^k$, $k = |\Gamma|$. For small θ, our hypothesis translates into $d(p^+(\theta), p^-(\theta)) = O(\theta^2)$, where d is the Euclidean metric on $\mathbb{R}^k$. Since l_ξ is a real analytic on $\mathscr{F}$, so is $\partial l_\xi / \partial t_\nu$, and we deduce that

$$\frac{\partial l_\xi}{\partial t_\nu}\big(p^+(\theta_n)\big) - \frac{\partial l_\xi}{\partial t_\nu}\big(p^-(\theta_n)\big) = O(\theta_n^2). \tag{4}$$

It follows that

$$\frac{\partial l_\xi}{\partial t_\mu}\big(p^+(\theta_n)\big) + \frac{\partial l_\xi}{\partial t_\nu}\big(p^+(\theta_n)\big) = O(\theta_n). \tag{5}$$

Using Proposition 2.1 again, we make the required conclusion by taking limits as $n \to \infty$. □

Theorem 1.4 follows immediately from Proposition 3.1 and Proposition 1.8, which are proved in Sections 5 (rational case) and 6 (general case).

Remark 3.2

Theorem 5.1 in [24] is effectively the special case of Proposition 3.1, in which μ and ν are both rational laminations whose supports are pants decompositions of S. However, the proof in [24] fails completely in the irrational case. Notice also that the proof of Theorem 1.4 rests heavily on the assumption that the limit group p exists; even if $p^{\pm}(\theta)$ remain close, the error terms $(\partial^2 \lambda_\xi / \partial t_\mu^2)(r^+(\theta))$, $(\partial^2 \lambda_\xi / \partial t_\nu^2)(r^-(\theta))$ might well become unbounded if $p^{\pm}(\theta) \to \partial\mathscr{F}$. (Wolpert's formula [29, Theorem 2.11] for the second derivatives shows that these terms contain a factor $1/\lambda_\xi$.)

Theorem 1.5 is proved by a similar method. This time we need only Proposition 5.1 from Sections 5 and 6.

Proof of Theorem 1.5

Suppose as usual that $\mu, \nu \in \mathscr{ML}$ fill up S, and suppose that we have a sequence of groups $G_n = G(\theta_n \mu, \phi_n \nu)$ with $\theta_n, \phi_n \to 0$ for which $\theta_n / \phi_n \to 0$. We have to show that the sequence G_n diverges. If not, then (by passing to a subsequence if necessary) we may suppose that G_n has an algebraic limit G_∞.

First we show that G_∞ must be quasi-Fuchsian. Write $p_n^\pm$ for the Fuchsian structures on $\partial\mathscr{C}^\pm(G_n)/G_n$. Let γ be any simple closed curve on S, and denote its geodesic representatives on $\partial\mathscr{C}^\pm(G)/G$ and in $\mathbb{H}^3/G$ by $\gamma^\pm$, γ^*, respectively. Here and elsewhere, we denote by $l_{\gamma^\pm}, l_{\gamma^*}$ the lengths of these curves in $\partial\mathscr{C}^\pm(G)/G$ and in $\mathbb{H}^3/G$. The comparison in Proposition 5.1 shows that $l_{\gamma^+}(p_n^+)$ either tends to zero or is bounded above by $Kl_{\gamma^*}(G_n)$ for some $K = K(\gamma) > 0$. Since the sequence G_n has an algebraic limit, there is a constant $K' = K'(\gamma)$ such that $l_{\gamma^*}(G_n) \leq K'$ for all n. This gives a uniform upper bound $l_{\gamma^+}(p_n^+) \leq K''(\gamma)$, from which we deduce that, up to a subsequence, the sequence of structures p_n^+ has a limit $p_\infty^+ \in \mathscr{F}$. By Sullivan's theorem (see [25], [7]), there is a uniform upper bound to the Teichmüller distance between the structure $\partial\mathscr{C}^+(G)/G$ and the hyperbolic structure $\omega^+(G)$ on the corresponding component $\Omega^+(G)/G$ of the conformal boundary at infinity for any quasi-Fuchsian group G. Thus, up to a subsequence, the sequence $\omega^+(G_n)$ also converges to a point in $\mathscr{F}$. The same is true of the sequence $\omega^-(G_n)$. By Bers' simultaneous uniformisation theorem, there is a quasi-Fuchsian group G'_∞ realising these limit structures on its conformal boundary. Also by Bers' theorem, the map $\mathscr{F} \times \mathscr{F} \to \mathscr{QF}$ is a homeomorphism with respect to the algebraic topology on $\mathscr{QF}$. Thus the groups G_n converge algebraically to G'_∞, and we conclude that $G'_\infty = G_\infty$ since G_∞ is also the algebraic limit of the same sequence.

To see that G_∞ is Fuchsian, write $\mathrm{pl}_n^\pm$, $\mathrm{pl}_\infty^\pm$ for the bending measures of G_n, G_∞, respectively. By Proposition 2.1, $\mathrm{pl}_n^\pm \to \mathrm{pl}_\infty^\pm$. Since our hypothesis implies that $\mathrm{pl}_n^\pm \to 0$, we deduce that $\mathrm{pl}_\infty^\pm = 0$. Thus each of $\partial\mathscr{C}^\pm(G_\infty)$ is contained in a single hyperbolic plane, from which it follows that the regular set of G_∞ contains at least two circular invariant domains. Since we have already shown that G_∞ is quasi-Fuchsian, it has exactly two invariant domains, each of which must be circular. Hence G_∞ is Fuchsian.

Now following exactly the method used to arrive at equation (3) in Proposition 3.1, we find

$$\theta_n \frac{\partial l_\xi}{\partial t_\mu}(p_n^+) + \phi_n \frac{\partial l_\xi}{\partial t_\nu}(p_n^-) = O(\theta_n^2 + \phi_n^2). \tag{6}$$

As before, let Γ be a finite set of curves which determine the analytic structure on $\mathscr{F}$. In a compact neighbourhood of G_∞, the noncuspidal injectivity radii of the structures $p_n^\pm$ are uniformly bounded below, and the lengths of the curves in Γ are uniformly bounded above. Thus Proposition 5.1 gives that $|l_{\gamma^*}(G_n) - l_{\gamma^+}(G_n)| \leq O(\theta_n^2)$ and $|l_{\gamma^*}(G_n) - l_{\gamma^-}(G_n)| \leq O(\phi_n^2)$. Hence, abbreviating $l_{\gamma^+}(p_n^+)$ as $l_\gamma(p_n^+)$, we have $|l_\gamma(p_n^+) - l_\gamma(p_n^-)| \leq O(\theta_n^2 + \phi_n^2)$ for all $\gamma \in \Gamma$. Since by choice the curves in Γ determine the real-analytic structure on $\mathscr{F}$, combining this with the fact that l_ξ is a real-analytic function on $\mathscr{F}$, we obtain (as in the proof of Proposition 3.1) that for any $\xi \in \mathscr{ML}$,

$$\frac{\partial l_\xi}{\partial t_\nu}(p_n^+) + \frac{\partial l_\xi}{\partial t_\nu}(p_n^-) = O(\theta_n^2 + \phi_n^2). \tag{7}$$

Together with (6), this gives

$$\theta_n \frac{\partial l_\xi}{\partial t_\mu}(p_n^+) + \phi_n \frac{\partial l_\xi}{\partial t_\nu}(p_n^+) = O(\theta_n^2 + \phi_n^2). \tag{8}$$

Dividing through by ϕ_n and taking limits, we deduce (again using Proposition 2.1) that

$$\frac{\partial l_\xi}{\partial t_\nu}(p_\infty) = 0$$

or, equivalently, using the asymmetry relations, that

$$\frac{\partial l_\nu}{\partial t_\xi}(p_\infty) = 0.$$

Since this holds for all $\xi \in \mathscr{ML}$, and since the tangent vectors $\partial/\partial t_\xi$ span the tangent space to $\mathscr{F}$ (see [14, Theorem 3.5]), we deduce that l_ν has a critical point at $p_\infty \in \mathscr{F}$, which is impossible (since not all derivatives along all possible earthquake paths $\mathscr{E}_\xi(t)$ can vanish; see, e.g., [14, p. 194]). □

4. A special example

Before proceeding to the proof of Proposition 1.8, we pause to examine one of the few examples in which one can write down exact formulae for the relationship between bending angles and lengths and hence explore the limit behaviour explicitly.† Namely, take S to be a once-punctured torus, and let $\alpha, \beta \in \pi_1(S)$ intersect exactly once. Thus $\pi_1(S)$ is the free group generated by α and β, and the commutator $\alpha\beta\alpha^{-1}\beta^{-1}$ represents a loop around the puncture. We study the case in which $\mathrm{pl}^+ \in [\alpha]$ and $\mathrm{pl}^- \in [\beta]$; in other words, the surfaces $\partial\mathscr{C}^\pm$ are bent along axes that project to α and β, respectively. Although this situation appears to be very special, quite similar geometry appears in the general case.

By [8, Lemma 4.6], if a geodesic γ is contained in the bending lamination, then its image in PSL(2, $\mathbb{C}$) has real trace. As shown in [22, p. 195], this gives the equations

$$\cos\left(\frac{\theta_\alpha}{2}\right) = \left(\cosh\frac{l_{\beta^*}}{2}\right)\ \tanh\left(\frac{l_{\alpha^*}}{2}\right), \qquad \cos\left(\frac{\theta_\beta}{2}\right) = \cosh\left(\frac{l_{\alpha^*}}{2}\right)\ \tanh\left(\frac{l_{\beta^*}}{2}\right), \tag{9}$$

relating the bending angles $\theta_\alpha, \theta_\beta$ to the lengths $l_{\alpha^*}, l_{\beta^*}$ of the geodesic representatives α^* and β^* of α, β in $\mathbb{H}^3/G$. Moreover, suitably chosen lifts Ax A, Ax B of α^* and β^* are mutually perpendicular at distance d, where

$$\cosh\left(\frac{d}{2}\right)\ \sinh\left(\frac{l_{\alpha^*}}{2}\right)\ \sinh\left(\frac{l_{\beta^*}}{2}\right) = 1. \tag{10}$$

†For some other examples in which explicit calculations can be made, see [5].

(Here $A, B \in G$ are translations by $l_{\alpha^*}, l_{\beta^*}$ along the respective axes.)

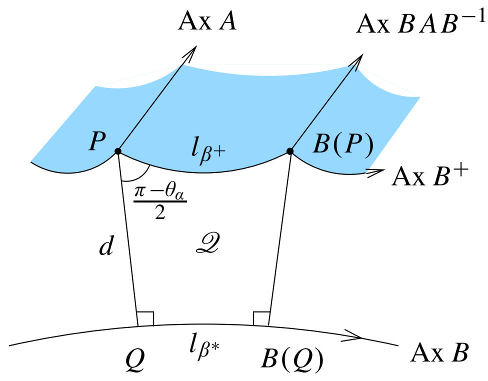

Figure 1. Configuration of bending axes on the once-punctured torus

Since $\text{pl}^+ \in [\alpha]$, the axis Ax A lies on $\partial\mathscr{C}^+$. Let PQ be the common perpendicular to Ax A and Ax B with $P \in$ Ax A and $Q \in$ Ax B (see Figure 1). Since B is purely hyperbolic, the quadrilateral $\mathscr{Q}$ with vertices P, Q and $B(P)$, $B(Q)$ is planar; call the plane containing these four points Π. It follows that the line $PB(P)$ is perpendicular to Ax A and Ax BAB^{-1} and is thus a lift of the shortest curve β^+ in the homotopy class of β on $\partial\mathscr{C}^+(G)/G$. In other words, $PB(P)$ projects to β^+. Note that $|PQ| = d$, and that since Π is orthogonal to Ax A, the line $PB(P)$ makes an angle $(\pi - \theta_\alpha)/2$ with PQ. By applying the quadrilateral formulae (see [1, Theorem 7.17.1]) to $\mathscr{Q}$, we obtain

$$\sinh d = \coth\left(\frac{l_{\beta^*}}{2}\right)\tan\left(\frac{\theta_\alpha}{2}\right), \qquad \cosh\left(\frac{d}{2}\right)\sinh\left(\frac{l_{\beta^*}}{2}\right) = \sinh\left(\frac{l_{\beta^+}}{2}\right). \tag{11}$$

LEMMA 4.1

Suppose that $a, b > 0$ are fixed and that $d = d(\theta)$ is the distance between Ax A *and* Ax B *in the group $G(a\theta, b\theta)$. Then $d \leq O(\theta)$ as $\theta \to 0$.*

Proof

Our hypothesis means that $\theta_\alpha = a\theta$ and $\theta_\beta = b\theta$. Suppose first that there is some subsequence along which $l_{\beta^+} \geq c > 0$. If, in addition, $l_{\beta^*} \geq c' > 0$, then equation (11) gives $d \leq O(\theta)$.

Otherwise, passing to a further subsequence, we may assume that $l_{\beta^*} \to 0$. From (11), we have

$$\tanh d = \frac{\tan\theta_\alpha/2}{\tanh l_{\beta^*}/2}\,\frac{\sinh l_{\beta^*}/2}{\sinh l_{\beta^+}/2},$$

from which, since l_{β^+} is bounded away from zero, it follows that $d \leq O(\theta)$.

By interchanging the roles of α and β, we conclude either that both $l_{\beta^+} \to 0$ and $l_{\alpha^-} \to 0$ or that $d \leq O(\theta)$ as $\theta \to 0$. Suppose that the first alternative applies. Then it is also certainly true that $l_{\alpha^*} \to 0$. However, l_{α^*} and l_{β^+} are the geodesic lengths of α and β on the Fuchsian structure $\partial\mathscr{C}^+/G$, and by the collar lemma, this situation is impossible. □

COROLLARY 4.2

Let S be a once-punctured torus with generators α, β, and suppose that $\mu = a\delta_\alpha$, $\nu = b\delta_\beta$. Then the group $G(\theta\mu, \theta\nu)$ converges to a Fuchsian group as $\theta \to 0$. Moreover, the hypotheses of Theorem 1.4 hold, so that the limit is the minimum on $\mathscr{F}$ of the function $al_\alpha + bl_\beta$.

Proof

From the lemma we have that $d \leq O(\theta)$ as $\theta \to 0$. We deduce from (11) and its analogue with α and β interchanged that both l_{α^*} and l_{β^*} are bounded away from 0; and then from (10) we deduce that they are both bounded above. This is sufficient to ensure (up to a subsequence) the existence of the algebraic limit of $G(\theta\mu, \theta\nu)$. (One way to see this is to use the Markov equation, which relates $\operatorname{Tr} AB$ to $\operatorname{Tr} A$ and $\operatorname{Tr} B$.) One also sees that $l_{\alpha^*}/l_{\alpha^+} \to 1$ (and similarly for β). Moreover, not only the axes l_{α^*} and l_{β^*}, but also l_{α^*} and l_{β^+}, and l_{α^-} and l_{β^*}, are orthogonal. This is enough to ensure that the limit of each of the two Fuchsian structures $\partial\mathscr{C}^\pm/G$ also exists and equals the limit of $G(\theta\mu, \theta\nu)$. The remaining details are left to the reader. □

In the above discussion, we made crucial use of the fact that $\theta_\alpha/\theta_\beta$ is bounded away from 0 and ∞ (in fact constant) as $\theta \to 0$. Without this hypothesis, the result fails. In fact, by rearranging equation (9), one obtains that

$$\sinh\left(\frac{l_{\alpha^*}}{2}\right) = \sin\left(\frac{\theta_\beta}{2}\right)\cot\left(\frac{\theta_\alpha}{2}\right), \qquad \sinh\left(\frac{l_{\beta^*}}{2}\right) = \sin\left(\frac{\theta_\alpha}{2}\right)\cot\left(\frac{\theta_\beta}{2}\right). \tag{12}$$

If only one of θ_α and θ_β converges to 0, then one of l_{α^*} and l_{β^*} diverges to ∞. If both θ_α and θ_β converge to 0, then $\sinh l_{\alpha^*} \sim \theta_\alpha/\theta_\beta$ and $\sinh l_{\beta^*} \sim \theta_\beta/\theta_\alpha$. If the ratio is unbounded either above or below, at least one of l_{α^*} and l_{β^*} diverges to ∞. Note, however, that in this case $1/\cosh d = \sinh l_{\alpha^*} \sinh l_{\beta^*} \to 1$, so we still get that $d \to 0$.

5. The main limit theorem

In this section we establish Proposition 1.8 in the case in which μ and ν are rational. The idea of the proof is as follows. First, in Proposition 5.1, we establish an upper bound $d \leq O(\theta)$ for the distance between any point on the lift of a closed geodesic to $\partial\mathscr{C}^{\pm}$ and the corresponding axis in $\mathbb{H}^3$, under the hypothesis that the length of the corresponding curve on $\partial\mathscr{C}^{\pm}/G$ is bounded below. Our estimate also controls the ratio of the lengths on $\partial\mathscr{C}^{\pm}/G$ and in $\mathbb{H}^3/G$. Then, in Proposition 5.8, we prove a *lower* bound $d \geq O(\theta)$ for the distance between any point on a bending line and the opposite side of $\partial\mathscr{C}$. In Proposition 5.10, we play off these two bounds against each other to deduce an upper bound on the lengths of all bending lines. This is sufficient to establish the existence of the limit. Another use of Proposition 5.1 also establishes the necessary estimate on the variation of length of curves in Γ.

PROPOSITION 5.1
Fix $L > 0$. Let $\mu \in \mathscr{ML}$ be fixed, and suppose that $G \in \mathscr{QF}(S)$ is such that $\mathrm{pl}^+(G) = \theta\mu$. *For any $\gamma \in \pi_1(S)$, let $\tilde{\gamma}^+$, $\tilde{\gamma}^*$ be lifts of γ^+, γ^* to $\mathbb{H}^3$ with the same endpoints on $\partial\mathbb{H}^3$. Suppose that $l_{\gamma^+} \geq L$. Then there are constants θ_0 and c_0 depending on L but not on G, μ, or γ, such that whenever $i(\gamma,\mu)\theta < \theta_0$ and $P \in \tilde{\gamma}^+$, then*

$$d(P,\tilde{\gamma}^*) \leq c_0 i(\gamma,\mu)\theta \qquad \text{and} \qquad l_{\gamma^*} \geq \big(1 - c_0(i(\gamma,\mu)\theta)^2\big) l_{\gamma^+}.$$

In this section we prove this result on the assumption that μ is rational. The extension to the general case is not hard and is done at the beginning of Section 6.

Remark 5.2

(1) This result certainly has applications beyond the present one. It will be noted in the proof that the Kleinian group G can be quite general and that all that is needed is that $\tilde{\gamma}^+$ lie on a pleated surface with bending angle $O(\theta)$; convexity is also not required.

(2) It is crucial in our statement that the constants c_0 and θ_0 not depend on the hyperbolic structure of $\partial\mathscr{C}^+$. One sees from the proof that θ_0 increases with increasing L. If μ is multiplied by a scalar $t > 0$, then the term $i(\gamma,\mu)$ scales accordingly.

(3) The result fails without the hypothesis of a lower bound on l_{γ^+}. In this situation, the distance between l_{γ^+} and l_{γ^*} may become infinite with no control on the ratio $l_{\gamma^+}/l_{\gamma^*}$. This can be seen by letting $\theta_\alpha \to 0$ and $\theta_\beta \to \pi/2$ in equation (12) in Section 4 and then examining (11).

(4) The following variant has been proved independently by Lecuire [16] (without the estimate of the distance between l_{γ^+} and l_{γ^*}). Suppose that $\epsilon \leq \pi/12$ and $i(\gamma,\mu) \leq \epsilon$. Then $l_{\gamma^+} \leq (1 + \tan\epsilon)(l_{\gamma^*} + 6\epsilon)$.

Let $\gamma \in \pi_1(S)$ have lifts $\tilde{\gamma}^+$, $\tilde{\gamma}^*$ as in the statement of the proposition. Pick $P \in \tilde{\gamma}^+$, and let $\hat{\gamma}$ denote the piecewise geodesic arc in $\mathbb{H}^3$ joining the points $\gamma^n(P)$, $n \in \mathbb{Z}$

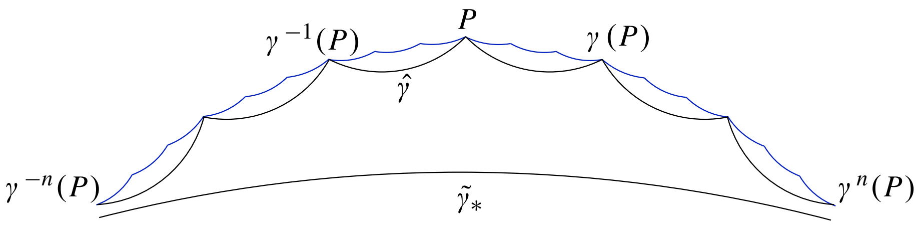

Figure 2. Configuration for Proposition 5.1

(see Figure 2). (By abuse of notation, we are also using γ to denote the element of G in the conjugacy class of γ which fixes the axis $\tilde{\gamma}^*$.) The idea is to estimate the distance between $\tilde{\gamma}^+$ and $\hat{\gamma}$, and then between $\hat{\gamma}$ and $\tilde{\gamma}^*$, while at the same time comparing their lengths. To do this we use two lemmas about piecewise geodesic arcs in $\mathbb{H}^3$ based on a nice idea in [4, Theorem 4.2.12]. First, here is a simple result about hyperbolic triangles.

LEMMA 5.3
Let ABC be a hyperbolic triangle with exterior angle $\phi < \pi/2$ at C. If $h = d(C, AB)$, then $\tanh h \le \sin\phi$.

Proof
Since $\phi < \pi/2$, the perpendicular from C to the geodesic containing AB lands within the segment AB; denote this endpoint by X. Since both of the angles $\angle ACX$ and $\angle BCX$ are less than $\pi/2$, the line through C and perpendicular to CX is outside the triangle ABC and hence makes an angle $\psi < \phi$ with CB.

Let E be the point at which the extension of CB meets $\partial\mathbb{H}^2$, and let Z be the foot of the perpendicular from E to the extension of CX. The extension λ of AB divides $\mathbb{H}^2$ into two half-planes; since EZ cannot cut λ, it must lie on the side not containing C, so that $|CZ| \ge |CX|$. By the angle of parallelism formula, $\tanh |CZ| = \sin\psi$. The result follows. □

Now for the estimates based on [4].† The following notation is convenient. Let σ be a piecewise geodesic arc in $\mathbb{H}^3$ with endpoints X and X'. For $P \in \sigma$, let $\upsilon(P) = \upsilon(P, \sigma)$ be the (positive) angle at P between the forward vector along σ at P and the forward vector along the line extending XP and pointing away from X. (Notice that $\upsilon(P)$ is still well defined, even if P is on a bending line; the function $\upsilon(P)$

†We note that the last sentence in the statement of [4, Theorem 4.2.12] is incorrect. The proof, however, is correct, and our version here indicates one way of proving what was clearly intended.

is discontinuous at such points.) Likewise, let $w(P) = w(P, \sigma)$ be the angle at P between the backward vector along σ at P and the forward vector along the line extending $X'P$ and pointing away from X'. The configuration is shown in Figure 3.

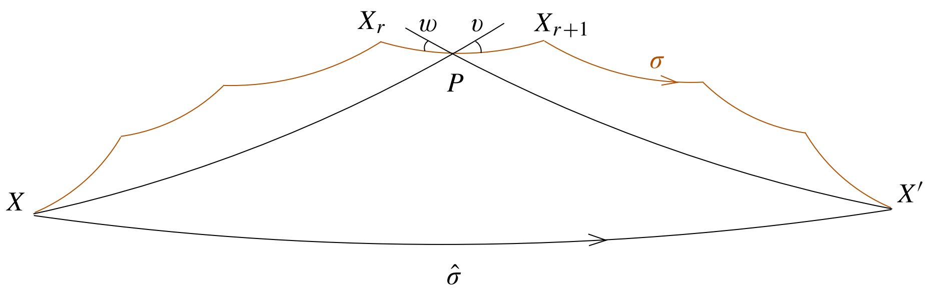

Figure 3. Configuration for Lemma 5.4

LEMMA 5.4

Let σ be a piecewise geodesic arc in $\mathbb{H}^3$ with endpoints X and X', and let $\hat{\sigma}$ be the $\mathbb{H}^3$-geodesic joining X to X'. Suppose that for all $P \in \sigma$, smooth or not, both the angles $v(P, \sigma)$, $w(P, \sigma)$ are bounded above by some angle $\phi < \pi/4$. Then $l_{\hat{\sigma}} \geq (\cos\phi) l_\sigma$ and $\tanh d(P, \hat{\sigma}) \leq \sin 2\phi$ for all $P \in \sigma$, where l_σ and $l_{\hat{\sigma}}$ are the lengths of σ and $\hat{\sigma}$, respectively.

Proof

Suppose the arc σ has successive bends at points $X = X_0, X_1, X_2, \ldots, X_k = X'$. Let us compare the geodesic distance $x = |XP|$ with the distance $t = \sum_{i=0}^{r-1} |X_i X_{i+1}| + |X_r P|$ measured along the broken arc σ. Obviously, x is a piecewise C^1-function of x which only fails to be differentiable at the bends X_i. It is not hard to check that on each open arc, $dx/dt = \cos v(P, \sigma)$. Thus the first part of the result follows by integrating (using the obvious continuity of x as P moves through each point X_i).

For the second part, our hypothesis implies that the exterior angle at P of the triangle PXX' is at most 2ϕ. An application of Lemma 5.3 gives the result. □

In the next two lemmas, we assume for simplicity that all bending angles are positive. This is not necessary, but, since we are dealing with convex hull boundaries, it is all that we need.

LEMMA 5.5

Fix $L > 0$. Then there exist $\theta_1 > 0$ and c_1, both depending only on L, with the following property. Suppose σ is a piecewise geodesic arc in $\mathbb{H}^3$ for which each segment *has a length of at least L and for which the angle at each bend is positive and at most ϕ for some $0 < \phi < \theta_1$. Then for any $P \in \sigma$, the angles $v(P, \sigma)$ and $w(P, \sigma)$ are at most $c_1\phi$.*

Proof
This can be deduced as in [4, Theorem 4.2.12]; for convenience, we give a slightly different version here. We work with $v(P)$; the argument for $w(P)$ is the same. As before, suppose σ meets successive bending lines in points $X = X_0, X_1, X_2, \ldots, X_k = X'$. Let P be a point on the open segment $X_r X_{r+1}$, so that $v(P) = v(P, \sigma)$ is the (positive) angle from the oriented line $X_r X_{r+1}$ and the extension of XP through P, oriented away from X. Clearly, $v(P)$ decreases as P moves along the arc from X_r to X_{r+1}. Let $b(X_r)$ denote the bending angle at X_r, and let $u(X_r)$ denote the angle at X_r between the forward vector along the extension of $X_{r-1}X_r$ and XX_r. Then $v(X_r) \le b(X_r) + u(X_r)$ with equality if the points X, X_{r-1}, X_r, X_{r+1} are coplanar. (This follows from the triangle inequality for angles on the sphere.) Writing $t = d(X_r, P)$ and $v = v(P)$, then as shown in [4], $dv/dt = -\sin v/\tanh|XP|$. Hence we can estimate the decrease in v along $X_r X_{r+1}$ by

$$v(X_r) - u(X_{r+1}) = -\int_{X_r}^{X_{r+1}} \frac{dv}{dt}\, dt \ge L \sin u(X_{r+1}) \ge Lu(X_{r+1})/2,$$

provided the initial angle $v(X_r)$ is less than some fixed α for which $\sin\alpha \ge \alpha/2$, say. Thus

$$u(X_{r+1}) \le \frac{2v(X_r)}{(2+L)},$$

and so, by our hypothesis on the bending angles,

$$v(X_{r+1}) \le \frac{2v(X_r)}{(2+L)} + \phi.$$

Choose $c_1(L) = (2+L)/L$, and choose $\theta_1 < \alpha/c_1$. We show inductively that if $\phi < \theta_1$, then the angle $v(P)$ is at most $c_1\phi$ for any point P in the closed subsegment of σ between X_0 and X_r. This is certainly true for $r = 1$; on the first open segment $v(P) = 0$, and $v(X_1)$ is the bending angle at X_1. Thus, by hypothesis, $v(X_1) \le \phi < c_1\phi$ (since $c_1 > 1$). Now suppose inductively that $v(X_r) \le c_1\phi$. By our choice of c_1, this implies $v(X_r) < \alpha$. Then, as above, $v(X_{r+1}) \le 2c_1\phi/(2+L) + \phi = c_1\phi$. The result follows. □

LEMMA 5.6
Let σ be a piecewise geodesic arc in $\mathbb{H}^3$ with positive bends at points $X_0, \ldots, X_k$. Then

$$\angle X_1 X_0 X_k + \angle X_{k-1} X_k X_0 \le \sum_{r=1}^{k-1} b(X_r),$$

where $b(X_r)$, as defined above, is the bending angle at X_r.

Proof

In the triangle $X_0X_rX_{r+1}$, let α_r be the interior angle at X_0, and let β_{r+1} be the interior angle at X_{r+1}. Note that $v(X_r)$ is the exterior angle at X_r, so that $\alpha_r+\beta_{r+1} < v(X_r)$ for $r = 1, \dots, k-1$. Setting $\beta_1 = 0$, we also have $v(X_r) \leq b(X_r) + \beta_r$ for $r = 1, \dots, k-1$, with equality if the points $X_0X_{r-1}X_rX_{r+1}$ are coplanar. Likewise, $\angle X_1X_0X_k \leq \sum_{r=1}^{k-1} \alpha_r$, while $\angle X_{k-1}X_kX_0 = \beta_k$. Summing from $r = 1$ to $k-1$ gives the result. □

Proof of Proposition 5.1

Assume that μ is rational. As illustrated in Figure 2, pick a point $P \in \tilde{\gamma}^+$, and denote one full translation length of $\tilde{\gamma}^+$ along $\partial\mathscr{C}^+$ from P to $\gamma(P)$ by σ^+. (Here γ denotes the particular choice of element in the conjugacy class of γ which fixes $\tilde{\gamma}^+$.) We first compare σ^+ to the $\mathbb{H}^3$-geodesic $\hat{\sigma}$ joining P to $\gamma(P)$. Since σ^+ is geodesic on $\partial\mathscr{C}^+$, the angle between successive geodesic segments of σ^+ is bounded above by the bending angle on $\partial\mathscr{C}^+$ (see, e.g., [10, Lemma 6.2]). Let Q be a point on one such segment X_rX_{r+1}. By Lemma 5.6, the angle between PQ and X_rX_{r+1} is bounded above by $\theta i(\gamma, \mu)$, as is the angle between $\gamma(P)Q$ and X_rX_{r+1}. Thus $v(P)$ and $w(P)$ are each bounded above by $2\theta i(\mu, \gamma)$. (We need the factor of 2 to cover the case in which $P = X_{r+1}$ is a bending point so that $v(P)$ is bounded above by the sum of the angle between $\gamma(P)Q$ and X_rX_{r+1} and the bending angle at X_{r+1}.)

Choose $\theta_1' < \pi/4$ small enough that for $|x| < \theta_1'$ we have $\cos x \geq 1 - x^2/2$ and such that $\tanh y \leq \sin 2x$ implies that $y \leq 3x$. Then by Lemma 5.4, provided $2i(\mu, \gamma)\theta < \theta_1'$, we have

$$l_{\hat{\sigma}} \geq \big(1 - 2(i(\mu, \gamma)\theta)^2\big) l_{\sigma^+} \qquad \text{and} \qquad d(Q, \hat{\sigma}) \leq 6i(\mu, \gamma)\theta,$$

where $l_{\hat{\sigma}}$ is the length of $\hat{\sigma}$, that is, the distance in $\mathbb{H}^3$ between P and $\gamma(P)$.

Now let $\hat{\gamma}_n$ denote the piecewise geodesic arc $\bigcup_{r=-n}^{n-1} \gamma^r(\hat{\sigma})$, so that $\hat{\gamma}_n$ joins the points $\gamma^r(P), \gamma^{r+1}(P)$ for $r = -n, \dots, n-1$. By Lemma 5.6, the bending angles between the segments of $\hat{\gamma}_n$ are at most $2i(\mu, \gamma)\theta$. Given $L > 0$, choose $c_1(L)$ and $\theta_1(L)$ as in Lemma 5.5. Replacing $\theta_1 = \theta_1(L)$ by $\min\{\theta_1(L), \theta_1'\}$ if necessary, apply the lemma to $\hat{\gamma}_n$ to see that whenever $2\theta i(\mu, \gamma) < \theta_1$, then for any $Q \in \hat{\gamma}_n$, the angles between $\gamma^{-n}(P)Q$ and $\hat{\gamma}_n$, and between $Q\gamma^n(P)$ and $\hat{\gamma}_n$, are at most $2c_1 i(\mu, \gamma)\theta$.

Applying Lemma 5.4 again shows that whenever $2\theta c_1 i(\mu, \gamma) < \theta_1$, any point on the geodesic arc joining $\gamma^{-n}(P)$ to $\gamma^n(P)$ is within distance $6c_1 i(\mu, \gamma)\theta$ of $\hat{\gamma}_n$ and that

$$d_{\mathbb{H}^3}\big(\gamma^{-n}(P), \gamma^n(P)\big) \geq 2n\big(1 - 2(c_1 i(\mu, \gamma)\theta)^2\big) l_{\hat{\sigma}}.$$

Since $\gamma^{-n}(P)$ and $\gamma^n(P)$ converge to the negative and positive fixed points of the axis $\tilde{\gamma}^*$, respectively, the arc joining $\gamma^{-n}(P)$ to $\gamma^n(P)$ converges to $\tilde{\gamma}^*$. Choosing c_0 appropriately, the result follows. □

We now turn to the lower bound discussed at the beginning of this section. A *support plane* Σ to $\partial\mathscr{C}$ is a complete hyperbolic plane in $\mathbb{H}^3$ which meets $\mathscr{C}$, with the property that all of $\mathscr{C}$ is contained in one of the two half-spaces cut out by Σ. We use the following easy facts.

LEMMA 5.7
Let Σ^+, Σ^- be support planes to $\partial\mathscr{C}^+$, $\partial\mathscr{C}^-$, respectively. Then Σ^+ and Σ^- are disjoint. Moreover, no bending line of $\partial\mathscr{C}^+$ along which there is a nonzero bending angle is tangent to Σ^- at infinity.

Proof
Denote by $D^\pm$ the open round disks in $\hat{\mathbb{C}}$ which are the ends at infinity of the half-spaces in $\mathbb{H}^3$ cut out by $\Sigma^\pm$ and not containing $\mathscr{C}$. We claim that neither disk $D^\pm$ contains any limit points of G. In fact, if there were a limit point in D^+, then by convexity there would be a line segment contained in $\mathscr{C}$ and meeting the half-planes on both sides of Σ^+, which is impossible.

Since G is quasi-Fuchsian, its regular set has two components $\Omega^\pm$. By the definition of $\partial\mathscr{C}^\pm$, we have $D^\pm \subset \Omega^\pm$. If Σ^+ and Σ^- meet, then so do D^+ and D^-, contradicting the fact that Ω^+ and Ω^- are disjoint. This proves the first claim.

Now let L be a bending line of $\partial\mathscr{C}^+$ along which the bending angle is $\phi > 0$. Let D_1, D_2 be the open disks in $\hat{\mathbb{C}}$ cut out by the support planes to $\partial\mathscr{C}^+$ meeting along L and not containing $\mathscr{C}$ as above. Then $\partial D_1, \partial D_2$ are circles intersecting at angle ϕ in the endpoints of L. Let Σ^- be a support plane of $\partial\mathscr{C}^-$ cutting out a disk D^- on $\hat{\mathbb{C}}$ as above. If L meets Σ^- on $\hat{\mathbb{C}}$, then ∂D^- is a circle through one of the endpoints of L, so that D^- must intersect at least one of the open disks D_1, D_2, contradicting the first statement. □

The idea is that there is not only a maximum but also a *minimum* distance between a bending line on one side of $\partial\mathscr{C}$ and any support plane of the other side because, as illustrated in Figure 4, a pair of support planes which meet at a definite angle are at a definite distance away from any other plane disjoint from them both. The following result, which holds for unrestricted bending angles, is mainly of interest when θ is small.

PROPOSITION 5.8
There exists a universal constant $c_2 > 0$ with the property that if the point P is on a

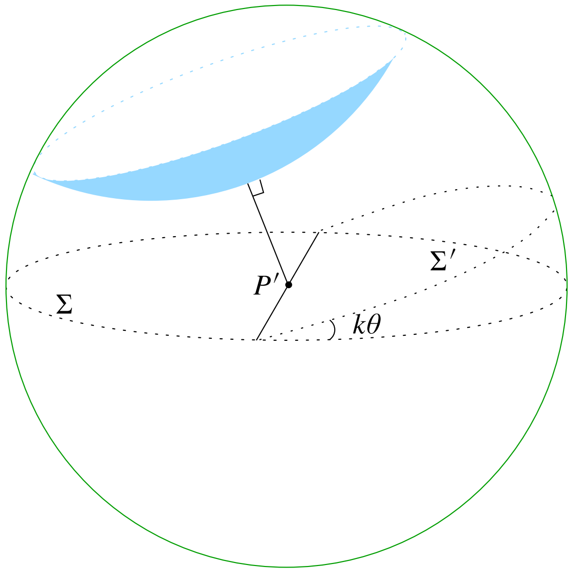

Figure 4. Support planes to $\partial\mathscr{C}^+$ and $\partial\mathscr{C}^-$ illustrating Proposition 5.8

bending line $\tilde{\alpha}$ in $\partial\mathscr{C}^+$ whose weight in the lamination $\mu = \sum_i a_i\alpha_i$ is $k > 0$, then $d(P, \partial\mathscr{C}^-) \geq c_2 k\theta$.

Proof

We seek a lower bound for $d(P, Q)$ where P is on a bending line $\tilde{\alpha}$ in $\partial\mathscr{C}^+$ as in the statement and $Q \in \partial\mathscr{C}^-$. Choose a support plane Σ^- of $\partial\mathscr{C}^-$ containing Q. By Lemma 5.7, $\tilde{\alpha}$ is disjoint from Σ^- and not asymptotic to Σ^- at infinity. Thus $\tilde{\alpha}$ and Σ^- have a common perpendicular δ with endpoints $P' \in \tilde{\alpha}$ and $Q' \in \Sigma^-$. Since $d(P, Q) \geq d(P', Q')$, it suffices to find a lower bound for $d(P', Q')$.

Consider the plane Π orthogonal to $\tilde{\alpha}$ and containing δ. Let Σ, Σ' be the support planes that meet along $\tilde{\alpha}$. Let λ, λ' be the lines in which Σ, Σ' meet Π, and let η be the line that joins the ends of λ, λ' at infinity, forming a triangle Δ with two ideal vertices and an exterior angle $k\theta$ at P'. By Lemma 5.7 again, no support planes of $\partial\mathscr{C}^-$ intersect Δ; hence $d(P', Q') \geq d(P', \eta)$. The perpendicular distance h from P' to η is given by $\tanh h = \sin(k\theta/2)$. The result follows. (Consider the cases $k\theta$ small and $k\theta$ bounded away from zero separately.) □

We can now play Propositions 5.1 and 5.8 against each other to get a bound on the lengths of the bending lines. The idea is that if a very long segment of the geodesic representative $\tilde{\alpha}^-$ of a bending line $\alpha \subset |\mu|$ is entirely contained in a flat piece of $\partial\mathscr{C}^-$, then it must get very close to the actual bending line $\tilde{\alpha}^* = \tilde{\alpha}^+$ on $\partial\mathscr{C}^+$, contradicting Proposition 5.8. We need the following lemma about skew quadrilaterals; this is [4, Theorem 2.4.6]. It follows from an explicit calculation of the distance between points on two fixed geodesics and from the fact that the distance function is convex.

LEMMA 5.9

Let $X_1X_2Y_2Y_1$ be a skew hyperbolic quadrilateral. Suppose that $d(X_i, Y_i) \leq v$ for $i = 1, 2$. Let $\eta > 0$ be given, and let Z be any point on X_1X_2 with $d(X_i, Z) \geq \eta$ for each i. Let $u = d(Z, Y_1Y_2)$ be the distance from Z to the line Y_2Y_1. Then $\sinh u \leq \sinh v / \cosh \eta$.

PROPOSITION 5.10

The lengths $l_{\mu^\pm}, l_{\nu^\pm}$ of the bending laminations on $\partial\mathscr{C}^\pm/G(\theta)$ are uniformly bounded above as $\theta \to 0$.

Proof

It will be enough to show that there is a uniform upper bound on l_{α^-} for any component α of $|\mu|$. By similar reasoning, we also obtain an upper bound on l_{β^+} for any component β of $|\nu|$ and, hence, a fortiori on l_{β^-}.

A lift $\tilde{\alpha}^-$ of α^- on $\partial\mathscr{C}^-$ is partitioned into a finite number of geodesic segments by the points at which it meets the bending lines $|\nu|$ on $\partial\mathscr{C}^-$. We claim that the length of each segment is uniformly bounded above as $\theta \to 0$. As usual, let $\tilde{\alpha}^\pm$ be lifts of $\alpha^\pm$ with the same endpoints on $\partial\mathbb{H}^3$.

We may as well suppose that $l_{\alpha^-} \geq 1$ so that we can apply Proposition 5.1 to $\partial\mathscr{C}^-$ and α, with $L = 1$, to show that $d(P, \tilde{\alpha}^+) \leq O(\theta)$ for all $P \in \tilde{\alpha}^-$. (The claim is that this inequality holds as $\theta \to 0$. Since we are dealing only with the fixed finite collection of curves in $|\mu|$, by reducing the constant θ_1 if necessary, we may replace the condition, required by Proposition 5.1, that $\theta i(\alpha, \nu) \leq \theta_1$ for each component $\alpha \subset |\mu|$, with the single condition $\theta \leq \theta_1$.) Let X_1, X_2 be successive points at which α^- meets $|\nu|$, and let Y_1, Y_2 be the feet of the perpendiculars from X_1 and X_2 to $\tilde{\alpha}^+$. We apply Lemma 5.9 to the skew quadrilateral X_1, X_2, Y_2, Y_1. We have just shown that $d(X_i, Y_i) \leq O(\theta)$. Let Z be the midpoint of Y_1Y_2, and let $u = d(Z, X_1X_2)$. Since $\tilde{\alpha}^*$ is a bending line and the segment from X_1 to X_2 is contained in $\partial\mathscr{C}^-$, Proposition 5.8 gives $u \geq O(\theta)$ whenever $\theta \leq \theta_1$, where we again adjust the constant θ_1 to take account of all the finitely many inequalities required by the proposition. Applying Lemma 5.9, we find $\sinh u \leq O(\theta)/\sinh y$, where $y = |Y_1Y_2|/2$. Given the lower bound on u, this is impossible if $y \to \infty$. We deduce that y is uniformly bounded above. Since Y_1Y_2 is the perpendicular projection of X_1X_2 through a distance $O(\theta)$, we also get a bound on X_1X_2. Summing over all segments gives a uniform upper bound on l_{α^-}, as required. □

COROLLARY 5.11
There exists $\theta_0 > 0$ *such that for* $\theta < \theta_0$*, the structures* $p^{\pm}(\theta)$ *lie in a compact set in* $\mathscr{F}(S)$.

Proof
This is [26, Corollary 2.3]. If the structures $p^{+}(\theta)$ were not in a compact set in $\mathscr{F}$, then we could find a subsequence converging to a point ξ in the Thurston boundary P$\mathscr{ML}$. Since the systems $|\mu|$ and $|\nu|$ together fill up the surface, ξ has a nonzero intersection number with at least one component δ of either $|\mu|$ or $|\nu|$; and hence $l_{\delta^+} \to \infty$ as $\theta \to 0$. This contradicts the above proposition and proves the claim. □

We can now prove the main result of this section.

Proof of Proposition 1.8 for rational μ, ν
By Corollary 5.11, for small θ, all of the structures $p^{\pm}(\theta)$ lie in a compact set K in $\mathscr{F}$. Choose a sequence $\theta_n \to 0$ along which $p^{+}(\theta_n) \to p^{+}_{\infty}$ and $p^{-}(\theta_n) \to p^{-}_{\infty}$ for points $p^{\pm}_{\infty} \in K$.

By compactness, there is a uniform lower bound to the noncuspidal injectivity radius of all surfaces in K, equivalently, a uniform lower bound to the length of all simple geodesics. Therefore we may apply Proposition 5.1 to see that $l_{\gamma^*} \geq (1 - O(\theta_n^2))l_{\gamma^{\pm}}$ as $\theta \to 0$ for any curve $\gamma \in \mathscr{S}$, where the constants depend only on $i(\gamma, \mu)$, $i(\gamma, \nu)$, and K. Writing G_n for $G(\theta_n)$, we deduce that $1 - O(\theta_n^2) \leq l_{\gamma^-}(G_n)/l_{\gamma^+}(G_n) \leq 1 + O(\theta_n^2)$ as $n \to \infty$. Since in K the lengths $l_{\gamma^{\pm}}(G_n)$ are also uniformly bounded above, we deduce that $|l_{\gamma^+}(G_n) - l_{\gamma^-}(G_n)| \leq O(\theta_n^2)$ as $n \to \infty$, with constants depending only on γ. Applying this to a fixed finite collection of curves γ_i whose lengths determine the real-analytic structure on $\mathscr{F}$, we deduce in particular that $p^{+}_{\infty} = p^{-}_{\infty}$.

To prove the convergence of the groups G_n to p^{+}_{∞}, note that as in the proof of Theorem 1.5 in Section 3, it follows from Sullivan's theorem that the conformal structures $\omega^{\pm}(G_n)$ on the conformal boundaries $\Omega^{\pm}(G_n)/G_n$ at infinity also lie in a compact set in $\mathscr{F}$. Using the complex analytic dependence in Bers' simultaneous uniformisation theorem, it follows that the groups G_n are contained in a compact set $K' \subset \mathscr{QF}$.

Now choose a sufficiently large finite set of curves δ_i so that their complex lengths λ_{δ_i} determine the complex analytic structure on $\mathscr{QF}$. The functions λ_{δ_i}, together with their first and second derivatives with respect to the twists t_μ and t_ν, are uniformly bounded on K'. Then equations (1) and (2) in Proposition 3.1 show that the values of the functions λ_{δ_i} at the points $p^{\pm}(\theta_n)$ and at G_n are uniformly close as $\theta \to 0$, and hence the algebraic limit of the groups G_n as $\theta \to 0$ is also p^+_∞. (Note that since p^+_∞ is geometrically finite, the limit is strong.) □

Proof of Theorems 1.2 and 1.4 in the rational case
Combining this result with Proposition 3.1, we see that the limit in Proposition 1.8 is in fact the minimum $M(\mu, \nu)$ and, hence, independent of the subsequence chosen. This completes the proof of Theorems 1.2 and 1.4 in the rational case. □

Notice that the compact set K in the above proof is *not* given uniformly in terms of θ but depends in an unspecified way on the point p^+_∞. This will cause us some grief in Section 7.

6. Extension to the irrational case

We now discuss the extension of Proposition 1.8 to the case in which μ, ν are irrational. The proof in the last section does not immediately extend; this is mainly because the constants involved in the final estimates in Proposition 5.10 depend heavily on the number of bending lines in $|\mu|$ and $|\nu|$. In order to control the "size" of a measured lamination $\xi \in \mathscr{ML}$, fix once and for all a set $\Gamma = \{\gamma_1, \dots, \gamma_k\} \subset \mathscr{S}$ of curves that fill up S, and set $||\xi||_\Gamma = \sum_j i(\gamma_j, \xi)$. Since the curves fill up, we have $||\xi||_\Gamma > 0$. Notice that $||\xi||_\Gamma$ is independent of the hyperbolic structure on S. (In the rational case, one can choose Γ to be the curves in $|\mu| \cup |\nu|$. With this choice, it is not hard to check that the constants involved in Proposition 5.10 depend only on $||\mu||$ and $||\nu||$.)

We begin by completing the proof of Proposition 5.1 for irrational μ.

Proof of Proposition 5.1
All we need to do is adapt the first part of the proof from Section 5 to the case in which $\mu \notin \mathscr{ML}_{\mathbb{Q}}$. As before, pick a point $P \in \tilde{\gamma}^+$, and denote the arc of $\tilde{\gamma}^+$ along $\partial\mathscr{C}^+$ from P to $\gamma(P)$ by σ^+. We want to compare σ^+ to the $\mathbb{H}^3$-geodesic $\hat{\sigma}$ joining P to $\gamma(P)$.

Recall that the bending measure and distance along any arc κ on $\partial\mathscr{C}^+$ is defined in terms of finite approximations called "roofs". To construct a roof, pick a finite number of support planes to $\partial\mathscr{C}^+$ at points along κ. These planes intersect in lines called "ridge lines". The path κ is approximated by a piecewise geodesic arc made up of segments in these support planes joining suitably chosen points in the ridge lines. The arc length and total bending angle along this finite approximation are measured in the obvious way. The distance and bending angle along κ are by definition the infima, over all possible roofs, of the corresponding finite approximations (see [7] and also [9] for more details).

In the current situation, we obtain the required type of estimate for any roof exactly as before, and the required comparisons

$$l_{\hat{\sigma}} \geq \big(1 - O((i(\mu,\gamma)\theta)^2)\big) l_{\sigma^+} \qquad \text{and} \qquad d(P,\hat{\sigma}) \leq O\big(i(\mu,\gamma)\theta\big)$$

for any $P \in \sigma^+$ follow. The remainder of the proof is exactly as before. □

We need the following extension of Proposition 5.8. Although we keep the same names, the constants involved are not exactly the same as those in the earlier version.

PROPOSITION 6.1
There exist $\epsilon_2, c_2 > 0$ with the following property: if $P \in \partial\mathscr{C}^+$ lies on a geodesic segment σ of length at most ϵ_2, then $d(P, \partial\mathscr{C}^-) \geq c_2 i(\sigma,\mu)\theta$.

The idea of this result is clear, but a careful proof requires some work. As in Proposition 5.8, we are mainly interested in the case when $i(\sigma,\mu)\theta$ is small. Note that if σ is not transverse to μ, then $i(\sigma,\mu) = 0$, and the result is vacuously true. The following two lemmas control changes of angle as we move short distances along a piecewise geodesic arc.

LEMMA 6.2
Let $Z \in \partial\mathbb{H}^3$, and let λ be an oriented line with endpoints distinct from Z. For $i = 1, 2$, let X_i be points on λ with $X_2 > X_1$, and let ϕ_i be the positive angle at X_i between the forward direction of λ and the extension of ZX_i through X_i. Then $\phi_2/\phi_1 > 1 - |X_1X_2|$.

Proof
Let W be the foot of the perpendicular from Z to λ. For any point $X \in \lambda$, define ϕ as in the statement, and let $t = |WX|$. By the angle of parallelism formula, $\tanh t = \cos\phi$. Differentiating, we find $d\phi/dt = -\sin\phi$. (Note that this is the limiting case of a similar formula used in the proof of Lemma 5.5.) Integrating along the arc from X_2 to X_1 gives $\phi_1 - \phi_2 \leq |X_1X_2| \sin\phi_1$, from which the result follows. □

Now let σ be a piecewise geodesic arc in $\mathbb{H}^3$ with a finite number of bends $X_0, \ldots, X_k$ and endpoints Z, Z' in $\partial\mathbb{H}^3$. As in the discussion just before Lemma 5.4, for $P \in \sigma$, let $v(P) = v(P,\sigma)$ be the (positive) angle at P between the forward vector along σ at P and the forward vector along the line extending ZP and pointing away from Z.

LEMMA 6.3
Suppose that σ is a piecewise geodesic arc on $\partial\mathscr{C}^+$ with initial and final points

$Z, Z' \in \partial\mathbb{H}^3$ and successive bends at points $X_0, X_1, \ldots, X_k \in \mathbb{H}^3$. Suppose that the angle between successive segments at X_i is ϕ_i. Then with the notation above,

$$v(X_r) \geq \Big(1 - \sum_{i=0}^{r} |X_i X_{i+1}|\Big)\Big(\sum_{i=0}^{r} \phi_i\Big).$$

Proof

We prove this by induction on r. For $r = 0$, the result follows from the definitions. Assume that $r > 0$ and that the result holds for $r - 1$. Setting $\epsilon_i = |X_{i-1}X_i|$, this means that

$$v(X_{r-1}) \geq \Big(1 - \sum_{i=0}^{r-1} \epsilon_i\Big)\Big(\sum_{i=0}^{r-1} \phi_i\Big).$$

Let ψ_i denote the angle between the extension of ZX_i and $X_{i-1}X_i$, so that $v(X_i) = \psi_i + \phi_i$. Using Lemma 6.2 with λ the geodesic extending the arc $X_{r-1}X_r$, we find $\psi_r / v(X_{r-1}) \geq (1 - \epsilon_r)$. Hence

$$v(X_r) = \phi_r + \psi_r \geq \phi_r + (1 - \epsilon_r)\Big(1 - \sum_{i=0}^{r-1} \epsilon_i\Big)\Big(\sum_{i=0}^{r-1} \phi_i\Big).$$

This last expression is easily seen to be greater than $(1 - \sum_{i=0}^{r} \epsilon_i)(\sum_{i=0}^{r} \phi_i)$, as required. $\square$

Finally, we need a lemma to control the angles at which the arc σ intersects bending lines.

LEMMA 6.4

Choose $\epsilon < \cosh^{-1}\sqrt{2}$. Suppose that λ, λ' are disjoint lines in the hyperbolic plane $\mathbb{H}$ (possibly meeting on $\partial\mathbb{H}$) and that $P \in \lambda$, $P' \in \lambda'$ are such that $|PP'| \leq \epsilon$. Then the line through P orthogonal to λ meets λ' in a point Q; moreover, $|PQ| < O(\epsilon)$ and $\angle PQP' \geq \pi/2 - O(\epsilon)$.

Proof

If the orthogonal to λ through P does not meet λ', then $d(\lambda, \lambda') \geq \cosh^{-1}(\sqrt{2})$, this number being the altitude of a triangle with angles $\pi/2, 0, 0$. This proves the first statement.

Write $|PQ| = x$ and $\angle PQP' = \phi$. Let ϕ_0 be the angle between PQ and the line joining Q to the endpoint of λ on $\partial\mathbb{H}$ on the same side of PQ as P'. Clearly $\phi \geq \phi_0$, and by the angle of parallelism formula, $\sin\phi_0 = 1/\cosh x$.

Let P'' be the foot of the perpendicular from P to λ'; then $h = |PP''| \leq \epsilon$. By trigonometry in the triangle PQP'', we have $\sin\phi = \sinh h / \sinh x$. Combining these observations, we find $\sinh h \geq \tanh x$, from which it follows that $x \leq O(\epsilon)$ and, hence, that $\phi \geq \pi/2 - O(\epsilon)$, as claimed. □

Proof of Proposition 6.1

First suppose that μ is rational. Let σ be a geodesic segment in $\partial\mathscr{C}^+$ of length ϵ (to be determined later). We begin by showing that we may assume that σ is more or less orthogonal to all the bending lines. Let the first and last bending lines cut by σ be λ', λ'', respectively.

We claim that we can always find an $\mathbb{H}^3$-geodesic λ through P and completely contained in $\partial\mathscr{C}^+$. This is obvious if P is on a bending line. If not, there is some $\mathbb{H}^3$-geodesic λ through P contained in a flat piece of $\partial\mathscr{C}^+$ and disjoint from all leaves of $|\mu|$ (possibly meeting leaves of $|\mu|$ on $\partial\mathbb{H}^3$).

Let H be the plane through P orthogonal to λ. We first show that we may replace σ by the segment σ_1 in $H \cap \partial\mathscr{C}^+$ joining λ' to λ''. In fact, applying Lemma 6.4, we see that σ_1 meets the same leaves as σ (so that $i(\sigma_1, \mu) = i(\sigma, \mu)$) and that the length of σ_1 is at most $O(\epsilon)$. Thus we may as well work in the plane H.

As usual, let $X = X_0, X_1, \ldots, X_k = X'$ denote the points at which σ_1 meets the bending lines of $\partial\mathscr{C}^+$. Let ϕ_i denote the angle at X_i between the segments $X_{i-1}X_i$ and X_iX_{i+1}, and let θ_i be the angle between the support planes that meet at X_i. We claim that we may as well replace $i(\sigma, \mu) = \sum \theta_i$ by $\sum \phi_i$. In fact, from Lemma 6.4, we see that σ_1 is almost orthogonal to the bending line through X_i, crossing at the angle ψ_i, say. These angles are related by the formula $\tan\phi_i/2 = \tan\theta_i/2 \sin\psi_i$. We deduce that $\phi_i > (1 - O(\epsilon))\theta_i$. By definition, $\sum_{i=0}^k \theta_i = i(\sigma, \theta\mu)$. Therefore $\sum_{i=0}^k \phi_i > i(\sigma, \mu)\theta/2$, say, for all sufficiently small ϵ.

Now let X_{-1} and X_{k+1}, respectively, be the bending points immediately preceding X_0 and immediately following X_k on the extension of the $\partial\mathscr{C}^+$ geodesic containing σ_1, and let Z, Z' be the points where the continuations of X_0X_{-1} and X_kX_{k+1} meet $\partial\mathbb{H}^3$. Suppose that P is on the arc $X_{r-1}X_r$, $0 < r \leq k$. As above, if P is not on a bending line, we may insert an extra line λ, containing P and disjoint from the other lines in $|\mu|$, and treat λ as a bending line with bending angle 0. Applying Lemma 6.3, we get $v(P) > (1-\epsilon)\sum_{i=0}^{r-1}\phi_i$ and, similarly, $w(P) > (1-\epsilon)\sum_{i=r}^{k}\phi_i$, where $w(P)$ is the angle at P between the forward vector along the line extending $Z'P$ and pointing away from Z' and the backwards direction along σ_1. Thus the angle $v(P) + w(P)$ between the lines ZP and PZ' at P is at least $(1 - \epsilon)i(\sigma, \mu)\theta/2$.

We would now like to follow the method of Proposition 5.8, replacing the support planes which meet along the bending line λ with the planes Σ, Σ' which contain the lines ZP and PZ' and are orthogonal to H. Let Σ^- be any support plane of $\partial\mathscr{C}^-$. In the proof of Proposition 5.8, we replaced PQ by the common perpendicular between λ and Σ^-. The fact that Σ^- may not be disjoint from Σ and Σ' causes problems, so instead we seek a lower bound on $d(P, \Sigma^-)$.

Although Σ^- may not be disjoint from Σ, Σ', we claim that Σ^- is disjoint from both lines ZP and PZ'. For if Σ^- meets PZ in $\mathbb{H}^3$, then Z is contained in the open disk on $\partial\mathbb{H}^3$ which is spanned by Σ^- and which contains no limit points (see Lemma 5.7). But Z is also on the boundary of any support plane to $\partial\mathscr{C}^+$ which contains the arc $X_{-1}X_0$. By Lemma 5.7, this is impossible. Now let Q be the point on $\Sigma^- \cap H$ nearest to P. As in the proof of Proposition 5.8, we obtain an estimate $|PQ| \geq ci(\sigma, \mu)\theta$ for a suitable universal constant c.

What we actually seek is a lower bound on $|PQ'|$ where Q' is the endpoint on Σ^- of the perpendicular from P to Σ^-. Notice that λ and Σ^- are disjoint. Let $\pi/2 - \alpha$ be the angle between PQ' and λ. The limiting case is when λ meets Σ^- on the sphere at infinity, in which case the angle of parallelism formula gives $\tanh|PQ'| = \sin\alpha$. Thus, in general, $\tanh|PQ'| \geq \sin\alpha$. Since λ is orthogonal to H, the angle between PQ' and H is α. This shows that if $\alpha \geq \pi/4$, say, then there is a universal lower bound $|PQ'| \geq \tanh^{-1}(1/\sqrt{2})$.

Now consider the plane J, which contains Q and which is orthogonal to the line $\Sigma^- \cap H$. Then J is orthogonal to both Σ^- and H; moreover, J contains the line PQ and hence also the line λ, which by construction is orthogonal to H. It is not hard to see that J also contains the perpendicular PQ' from P to Σ^-; in fact, Q' is the footpoint of the perpendicular from P to the line $\Sigma^- \cap J$.

Thus the triangle $PQ'Q$ is contained in J, and $\angle Q'PQ$ is the angle α between PQ' and H. As shown above, either there is a universal lower bound to $|PQ'|$, or $\angle Q'PQ < \pi/4$. In the first case, we are done; in the second, the bound $|PQ| \geq ci(\sigma, \mu)\theta$, together with the equation $\tanh|PQ'| = \cos Q'PQ \tanh|PQ|$, gives a bound $|PQ'| \geq c_2 i(\sigma, \mu)\theta$ for some universal c_2, as required.

Finally, we need to deal with the case in which μ is irrational. Since none of the above estimates depend on the number of bending lines which meet σ, approximating μ by finite laminations, as explained in the part of the proof of Proposition 5.1 at the beginning of this section, will work. □

As in the rational case, we are now set to play the upper and lower bounds against each other. We need the following lemma on intersection numbers.

LEMMA 6.5
Suppose that μ and ν fill up S. Then there exists $c_3 > 0$ such that $i(\gamma, \mu) + i(\gamma, \nu) > c_3$ for all $\gamma \in \mathscr{S}$.

Proof
Since the result depends only on intersection numbers, we can work entirely with

a fixed hyperbolic structure $p_0 \in \mathscr{F}$ whose noncuspidal injectivity radius is $\rho_0 > 0$, say. If the result is false, then we can find a subsequence $\gamma_n \in \mathscr{S}$ such that $i(\gamma_n, \mu) + i(\gamma_n, \nu) \to 0$. Passing to a further subsequence, we may assume that there is a sequence $h_n > 0$ such that $h_n \delta_{\gamma_n} \to \xi$ in $\mathscr{ML}$, where $l_\xi(p_0) = 1$. Then $h_n l_{\gamma_n} \to 1$. Since $l_{\gamma_n} \geq \rho_0$, we have $h_n \leq 2/\rho_0$. Thus $i(h_n\gamma_n, \mu) + i(h_n\gamma_n, \nu) \to 0$; and, hence, taking limits, $i(\xi, \mu) + i(\xi, \nu) = 0$. Since μ and ν fill up S, this is impossible. □

Now we can establish a uniform lower bound to the noncuspidal injectivity radii of the structures $p^{\pm}(\theta)$. Write $C = C(\mu, \nu, \Gamma) = ||\mu||_\Gamma + ||\nu||_\Gamma$.

First, we note a useful fact about surfaces (see, e.g., [19, pp. 610–611]). For the reader's convenience, we furnish a complete proof.

LEMMA 6.6

Let S be a hyperbolic surface. Then there exist $k_0 > 0$ and $A > 0$ (depending only on the genus of S) such that if $k < k_0$ and if x is in the k-thick part of S for which the injectivity radius is at least k, then there are two essential primitive and nonhomotopic loops based at x of length at most A/k.

Proof

Provided we take k_0 less than the Margulis constant, each component of the k-thick part of S is itself a surface of negative Euler characteristic, possibly with boundary.

First we bound the diameter of the component X containing x. Clearly, we may assume the $(k/2)$-neighbourhood of the boundary ∂X is a union of annuli. It is sufficient to bound the diameter of $X' = X - \mathscr{N}_{k/2}(\partial X)$. Write $l(\alpha) = O(1/k)$ to mean that there exist $k_0 > 0$ and $A > 0$ such that $l(\alpha) < A/k$ for all $k < k_0$. Take $y, y' \in X'$, and consider the shortest path γ in X' from y to y'. The $(k/2)$-neighbourhood $\mathscr{N}_{k/2}(\gamma)$ of γ is contained in X; we claim it is embedded. To see this, for $z \in \gamma$, let $U(z)$ be the intersection of the segment perpendicular to γ through z with $\mathscr{N}_{k/2}(\gamma)$. Then $U(z)$ is embedded in X; we call it the transversal to $\mathscr{N}_{k/2}(\gamma)$ through z. If $\mathscr{N}_{k/2}(\gamma)$ is not embedded, there exist $z, z' \in \gamma$ with $d(z, z') > k$ for which $U(z) \cap U(z') \neq \emptyset$. Removing the segment from z to z' and adding in short segments in $U(z)$ and $U(z')$, we can shorten γ by an amount $d(z, z') - k > 0$, contrary to hypothesis. Since $\mathscr{N}_{k/2}(\gamma)$ has area $2l(\gamma)\sinh k/2$, and since the area of X is bounded above in terms only of the genus of S, we obtain an estimate $l(\gamma) = O(1/k)$.

If X has two or more boundary components (each of which necessarily has length $2k$), by taking the shortest paths from x to each of these components we obtain nonhomotopic loops in X based at x whose lengths are bounded as required.

Otherwise, let α be the shortest nonseparating closed loop in X'. We claim that

$\mathcal{N}_{k/2}(\alpha)$ is embedded in S. If not, we can find two intersecting transversals $U(z)$, $U(z')$ to α and, hence, create two disjoint loops, each of which is strictly shorter than α. At least one of these must be homotopically nontrivial and nonseparating, contradicting the definition of α. Cutting X along α, we obtain a surface with at least two boundary components, each being of length at most $O(1/k)$. The two loops in $X-\alpha$ based at x and homotopic to these boundary components are nonhomotopic. As before, the bound on diameter shows that they are of length $O(1/k)$, as required. □

We also need an estimate of lengths in a 3-dimensional Margulis tube.

LEMMA 6.7
Suppose that γ_* is a geodesic of length η in a hyperbolic 3-manifold M. Let x be a point at distance h from γ_*, and let $l(h)$ be the length of the geodesic through x homotopic to γ. Then $\sinh(l(h)/2) \geq \sinh(\eta/2)\cosh h$.

Proof
This is a routine computation that can conveniently be done in the upper half-space model of $\mathbb{H}^3$ by lifting the axis of γ_* to the vertical axis $V = \{(0,0,t) : t > 0\}$ and lifting x to the point A in the plane $\{(s,0,t) : s \in \mathbb{R}, t > 0\}$ whose perpendicular projection onto V is $(0,0,1)$. Then $A = (\cos\phi, 0, \sin\phi)$, where $\cot\phi = \sinh h$. Suppose that γ_* is represented by the loxodromic g which translates by complex distance $\eta + i\psi$ along V. Then $g(A)$ has coordinates $(e^\eta\cos\phi\cos\psi, e^\eta\cos\phi\sin\psi, e^\eta\sin\phi)$. Using the formula

$$\cosh d_{\mathbb{H}^3}(A,B) = 1 + \frac{|A-B|^2}{2\Im A\Im B},$$

where $|\cdot|$ denotes Euclidean distance in $\mathbb{R}^3$, we find

$$\cosh l(h) = 1 + 2\sinh^2\frac{\eta}{2}\cosh^2 h + (1-\cos\psi)\sinh^2 h.$$

Thus

$$\sinh^2\frac{l(h)}{2} \geq \sinh^2\frac{\eta}{2}\cosh^2 h,$$

which gives the result. □

COROLLARY 6.8
There exists a constant R_0 with the following property. Suppose that M is a hyperbolic 3-manifold and that $R > R_0$. Then there exists $\eta = \eta(R) > 0$ such that if γ_* is a geodesic in M with $l_{\gamma_*} < \eta$, and if x is on a loop homotopic to γ_* and of length at most $(1/8)e^{-R}$, then x is inside the Margulis tube T^* of γ_*, and $d(x,\partial T^*) \geq R$.

Proof

By [3, Theorem 1], there exists η_0 such that in any discrete group, the radius $r = r(\eta)$ of the Margulis tube about a geodesic of length $\eta < \eta_0$ satisfies $\cosh^2 r \geq (4\sinh^2(\eta/2))^{-1} - 1/2$. Thus $\cosh r \sinh(\eta/2) \geq 1/2$.

By Lemma 6.7, a loop homotopic to γ through a point x in T^* at distance at most R from ∂T^* must have a length of at least $l(r - R)$, where $\sinh(l(r-R)/2) \geq \sinh(\eta/2)\cosh(r - R)$. Combining this with the previous inequality, we get $\sinh(l(r-R)/2) \geq (1/2)\cosh(r-R)/\cosh r > (1/4)e^{-R}$. For $l(r-R) < l_0$, say, we obtain $l(r-R) > (1/8)e^{-R}$. Thus, if there is a loop through x homotopic to γ_* of length at most $(1/8)e^{-R}$, we must have $d(x, \partial T^*) \geq R$. □

PROPOSITION 6.9

Suppose that $\mu, \nu \in \mathscr{ML}$ and that $G(\theta) = G(\theta\mu, \theta\nu) \in \mathscr{QF}(S)$, and let $p^+(\theta)$ denote the Fuchsian structure on $\partial\mathscr{C}^+/G(\theta)$. Then there exist $\rho_* > 0$ and $\theta_3 > 0$, depending only on $C(\mu, \nu, \Gamma)$, such that $l_\gamma(p^+(\theta)) > \rho_*$ for all $\gamma \in \mathscr{S}$ whenever $\theta < \theta_3$.

Proof

Independently of the hyperbolic structure on S, we may use the Margulis lemma to choose $\rho > 0$ such that the ρ-thin parts of S where the injectivity radius is less than ρ are annular neighbourhoods of short curves or horospherical neighbourhoods of cusps and such that if $\delta_1, \delta_2 \in \mathscr{S}$ and $i(\delta_1, \delta_2) > 0$, then $l_{\delta_i} > \rho$ for at least one i. Further reducing ρ if necessary, we may also assume that $\rho < \epsilon_2$, chosen as in Proposition 6.1. As usual, for $\delta \in \mathscr{S}$, we write $l_{\delta^\pm}$ for $l_\delta(p^\pm(\theta))$ and l_{δ^*} for the length of the geodesic homotopic to δ in the manifold $\mathbb{H}^3/G(\theta)$.

Suppose that $\omega \in \mathscr{S}$ is such that $l_{\omega^+} < \rho$ for some θ. Since the curves in Γ fill up S, we must have $i(\omega, \delta) > 0$ for some $\delta = \delta(\theta) \in \Gamma$. By the choice made in the preceding paragraph, we have $l_{\delta^+} > \rho$. Applying Proposition 5.1 to δ and $\partial\mathscr{C}^+(\theta)$, we see that $d(P, \tilde{\delta}^*) \leq c_0(\rho) i(\delta, \mu)\theta$ for all $P \in \tilde{\delta}^+$ and, moreover, that $l_{\delta^*} \geq \rho/2 > 0$ whenever $i(\delta, \mu)\theta < \theta_0$ for some $\theta_0 = \theta_0(\rho) > 0$ that depends only on ρ. Thus we can apply Proposition 5.1 again to δ and $\partial\mathscr{C}^-$ to show that $d(Q, \tilde{\delta}^*) \leq c_0(\rho/2) i(\delta, \nu)\theta$ for all $Q \in \tilde{\delta}^-$ and $i(\delta, \nu)\theta < \theta_0(\rho/2)$. Combining these results and observing that perpendicular projection from $\tilde{\delta}^\pm$ to $\tilde{\delta}^*$ is surjective, we deduce that $d(P, \tilde{\delta}^-) \leq c_0(\rho/2)(i(\delta, \mu) + i(\delta, \nu))\theta \leq c_0(\rho/2)C\theta$ for all $P \in \tilde{\delta}^+$, provided that $\theta < \theta_0(\rho/2)/C$, where $C = C(\mu, \nu, \Gamma)$.

On the other hand, since μ, ν fill up S, by Lemma 6.5, we have $i(\omega, \mu) + i(\omega, \nu) > c_3$, so that either $i(\omega, \mu) > c_3/2$ or $i(\omega, \nu) > c_3/2$. Assume that the first case holds. Let σ be a segment along the lift of the axis of $\tilde{\omega}^+$ of length ρ. Then σ contains (roughly) ρ/l_{ω^+} periods of ω, so that $i(\sigma, \mu) > \rho i(\omega, \mu)/l_{\omega^+}$.

By Proposition 6.1, since we also arranged $\rho < \epsilon_2$, for any point $P \in \sigma$, we have $d(P, \partial\mathscr{C}^-) \geq c_2 i(\sigma, \mu)\theta$. Comparison with the previous estimate, applied to the intersection point P_0 of $\tilde{\omega}^+$ and $\tilde{\delta}^+$ on $\partial\mathscr{C}^+$, gives $l_{\omega^+} \geq c_2 c_3 \rho/(2Cc_0(\rho/2))$ whenever $\theta < \theta_0(\rho/2)/C$.

If $i(\omega, \mu) < c_3/2$, then $i(\omega, \nu) > c_3/2$. Moreover, either $l_{\omega^-} \geq \rho$ or, repeating the above arguments, $l_{\omega^-} \geq c_2 c_3 \rho/(2Cc_0(\rho/2))$ whenever $\theta < \theta_0(\rho/2)/C$. In other words, for small enough θ, there exists $k > 0$ such that $l_{\omega^-} > k$. We want to transfer this inequality to $\partial\mathscr{C}^+/G$. We are assuming that $l_{\omega^+} < \rho$; so, as in the second paragraph of this proof, if P_0 is the intersection point $\tilde{\omega}^+ \cap \tilde{\delta}^+ \in \partial\mathscr{C}^+$, there exists a point $Q \in \partial\mathscr{C}^-$ with $d(Q, P_0) < C(c_0(\rho/2)\theta$ whenever $\theta < \theta_0(\rho/2)/C$.

We now use a standard line of argument with Margulis tubes (see, e.g., [19, pp. 610–611]. Take k_0, A, as in Lemma 6.6. We are assuming $l_{\omega^-} > k$; by reducing k, if necessary, we can assume that $k < k_0$. Set $R = 2A/k$. By Corollary 6.8, we can choose $\eta = \eta(k) > 0$ such that if $l_{\omega^*} < \eta$ and if T^* is the Margulis tube around ω^* in $\mathbb{H}^3/G$, and if $z \in \mathbb{H}^3/G$ lies on a loop homotopic to ω^* and of length at most $e^{-R}/8$, then $z \in T^*$ and $d(z, \partial T^*) > R$. Let x denote the projection of Q to $\mathbb{H}^3/G$. We show that if $l_{\omega^+} < \min(\eta, e^{-R}/16)$, there is a loop based at x and homotopic to ω^* that is of length at most $e^{-R}/8$.

Let β denote the projection of the arc joining Q to P_0, so that β joins x to the projection y of P_0. The loop that follows β from x to y, then follows the loop ω^+, and finally follows back along β from y to x, has length at most $4Cc_0(\rho/2)\theta + l_{\omega^+}$. For $\theta < e^{-R}/(64Cc_0(\rho/2))$, under our assumption that $l_{\omega^+} < e^{-R}/16$, this is bounded above by $e^{-R}/8$. We conclude that $x \in T^*$ and that $d(x, \partial T^*) > R$.

Still keeping our assumption on l_{ω^+}, we claim that x is in the $(k/2)$-thick part of $\partial\mathscr{C}^-/G$. Indeed, suppose that γ^- is any primitive essential loop on $\partial\mathscr{C}^-/G$ based at x and of length bounded above by k. Since we can certainly assume that $k < R$, and since $x \in T^*$ with $d(x, \partial T^*) > R$, the curve γ^- is contained in T^* and, hence, must be homotopic to ω^*. Thus γ^- must be homotopic to ω^-, which is impossible since $l_{\omega^-} > k$.

From Lemma 6.6, we conclude that there are two primitive essential nonhomotopic loops on $\partial\mathscr{C}^-/G$ based at x and of length bounded above by $2A/k$. Since $d(x, \partial T^*) > R = 2A/k$, both loops are contained in T^*, which is impossible.

Thus we conclude that $l_{\omega^+} \geq \min(\eta, e^{-R}/16)$. We have shown that there exist constants θ_3, ρ_* depending only on $C(\mu, \nu, \Gamma)$ such that $l_{\omega^+} \geq \rho_*$ whenever $\theta < \theta_3$. For later purposes, we remark that θ_3 and ρ_* depend only on the upper bound for $C(\mu, \nu, \Gamma)$. □

The following corollary will be useful. (This can easily be strengthened to an assertion about the maximum distance between $\partial\mathscr{C}^+$ and $\partial\mathscr{C}^-$, but we do not need this.)

COROLLARY 6.10

Suppose that $P \in \partial\mathscr{C}^+$ lies on one of the curves γ_i^+, $\gamma_i \in \Gamma$. Then there exist constants c_4, θ_4 such that $d(P, \partial\mathscr{C}^-) < c_4 C(\mu, \nu, \Gamma)\theta$ whenever $\theta < \theta_4/C$.

Proof

Suppose that $P \in \partial\mathscr{C}^+$, and write $\gamma = \gamma_i$. Choose ρ_* as in Proposition 6.9. Since $l_\gamma > \rho_*$, by Proposition 5.1, and for sufficiently small θ, we have $d(P, Q) < i(\mu, \gamma)O(\theta)$ for some $Q \in \tilde{\gamma}^*$, and $l_{\gamma^*} > \rho_*/2$. Then $l_{\gamma^-} > \rho_*/2$. Noting that perpendicular projection from $\tilde{\gamma}^-$ to $\tilde{\gamma}^*$ is surjective, we can apply Proposition 5.1 again to $\partial\mathscr{C}^-$ to deduce that there is a point $P' \in \tilde{\gamma}^-$, so that $d(P', Q) < i(\nu, \gamma)O(\theta)$. Combining these results gives $d(P, \partial\mathscr{C}^-) < C(\mu, \nu, \Gamma)O(\theta)$. Moreover, it is easy to see that all the required inequalities hold, provided that $\theta < \theta_4/C$ for some suitably chosen θ_4, as claimed. □

We now need to get upper bounds on the lengths $l_{\mu^\pm}$ and $l_{\nu^\pm}$. To do this, it is convenient to organise things so that the laminations μ and ν are confined to narrow paths on the surface. As sketched by Thurston [27, Section 8.9, p. 204; p. 8.52 in original version], given a geodesic lamination $\mu \in \mathscr{ML}$, one can find $\epsilon > 0$ and a train track τ, such that the ϵ-neighbourhood $N_\epsilon(\tau)$ is just the product of an open interval with τ (i.e., a "strip with switches") and such that $|\mu| \subset N_\epsilon(\tau)$. It is important for us to understand the dependence of ϵ on the geometry of S and μ, so we state this precisely in the following proposition.

PROPOSITION 6.11

Suppose that $\mu \in \mathscr{ML}(S)$ and that S_0 is a hyperbolic structure on the surface S such that the noncuspidal injectivity radius $\operatorname{inj}(S_0^C) > \rho$. Then there exist $\epsilon > 0$, depending only on ρ and the topology of S, and a train track τ, such that $N_\epsilon(\tau)$ is homeomorphic to τ and such that every leaf of μ is contained in $N_\epsilon(\tau)$. There is a fixed upper bound on the number of switches and branches of τ.

Proof

The proof is explained in detail in [23, Theorem 1.6.5]; however, the dependence on $\operatorname{inj}(S_0^C)$ is not spelled out. The idea is that the complement of $|\mu|$ in S consists of a finite number of ideal polygons (possibly with punctures). The area of the ϵ-neighbourhood U_ϵ of the boundary of any such polygon tends to zero with ϵ.

On the other hand, the bound on injectivity radius means that any disk D_ρ of radius ρ and contained in S_0^C is embedded. Since such a disk has definite area, it cannot be contained in U_ϵ as $\epsilon \to 0$. Thus D_ρ must intersect U_ϵ in thin tubular neighbourhoods of possibly branched 1-manifolds.† Now use the fact that we may choose the neighbourhoods of the cusps such that every simple geodesic, in particular every leaf of $|\mu|$, is entirely contained in S_0^C. Clearly, the bound on ϵ depends only on ρ and not on μ. The bounds on the number of branches and switches come from the obvious bounds on the number of sides and cusps of the complementary regions to μ, which are again independent of μ. □

We call a train track τ chosen as in Proposition 6.11 an *ϵ-thin train track*, and we say that μ is *carried* by τ. It is also part of the above construction that $N_\epsilon(\tau)$ is foliated by arcs of length at most 2ϵ transverse (and approximately orthogonal) to the branches. If b is a branch of τ, we write $N_\epsilon(b)$ for the union of the leaves transverse to b. Each of these arcs is a transversal to μ and carries the same transverse weight, which we denote $\mu(b)$. We emphasise that the topology of τ and the weights of each of its branches may well change as S varies.

Now as in Section 5, we are seeking an upper bound on the length of μ. For fixed ϵ, the branch $N_\epsilon(b)$ has a definite width, and thus its length is uniformly bounded above. Thus the only way in which the lamination μ can get very long is to have large weight concentrated in thin strips, in other words, for the weights $\mu(b)$ to get large. We rule out this possibility by once again playing off the upper and lower bounds on the distance between points on $\partial\mathscr{C}^+$ and $\partial\mathscr{C}^-$.

Remark 6.12

Of course, all the bounds under discussion depend on the choice of μ and ν. From now on, we understand "uniformly bounded as $\theta \to 0$" to mean that there are uniform bounds depending only on $C(\mu, \nu, \Gamma)$ which are valid provided that $\theta < \theta_5/C$ for some constant $\theta_5 > 0$ independent of C. Expressions such as $f(\theta) \leq O(\theta)$ are interpreted in a similar way. Inspection of the proofs shows that the bounds actually depend only on an upper bound for C.

PROPOSITION 6.13

Choose ρ_* as in Proposition 6.9, and fix ϵ_* depending on ρ_* as in Proposition 6.11, so that $2\epsilon_* < \epsilon_2$ as in Proposition 6.1. Suppose that $\mu \in \mathscr{ML}(S)$ is carried on some ϵ_*-thin train track τ on the surface $\partial\mathscr{C}^+/G(\theta)$. Then the weight $\mu(b)$ of a branch of τ is uniformly bounded above as $\theta \to 0$.

Proof

As usual, let Γ be our fixed set of curves which fill up S. First assume that $N_{\epsilon_*}(b) \cap$

†Figure 1.6.3 in [23] is somewhat deceptive since examination of the constants shows that for this to work, one must take $\epsilon = O(\rho^2)$ so that typically D_ρ will contain $O(1/\sqrt{\epsilon})$ such strips, not one as in the picture.

$\Gamma \neq \emptyset$. For each θ, pick $P = P(\theta) \in N_{\epsilon_*}(b) \cap \tilde{\gamma}_i^+$ for some i. By Corollary 6.10, we have $d(P, \partial\mathscr{C}^-) \leq O(\theta)$ with uniform constant (depending on $C(\mu, \nu, \Gamma)$) independent of θ. On the other hand, since the transverse measure of an arc of length $2\epsilon_*$ containing P is $\mu(b)$, by Proposition 6.1 we have $d(P, \partial\mathscr{C}^-) > \mu(b)O(\theta)$, also with a uniform constant. Comparing these two inequalities gives a uniform upper bound on $\mu(b)$.

Now assume that $N_{\epsilon_*}(b) \cap \Gamma = \emptyset$. In this case, $N_{\epsilon_*}(b)$ is contained in a component R of $S - \Gamma$ which topologically is either a disk or punctured disk. The boundary ∂R consists of a bounded number of finite arcs α_j, each contained in γ_i for some i. We need to guard against the possibility that leaves of μ wrap around many times inside R, assigning unduly large weight to $N_{\epsilon_*}(b)$.

Suppose first that R is simply connected. Let λ be a connected component of the intersection of some leaf of $|\mu|$ with R. Since ϵ_* is certainly less than the noncuspidal injectivity radius, it is easy to see that λ cannot return within distance ϵ_* of itself. From this we deduce that λ has finite length and that it intersects any transversal T to $N_{\epsilon_*}(b)$ at most once. It follows that there is a well-defined map from $T \cap |\mu|$ to ∂R which sends the point $x \in T$ to the point at which the component of $|\mu| \cap R$ through x meets ∂R. The map is injective when lifted to the closure of $\tilde{R}$ in $\mathbb{H}^2$. Pushing forward the measure of T to $\partial\tilde{R}$ and using the injectivity, we find $\mu(b) \leq \sum_j i(\mu, \alpha_j)$. Since $\alpha_i \subset \gamma_i$, and since a segment of γ_i can appear at most twice on $\partial\tilde{R}$, we have $\mu(b) \leq 2\sum_i i(\mu, \gamma_i) = 2||\mu||$.

Now suppose that R contains a puncture P. We claim that there is no circuit in τ encircling P; for otherwise, by following the boundary leaf of $|\mu|$ nearest to P around this circuit, we would have a leaf of $|\mu|$ parallel to a loop around P, which is impossible. Thus we can find an arc α joining P to ∂R with $\operatorname{Int}\alpha \subset R$ and $\alpha \cap |\mu| = \emptyset$. Now repeat the above argument working in $R - \alpha$. □

COROLLARY 6.14

Suppose that $\mu, \nu \in \mathscr{ML}$ and that $G(\theta) = G(\theta\mu, \theta\nu) \in \mathscr{QF}(S)$. Then there is a uniform upper bound on the lengths $l_{\mu^+} = l_{\mu^+}(p^+(\theta))$ as $\theta \to 0$.

Proof

Fix ϵ_* as in Proposition 6.11, and find an ϵ_*-thin train track τ on the surface $p^+(\theta)$ carrying μ. The length of μ on $p^+(\theta)$ is clearly estimated from above by $\sum_i \mu(b_i) l_{b_i}$, where $l_{b_i} = l_{b_i}(p^+(\theta))$ is the length of the branch b_i and $\mu(b_i) = \mu(b_i)(p^+(\theta))$ is its weight. Now the strip of width $2\epsilon_*$ about b_i is embedded on S, so since ϵ_* is fixed independent of θ, the length l_{b_i} is uniformly bounded above by an area estimate. The result follows. □

Proof of Proposition 1.8 for general μ, ν

By Corollary 6.14, the lengths $l_{\mu^+}(\theta)$ and $l_{\nu^-}(\theta)$ are uniformly bounded above. We claim that this implies that the hyperbolic structures $p^{\pm}(\theta)$ of $\partial\mathscr{C}^{\pm}/G(\theta)$ both converge in $\mathscr{F}$ and that their limits coincide.

If the structures $p^{+}(\theta)$ are not in a compact set in $\mathscr{F}$ as $\theta \to 0$, then we can find a sequence $p^{+}(\theta_n)$ whose limit is some projective lamination $[\xi] \in \mathrm{P}\mathscr{ML}$ as $\theta_n \to 0$. This means that there exists a lamination $\xi \in [\xi]$ and a sequence $h_n \to 0$ such that $h_n l_{\gamma^+}(\theta_n) \to i(\gamma, \xi)$ for all $\gamma \in \mathscr{S}$. Now as in the proof for rational μ, ν at the end of Section 5, it follows from Propositions 5.1 and 6.9 that $l_{\gamma^-}(\theta)/l_{\gamma^+}(\theta) \to 1$ as $\theta \to 0$. Thus $h_n l_{\gamma^-}(\theta_n) = h_n l_{\gamma^+} \cdot l_{\gamma^-}/l_{\gamma^+} \to i(\gamma, \xi)$, and so $p^{-}(\theta_n)$ converges to $[\xi] \in \mathrm{P}\mathscr{ML}$ also.

From the definition of convergence to $\mathrm{P}\mathscr{ML}$, the fact that $l_{\mu^+}(\theta_n)$ remains bounded implies that $i(\mu, \xi) = 0$; likewise, $i(\nu, \xi) = 0$. However, since μ, ν fill up S, this is impossible. We conclude that the structures $p^{+}(\theta)$ lie in a compact set in $\mathscr{F}$. Then we can pick a sequence $\theta_n \to 0$ such that $p^{+}(\theta_n) \to p^{+}_{\infty}$ and $p^{-}(\theta_n) \to p^{-}_{\infty}$ with $p^{\pm}_{\infty} \in \mathscr{F}$. The same length comparison $l_{\gamma^-}/l_{\gamma^+} \to 1$ shows that $p^{+}_{\infty} = p^{-}_{\infty}$. By Proposition 3.1, this limit is independent of the subsequence, so our claim follows.

We can now use the lower bound on lengths from Proposition 6.9 to conclude the proof exactly as we did in the rational case. □

Proof of Theorems 1.2 and 1.4 in the general case

Establishing Proposition 6.9 completes the proof of Theorems 1.2 and 1.4 in the general case. □

7. Diagonal limits

In this final section we discuss the question of diagonal limits. Here is a simple example which shows that care is needed. Let $\{\alpha_1, \dots, \alpha_k\}$ and $\{\beta_1, \dots, \beta_l\}$ be two systems of disjoint curves which fill up S but are such that $\{\alpha_2, \dots, \alpha_k, \beta_1, \dots, \beta_l\}$ do not. Fix coefficients $a_2, \dots, a_k, b_1, \dots, b_l > 0$, and choose a sequence $h_n \to 0$. Define $\mu_n = h_n\alpha_1 + \sum_{i=2}^{k} a_i\alpha_i$ and $\nu_n \equiv \nu = \sum_{i=1}^{l} b_i\beta_i$. Obviously, $\mu_n \to \mu_\infty = \sum_{i=2}^{k} a_i\alpha_i$ in $\mathscr{ML}(S)$.

Now consider the sequence of groups $G_n = G(\theta_n\mu_n, \theta_n\nu)$, where $\theta_n \to 0$ as $n \to \infty$. Since μ_∞, ν do not fill up S, the length function $l_{\mu_\infty} + l_\nu$ does not have a minimum on $\mathscr{F}(S)$ (see [14, p. 194]), and so the sequence G_n cannot have a Fuchsian limit. In fact, because the ratio between the weights on α_1 and each of the curves in $|\nu|$ tends to zero, the constants in the estimate for the lower bound on the distance between a bending line and the opposite side of $\partial\mathscr{C}$ in Proposition 5.8 become arbitrarily small. Thus the length bound in Proposition 5.10 fails; in other words, $l_{\alpha_1} \to \infty$, and the groups G_n diverge. (For the limiting behaviour along lines of minima, see [6].)

The final important step in our proof of Theorem 1.2 was establishing the length bounds on l_{μ^+} and l_{ν^-}, from which we deduced that the corresponding surfaces lay in a compact set in $\mathscr{F}(S)$. However, our example shows that to prove Theorem 1.6, it is no use just establishing uniform upper bounds on the lengths $l_{\mu_n}(\theta_n)$ and $l_{\nu_n}(\theta_n)$. In fact, it is easy to adjust the coefficients h_n so as to produce a sequence $p_n \in \mathscr{F}$ such that the lengths $l_{\mu_n}(p_n)$ and $l_\nu(p_n)$ are uniformly bounded above but also such that $l_{\alpha_1}(p_n) \to \infty$, so that the sequence p_n exits every compact set in $\mathscr{F}(S)$. To resolve this, we need to do more work to get a uniform upper bound on the lengths of the curves Γ. We shall invoke the convergence of the laminations through the following improved version of Lemma 6.5, which allows us to avoid the technical difficulty that it is not clear how to control the behaviour of transversals to μ as we transfer from surface to surface—a priori a narrow strip containing heavy weight on one structure might become extremely wide on another. We keep the same name for the constant c_3, although the actual value may have changed.

LEMMA 7.1
Let μ and ν fill up S, and suppose that $\mu_n \to \mu, \nu_n \to \nu$ in $\mathscr{ML}$. Then there exists $c_3 > 0$ such that $i(\gamma, \mu_n) + i(\gamma, \nu_n) > c_3$ for all $\gamma \in \mathscr{S}$ and all n.

Proof
The proof is almost the same as the previous version. Once again we can work entirely with a fixed hyperbolic structure $p_0 \in \mathscr{F}$ whose noncuspidal injectivity radius is the fixed value $\rho_0 > 0$. If the result is false, then we can find a sequence $\gamma_n \in \mathscr{S}$ such that $i(\gamma_n, \mu_n) + i(\gamma_n, \nu_n) \to 0$. As before, passing to a further subsequence, we find $h_n > 0$ such that $h_n\delta_{\gamma_n} \to \xi$ in $\mathscr{ML}$ and $h_n \le 2/\rho_0$. Then $i(h_n\gamma_n, \mu_n) + i(h_n\gamma_n, \nu_n) \to 0$, but also $i(h_n\gamma_n, \mu_n) + i(h_n\gamma_n, \nu_n) \to i(\xi, \mu) + i(\xi, \nu)$. Since μ and ν fill up S, this is impossible. □

Proof of Theorem 1.6
We have to study the behaviour of the groups $G_n = G(\theta_n\mu_n, \theta_n\nu_n)$ as $\mu_n \to \mu$, $\nu_n \to \nu$, and $\theta_n \to 0$. Observe that the constants involved in the proof of Theorem 1.2 depended only on the topology of S and the intersection numbers $i(\mu, \gamma)$ and $i(\nu, \gamma)$ for γ in the fixed finite set Γ. In particular, inspection of the proof of Proposition 6.9 shows that the lower bound ρ_* on the noncuspidal injectivity radius depends on the universal Margulis constant, upper bounds for the norm $C(\mu, \nu, \Gamma) = ||\mu||_\Gamma + ||\nu||_\Gamma$, and the constant c_3 of Lemma 6.5. If $\mu_n, \nu_n \to \mu, \nu$ in $\mathscr{ML}$, then $i(\mu_n, \delta) \to i(\mu, \delta)$ and $i(\nu_n, \delta) \to i(\nu, \delta)$ for all $\delta \in \pi_1(S)$. Since Γ is a fixed finite set, and since we have just shown that the constant c_3 is independent of n, we deduce that all bounds in question are uniform as $n \to \infty$. In particular, the lower bound ρ_* can be

chosen uniform for all the structures $p^{\pm}(\theta_n)$ on the surfaces $\partial\mathscr{C}^{\pm}/G_n$. This justifies the continued use of the terminology "uniformly bounded as $\theta \to 0$" in the same sense as in Remark 6.12.

From the results of [14], we know that $M(\mu_n, \nu_n) \to M(\mu, \nu)$. Thus to prove diagonal convergence, we just need a uniform estimate on the convergence of G_n to $M(\mu_n, \nu_n)$. From Proposition 5.1 and the uniform lower bound ρ_*, we get

$$\left|\frac{l_\delta(p^+(\theta_n))}{l_\delta(p^-(\theta_n))}\right| < 1 + O(\theta_n^2)$$

for any curve $\delta \in \pi_1(S)$, with uniform constants depending only on δ,μ and ν.

We show in Proposition 7.2 that the lengths of any fixed curve $\delta \in \mathscr{S}$ on either surface $p^{\pm}(\theta_n)$ have a uniform upper bound as $n \to \infty$. Thus we obtain

$$\left|l_\delta\big(p^+(\theta_n)\big) - l_\delta\big(p^-(\theta_n)\big)\right| < O(\theta_n^2)$$

with constants depending on δ. Taking a large enough finite set of curves δ to determine the analytic structure on $\mathscr{F}$, it follows that both structures $p^{\pm}(\theta_n)$ lie in a compact subset of $\mathscr{F}$.

Now we can use exactly the same argument via Sullivan's theorem, as in the proof of Proposition 1.8 in Section 5, to see that the groups G_n are contained in a compact set in $\mathscr{QF}$. Exactly as in that proof, we deduce from equations (1) and (2) in Proposition 3.1 that, for any finite set of curves δ_i, the complex lengths λ_{δ_i} evaluated at the points $p^{\pm}(\theta_n)$ and at G_n are uniformly close. Thus G_n converges uniformly to both $p^{\pm}(\theta_n)$ as $\theta \to 0$, and the result follows.

In view of the above, the main work remaining in proving Theorem 1.6 is establishing the following.

PROPOSITION 7.2
Suppose that μ, ν fill up S. Let $\gamma \in \mathscr{S}$ be fixed, and let $\mu_n, \nu_n \to \mu, \nu$ in $\mathscr{ML}$ be such that μ_n, ν_n fill up S for all n. Suppose that $\theta_n \to 0$. Then the lengths $l_\gamma(p^{\pm}(\theta_n))$ are uniformly bounded above as $n \to \infty$ (with a bound depending only on γ, μ, ν, and the topological type of S).

As indicated by our counter example, convergence may fail if the curve γ only meets μ_n or ν_n branches of vanishingly small weight. In fact, a closed loop in μ_n of very small weight may itself become extremely long; this is avoided only by the hypothesis that the limit laminations μ and ν themselves fill up the surface. Another possibility is that γ might meet a loop of definite weight and bounded length, but by wrapping around it many times, it could itself become extremely long. Thus, to prove Proposition 7.2, we show first (Proposition 7.3) that every segment of γ of some definite length must meet some bending line of definite weight, and second (Proposition 7.5) that it must meet the relevant bending line sufficiently transversally to make a definite contribution to the positive number $i(\mu_n, \gamma) + i(\nu_n, \gamma)$.

PROPOSITION 7.3

Let μ_n, ν_n, μ, ν be as above. Fix ϵ_* as in Proposition 6.11, and for each n, suppose that $\tau(\mu_n)$, $\tau(\nu_n)$ are ϵ_*-thin train tracks on $p^+(\theta_n)$ carrying μ_n and ν_n, respectively. Then there exist uniform constants $L_1, k_1 > 0$ such that if σ is any geodesic segment on $p^+(\theta_n)$ contained in a complete simple geodesic and of length at least L_1, then there is a point $P \in \sigma$ such that $P \in |\mu_n^+| \cup |\nu_n^+|$ and such that P is contained in a $\tau(\mu_n)$ or $\tau(\nu_n)$ branch of transverse weight at least k_1.

Proof

In the statement, μ_n, ν_n of course refer to the representatives of these laminations on $p^+(\theta_n)$. As already observed, the number of branches of $\tau(\mu_n)$ and $\tau(\nu_n)$ has a uniform upper bound depending only on the topology of S. Moreover, it is easy to see that any component of $N_\epsilon(\tau(\mu_n)) \cap N_\epsilon(\tau(\nu_n))$ contains a ball of radius at least ϵ, so that by an area argument, the total number of intersection points of $\tau(\mu_n)$ and $\tau(\nu_n)$ has an upper bound independent of n. It follows that there is also a uniform upper bound to the number of sides of each complementary region of $\tau(\mu_n) \cup \tau(\nu_n)$. As in the proof of Corollary 6.14, by area considerations there is a uniform upper bound l_0 to the length of any branch of $\tau(\mu_n)$ or $\tau(\nu_n)$. Now each complementary region is either simply connected or a once-punctured disk. Thus every simple arc crossing a complementary region is homotopic to a path along the boundary but not fully encircling the boundary. (No simple geodesic can completely encircle the puncture.) This gives a uniform upper bound l_1 to the length of the intersection of a simple geodesic with each complementary region; we may as well assume that $l_1 > l_0$.

Now let $\sigma : [0, T] \to \partial\mathscr{C}^+(\theta_n)$ be a geodesic segment parameterised for convenience by arc length. We associate a crude symbol sequence to σ as follows. First, after removing a transverse arc through each switch of $\tau(\mu)$, the open neighbourhood $N_\epsilon(\tau(\mu_n))$ is disconnected into a finite number of open sets V_i, one for each branch of μ. Disconnect $N_\epsilon(\tau(\nu_n))$ into sets W_j a similar way, and let $\mathscr{B}$ denote the set of all components of the resulting dissection of $N_\epsilon(\tau(\mu_n)) \cup N_\epsilon(\tau(\nu_n))$. Thus a set in $\mathscr{B}$ is a component of either $V_i \setminus N_\epsilon(\tau(\nu_n))$, $W_j \setminus N_\epsilon(\tau(\mu_n))$, or $V_i \cap W_j$. As we have seen, the size of $\mathscr{B}$ is uniformly bounded above by some $M \in \mathbb{N}$. The points at which σ meets ∂Y for any $Y \in \mathscr{B}$ give a partition of σ at the points $0 = t_0 < t_1 < \cdots < t_m = T$. Thus each open arc $\sigma(t_i, t_{i+1})$ is contained in either a component $Y \in \mathscr{B}$ or a complementary region of $\tau_{\mu_n} \cup \tau_{\nu_n}$. We associate to σ the sequence $e_1 \cdots e_m$, where $e_i = Y$ if $\sigma(t_{i-1}, t_i) \subset Y$ and $e_i = X$ if $\sigma(t_i, t_{i+1})$ is contained in a complementary region.

Observe that from the definition, no symbol is immediately followed by itself. Also note that $T = l_\sigma \le m l_1$.

Now consider any simple geodesic segment σ with length $l_\sigma \ge (2M+1)l_1$. Suppose that its symbol sequence is $e_1 \cdots e_m$. Then $(2M+1)l_1 \le l_\sigma \le m l_1$, so that $M + 1 \ge [m/2]$. Since the symbol X never follows itself, at least $[m/2]$ symbols from $e_1 \cdots e_m$ belong to $\mathscr{B}$; and, hence, some symbol $b \in \mathscr{B}$ occurs twice. That is, some subarc $\sigma' \subset \sigma$ runs from the component Y to itself. Assume that σ' is a minimal segment of this type, in the sense that the length of its symbol sequence is least possible, so that, in particular, this length is at most $2M + 1$. Let σ_Y be the geodesic arc joining the first point at which σ' leaves ∂Y to the next point at which it reenters it. Since Y is geodesically convex, $\sigma_Y \subset Y$. Thus $\sigma'' = \sigma' \cup \sigma_Y$ is a loop of length at most $(2M+2)l_1$.

By Lemma 7.1, we have $i(\sigma'', \mu_n) + i(\sigma'', \nu_n) > c_3$. It is clear that $i(\sigma'', \mu_n) = i(\sigma', \mu_n) + i(\sigma_Y, \mu_n)$ (and similarly for ν_n). Thus either $i(\sigma_Y, \mu_n) + i(\sigma_Y, \nu_n) > c_3/2$ or $i(\sigma', \mu_n) + i(\sigma', \nu_n) > c_3/2$. In the first case, we see that Y lies in a branch of either μ_n or ν_n of weight at least $c_3/4$. In the second case, since σ' contains at most $(2M+1)$ elements in its symbol sequence, we see that it must meet at least one branch $b' \in \mathscr{B}$ of μ_n- or ν_n-weight at least $c_3/2(2M+1)$. Thus, in all cases, σ' contains some point that lies in a μ_n- or ν_n-branch of weight at least $c_3/2(2M+1)$. Setting $L_1 = (2M+1)l_1$ and $k_1 = c_3/2(2M+1)$ gives the result. □

In the proof of the next proposition, we need to transfer the lamination ν from $\partial\mathscr{C}^+$ to $\partial\mathscr{C}^-$. The following lemma shows that we can do this without serious loss of control. For clarity, we denote the copies of a lamination ξ on $\partial\mathscr{C}^\pm$ by $\xi^\pm$, respectively.

LEMMA 7.4

Let λ^+ be a leaf of the lamination ν_n^+ lifted to the surface $\partial\mathscr{C}^+(\theta_n)$, and let λ^- be the corresponding leaf of ν_n^- on the surface $\partial\mathscr{C}^-(\theta_n)$, so that $\lambda^\pm$ have the same endpoints on $\partial\mathbb{H}^3$. Then for any point $P \in \lambda^+$, we have $d(P, \lambda^-) \le c_5\theta_n$, with a uniform constant c_5 as $n \to \infty$.

Proof

The idea, obviously, is to imitate the proof of Proposition 5.1. To do this, we need to see that there is a uniform upper bound to the μ_n^+-mass of any geodesic segment on $\partial\mathscr{C}^+(\theta_n)$ of definite length at most 1, say. In fact, if τ is an ϵ_*-thin train track carrying μ_n^+, by Proposition 6.13 there is a uniform upper bound to the weight of each μ_n^+-branch. Moreover, away from ϵ_*-balls around the switches, there is by the construction of τ a uniform lower bound to the distance between leaves of μ_n^+ contained in distinct branches. This gives the required bound.

Now pick equally spaced points $P_m, m \in \mathbb{Z}$ at unit distance apart along λ^+. The above discussion gives a uniform upper bound to $i(\sigma_m, \mu_n)$, where σ_m is the segment of λ^+ from P_m to P_{m+1}. Thus we may argue exactly as in the proof of Proposition 5.1 to show that all points on λ^+ are at most a uniform distance $O(\theta)$ away from the corresponding leaf λ^-. □

PROPOSITION 7.5
Let $\gamma \in \mathscr{S}$. Then there exist $L_0, C_0 > 0$, depending only on $i(\gamma, \mu), i(\gamma, \nu)$, and $i(\nu, \mu)$, such that if σ is any geodesic segment contained in γ^+ of length at least L_0 on any of the hyperbolic surfaces $p^{\pm}(\theta_n)$, then $i(\sigma, \mu_n) + i(\sigma, \nu_n) > C_0$ whenever $\theta_n < c/i(\mu, \gamma)$ for some constant $c > 0$ independent of γ, μ and ν.

Proof
If the result is false, then we can find a structure $p^+(\theta_n)$, say, on which γ has an arbitrarily long segment σ for which $i(\sigma, \mu_n)+i(\sigma, \nu_n)$ is arbitrarily small. The argument will follow the same lines as that of Proposition 5.10.

Consider such a segment σ, where the choice of constants will be determined later, and let $\hat{\sigma}$ be the $\mathbb{H}^3$-geodesic joining its endpoints X and X'. Clearly, we may assume that $L_0 > 1$, say; then, as in the first part of the proof of Proposition 5.1, there is a universal constant c such that $d(P, \hat{\sigma}) < ci(\sigma, \mu_n)\theta$ for all $P \in \sigma$ and such that $l_{\hat{\sigma}} > (1 - ci(\sigma, \mu_n)^2\theta^2)l_\sigma > l_\sigma/2$, say, for all small enough θ. (The condition on θ is really that $i(\sigma, \mu_n)\theta$ be sufficiently small; since we are assuming $i(\sigma, \mu_n)$ is small, it is enough to assume that θ is bounded above by a suitable constant.) Choose L_1 as in Proposition 7.3. Also by Proposition 5.1, if $P \in \tilde{\gamma}^+$, then $d(P, \tilde{\gamma}^*) < ci(\gamma, \mu)\theta$ whenever $\theta < c'/i(\gamma, \mu)$ for some universal $c' > 0$. Thus, by Lemma 5.9, given $h > 0$, we can find $L_2 = L_2(h)$ such that, if $l_\sigma > L_2$, then $d(Q, \tilde{\gamma}^*) < h\theta$ for all points Q on a segment $\hat{\sigma}' \subset \hat{\sigma}$ of length at least L_1. Since perpendicular projection from σ to $\hat{\sigma}$ is surjective, we can find a subarc of σ_1 of σ-length at least L_1 for which $d(Q, \tilde{\gamma}^*) < (h + ci(\sigma, \mu_n))\theta$ for all $Q \in \sigma_1$.

Since perpendicular projection from $\tilde{\gamma}^-$ to $\tilde{\gamma}^*$ is surjective, we can find $Y \in \partial\mathscr{C}^-$ such that Y is within distance $O(\theta)$ of the footpoint of the perpendicular from X to $\tilde{\gamma}^*$ and, hence, such that $d(X, Y) < O(\theta)$. Similarly, choose $Y' \in \partial\mathscr{C}^-$ with $d(X', Y') < O(\theta)$. Now, arguing as before, we can find a point R in the $\partial\mathscr{C}^-$-arc from Y to Y' such that $d(Q, R) < \big(2h + c(i(\sigma, \mu_n) + i(\sigma, \nu_n))\big)\theta$.

By Proposition 7.3, the arc σ_1 contains a point P that lies within distance at most ϵ_* of a point in a branch of either μ_n^+ or ν_n^+ and of weight at least k_1. Suppose first that this is a μ_n^+-branch. Then by Proposition 6.1, there is a constant $c_2 > 0$ such that $d(P, \partial\mathscr{C}^-) > c_2k_1\theta$. Choose h as above with $2h < c_2k_1/2$, and then use our hypothesis to choose σ with $l_\sigma > L_2(h)$ and $i(\sigma, \mu) + i(\sigma, \nu) < c_2k_1/2$. Comparison with the estimate $d(Q, R) < \big(2h + c(i(\sigma, \mu_n) + i(\sigma, \nu_n))\big)\theta$ gives a contradiction.

If P is in a branch of ν_n^+ of weight at least k_1, we can clearly make a similar argument, provided that we can find a point $P' \in \partial\mathscr{C}^-$ near P and within distance ϵ_2 of a ν_n^--branch of definite weight. From Lemma 7.4 we see that we can find a transversal T' to ν_n^- of $\partial\mathscr{C}^-$-length at most $2\epsilon_* + 2c_5\theta$ and with $\nu_n^-(T') > k_1$. Provided θ is sufficiently small, we have the result. □

Proof of Proposition 7.2
From Proposition 7.5, we easily obtain the bound

$$l_{\gamma^+}\big(p^+(\theta_n)\big) \leq 2\big(i(\gamma, \mu) + i(\gamma, \nu)\big)\frac{L_0}{C_0},$$

which holds whenever $\theta < c/(i(\gamma, \mu) + i(\gamma, \nu))$ for some universal c. □

This completes the proof of Theorem 1.6. □

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