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On Kerckhoff Minima and Pleating Loci for Quasi-Fuchsian Groups

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Abstract. We show how Kerckhoff's results on minima of length functions on Teichmüller space can be used to analyse the possible bending loci of the boundary of the convex hull for quasi-Fuchsian groups near to the Fuchsian locus.

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1. Introduction

The geometry of a hyperbolic three manifold $\mathbb{H}^3/G$ is carried by its convex core, the smallest closed set containing all closed geodesics. If G is geometrically finite, the boundary $\partial\mathcal{C}/G$ of the convex core is a *pleated surface* homeomorphic to the Riemann surfaces Ω/G at infinity. In a series of papers [9–12], see also [23], the author and L. Keen investigated the geometry of $\partial\mathcal{C}/G$ and in particular its pleating locus which, equipped with the natural *bending measure*, is a measured geodesic lamination on each component of Ω/G. We restrict here to quasi-Fuchsian groups, so that Ω/G has two connected components $\Omega^{\pm}/G$, each homeomorphic to a fixed hyperbolic surface S, possibly with punctures. We denote the (measured) bending laminations on the two components of $\partial\mathcal{C}/G$ by $pl^{\pm}$ respectively. The object of this paper is to investigate the pair $([pl^+], [pl^-])$ for groups near the Fuchsian locus $\mathcal{F}$. (Here $[\mu]$ denotes the projective measure class of the measured lamination μ.) The key tool is the observation that a necessary condition for a closed geodesic axis to be in the support of $pl^+ \cup pl^-$ is that its trace be real. Since the trace is a holomophic function on quasi-Fuchsian space $\mathcal{QF}$, and is obviously real valued on $\mathcal{F}$, one would expect the possible laminations to be related to critical points of length functions on $\mathcal{F}$. There is one obvious constraint: it is not hard to show that the laminations $([pl^+], [pl^-])$ always fill up S. In [15], Kerckhoff showed that for given measured laminations μ, ν which fill up S, and $k \in (0, \infty)$, the function $l_\mu + kl_\nu$ is convex along earthquake paths and has a unique minimum on $\mathcal{F}$. As k varies, these minima are distinct and form the *line of minima* $\mathcal{L}_{\mu,\nu}$ for μ, ν. Let μ, ν be measured laminations on S. We denote by $\mathcal{P}_{\mu,\nu}$ the set of points in $\mathcal{QF}$ for which $[pl^+] = [\mu], [pl^-] = [\nu]$. We call this set the *pleating variety* for μ and ν.

In the special case of the once punctured torus, we proved in [11] and [12] that if μ, ν are measured laminations whose support fills up S, then the closure of $\mathcal{P}_{\mu,\nu}$ intersects $\mathcal{F}$ precisely along the line of minima for μ, ν. We conjecture that the same result is true for an arbitrary hyperbolic surface S. In this paper we prove the following partial results in support of this claim. A measured lamination is *rational* if its support consists entirely of closed curves. We prove in Theorem 5.1 that if μ, ν are rational laminations whose support fills up S, then $\overline{\mathcal{P}_{\mu,\nu}} \cap \mathcal{F}$ is contained in the line of minima for μ, ν. In Theorem 7 we give a sufficient condition for a point p in $\mathcal{L}_{\mu,\nu}$ to be in the closure of $\mathcal{P}_{\mu,\nu}$, in terms of the Jacobian matrix $\left(\partial l_{\alpha_i}/\partial l_{\beta_j}\right)$, where α_i, β_j are the curves in the support of μ and ν. As discussed in more detail in Section 6, it may well be possible to combine our results with the methods of Bonahon and Otal [19] to prove the complete conjecture. It would be preferable, however, to find a more direct and elementary proof.

There are two difficulties in extending results from the punctured torus to the general case. The condition that the trace be real is not in general sufficient to ensure that an axis be a bending line. One needs in addition that the associated pleated surface be embedded and convex. The proof in [11] that if S is a once punctured torus, then near $\mathcal{F}$, the pleated surface realising an axis with real trace is embedded, extends without difficulty. However, the convexity condition needs more care. For the punctured torus case it is automatic: since (for rational laminations) there is only one bending line, there is only one direction in which to bend. When there are several nonconjugate bending lines, we have to ensure that all the bending angles have the same sign. The second difficulty is that if a lamination has several components, we need to ensure that the length of each component separately is real. The condition that the length of the lamination itself be real is not enough. This last point seems to present a serious difficulty in establishing the complete generalisation of the punctured torus result above. In general, although in one complex variable one can move away from a critical point in at least two directions keeping a single function real valued, the corresponding result in several variables is false. This point is discussed further in Section 6. Starting from a Fuchsian group, one can use Thurston's quakebend construction to bend along a measured geodesic lamination μ. This construction fixes the length l_μ and, at least for small values of the bending measure, one obtains a one parameter family of quasi-Fuchsian groups whose bending measure is in the class of pl^+ is the given lamination μ. It is natural to ask how to control the lamination $[pl^-]$, and whether all possible pairs $([pl^+], [pl^-])$ occur in this way. In view of the above results, this means that we should investigate how the Kerckhoff lines of minima meet the locus $l_\mu(p) = c$. When S is a once punctured torus it is not hard to show, see [12], that for given measured laminations μ, ν which fill up S, the Kerckhoff line $\mathcal{L}_{\mu,\nu}$ intersects $\{p: l_\mu(p) = c\}$ in a unique point p_ν. Thus bending away from p_ν, we can construct groups in the pleating variety $\mathcal{P}_{\mu,\nu}$. In higher genus the situation is a little more subtle. If S has genus g and b punctures, then Fuchsian space $\mathcal{F}$ has dimension $2d = 6g - 6 + 2b$. For each simple closed curve γ and $c > 0$, the horocycle $\mathcal{H}_{\gamma,c} = \{p \in \mathcal{F}: l_\gamma(p) = c\}$ is homeomorphic to $\mathbb{R}^{2d-1}$. The space $\mathrm{P}\mathcal{ML}$ of

projective measured laminations which forms the Thurston boundary of $\mathcal{F}$ is a sphere, also of dimension $2d-1$. It is not hard to see that the subset $\mathrm{P}\mathcal{ML}_\gamma$ of measured laminations $[\nu]$ such that γ, ν fill up S is homeomorphic to $\mathbb{R}^{2d-1}$.

Generalising the once punctured torus result, we prove in Theorem 7.2 that for each $c > 0$ the Kerckhoff line $\mathcal{L}_{\gamma,\nu}$ intersects $\mathcal{H}_{\gamma,c}$ in a unique point p_ν, and that the map $\nu \longmapsto p_\nu$ is a homeomorphism. Assuming our Conjecture 6.5, this would mean that for any γ, c and any $\nu \in \mathcal{ML}_\gamma$ we can find groups in $\mathcal{P}_{\gamma,\nu}$ for which $l_\gamma = c$. We also consider the problem of bending away fixing the lengths of a maximal set of d disjoint closed curves $\{\alpha_i\}$, thus restricting to a d-dimensional submanifold $\mathcal{E} \subset \mathcal{F}$. We now have the freedom to vary the ratios of the bending angles along the bending lines α_i. This means that the projective class of the bending lamination pl^+ varies in a $d-1$-dimensional simplex $\mathcal{Q}$. For each $p \in \mathcal{E}$ and $\mathbf{s} \in \mathcal{Q}$, there is a unique lamination $\nu = \Psi(p, \mathbf{s})$ such that the line of minima $\mathcal{L}_{\mu,\nu}$ meets $\mathcal{E}$ in p, where $\mu = \sum_i s_i \delta_{\alpha_i} \in \mathcal{Q}$. We show that Ψ is a homeomorphism from $\mathcal{E} \times \mathcal{Q}$ to a subset of the set $\mathcal{ML}_*$ of laminations ν such that $\nu, \alpha_1, \ldots, \alpha_d$ fill up S. However we also prove in Theorem 7.3 the rather surprising negative result that Ψ is *not* surjective. In other words, for any choice of fixed lengths $l_{\alpha_i} = c_i, i = 1, \ldots, d$, there are laminations $\nu \in \mathcal{ML}_*$ for which it is *impossible* to bend on the lines α_i obtaining groups with $pl^- \in [\nu]$.

The paper is organized as follows. Section 2 contains background on quasi-Fuchsian groups, geodesic laminations, and real and complex Fenchel Nielsen coordinates. In Section 3, we discuss pleated surfaces and the convex hull boundary and prove various basic facts about the possible pleating loci. Section 4 summarises the results we need from Kerckhoff about minima of length functions. In Section 5, we prove Theorem 5.1, and in Section 6 we prove Theorem 7. Our results about bending away from horocycle and shearing planes are in the final Section 7. The results in this paper are heavily based on the results of Kerckhoff [15] who deemed it prudent, see his discussion on p. 189, to restrict to surfaces without punctures. We wanted to include punctures since otherwise we would be excluding the case of once and twice punctured tori, which provide the most easily investigated examples for which the dimension d is 1 and 2, respectively. It is indeed true that many of the results on which [15] and, hence, our paper are based are stated only in the context of closed surfaces, however we have been unable to detect any point at which the results we use fail. The main difference is that at various points where one has to consider simply connected complementary components, one has now in addition to allow once punctured disks.

2. Background

2.1. QUASI-FUCHSIAN GROUPS

A Kleinian group G is a discrete subgroup of the isometry group $PSL(2, \mathbb{C})$ of hyperbolic 3-space $\mathbb{H}^3$. Such a group also acts by conformal automorphisms on

the sphere at infinity $\hat{\mathbb{C}}$. The limit set $\Lambda(G) \subset \hat{\mathbb{C}}$ is the set accumulation points of fixed points of G, and the regular set $\Omega(G)$ is its complement. A Kleinian group is *Fuchsian* if $\Lambda(G)$ is a round circle.

Let S be an oriented surface of negative Euler characteristic, homeomorphic to a closed surface with at most a finite number of points removed. A finitely generated Kleinian group is *quasi-Fuchsian* if $\mathbb{H}^3/G$ is homeomorphic to the product of such a surface S with the open interval $(0, 1)$, and if $\Omega(G)$ has exactly two simply connected G-invariant components $\Omega^{\pm}$. Equivalently, $G = \pi_1(S)$ and $\Lambda(G)$ is a topological circle. In this situation, the quotients $\Omega^{\pm}/G$ are Riemann surfaces, both homeomorphic to S. After filling in punctures, the surfaces $\Omega^{\pm}/G$ compactify the ends of $\mathbb{H}^3/G$ and $(\Omega \cup \mathbb{H}^3)/G$ is homeomorphic to $S \times [0, 1]$. The representation space $\mathcal{R}(S)$ is the set of representations $\rho\colon \pi_1(S) \to PSL(2, \mathbb{C})$ such that the images of loops around boundary components are parabolic, modulo conjugation in $PSL(2, \mathbb{C})$. A representation $\rho \in \mathcal{R}(S)$ is *quasi-Fuchsian* if $\rho(\pi_1(S))$ is a quasi-Fuchsian group with $\mathbb{H}^3/G \simeq S \times (0, 1)$; in particular, ρ is faithful and its image is discrete.

Let G_0 be a fixed Fuchsian group with Ω^+/G_0 homeomorphic to S. Let ρ_0 be the associated representation $\pi_1(S) \to PSL(2, \mathbb{C})$. Any other quasi-Fuchsian group G with $\Omega^+/G_0 \simeq S$, and corresponding representation ρ, can be obtained as a quasiconformal deformation of G_0. This means that there is a quasiconformal homeomorphism ϕ of $\hat{\mathbb{C}}$ which conjugates ρ_0 and ρ, in other words $\phi(\rho_0(\gamma)(z)) = \rho(\gamma)(\phi(z))$ for all $\gamma \in \pi_1(S)$ and $z \in \hat{\mathbb{C}}$. Notice that the map ϕ is conformal if and only if G_0 and G are conjugate in $PSL(2, \mathbb{C})$. The group G is *marked* by the homeomorphism ϕ. *Quasi-Fuchsian space* $\mathcal{QF}(S)$ is the space of marked quasi-Fuchsian groups, modulo conjugation in $PSL(2, \mathbb{C})$. It has a holomorphic structure induced from the natural holomorphic structure of $SL(2, \mathbb{C})$. *Fuchsian space* $\mathcal{F} = \mathcal{F}(S)$ is the subset of $\mathcal{QF}(S)$ such that the components $\Omega^{\pm}$ are round disks. Thus a point p in $\mathcal{F}$ corresponds to a hyperbolic structure $S(p) = \mathbb{H}^2/\rho(\pi_1(S))$ on S, in which the holonomy round each puncture is parabolic. As is well known, if S has genus g and b punctures, then $\mathcal{F}$ is homeomorphic to $\mathbb{R}^d$ where $d = 3g + b - 3$. It follows from Bers' simultaneous uniformization theorem that $\mathcal{QF}$ can be parameterized by points in $\mathcal{F} \times \overline{\mathcal{F}}$, (where $\overline{\mathcal{F}}$ denotes structures with the opposite orientation to those in $\mathcal{F}$) and hence is a ball in complex space of the same dimension $\mathbb{C}^{2d}$.

For further details on these definitions, good references are [17, 18].

2.2. FENCHEL NIELSEN COORDINATES

Throughout this paper, we shall make central use of Fenchel Nielsen and complex Fenchel Nielsen coordinates for $\mathcal{F}$ and $\mathcal{QF}$ respectively. Both are defined with respect to a fixed *pants decomposition* of S, that is, a collection $\mathcal{A} = \{\alpha_i\}$ of disjoint simple closed curves which cut it into three holed spheres $\mathcal{P} = \{\Pi_j\}$, where we allow that some of the holes may be punctures. We exclude from $\mathcal{A}$ peripheral loops round punctures, so that each $\alpha \in \mathcal{A}$ is a boundary component of exactly two of the pants $\Pi, \Pi' \in \mathcal{P}$, where possibly $\Pi = \Pi'$. A pants decomposition always contains exactly

d disjoint nonperipheral curves, where $d = 3g + b - 3$ as above. The Fenchel Nielsen coordinates of a hyperbolic structure $\mathbb{H}^2/G$ on S (represented by a point $p \in \mathcal{F}$) relative to the pants decomposition $\mathcal{A}$ consist of the lengths $l_{\alpha_i} \in \mathbb{R}^+$ of the nonperipheral curves $\{\alpha_i\}$, together with twists $t_{\alpha_i} \in \mathbb{R}$ which measure the relative positions along $\{\alpha_i\}$ in which the pants Π, Π' are glued to form $\mathbb{H}^2/G$. So as to be able to distinguish the effect of Dehn twists about α, one needs to choose the common perpendiculars relative to fixed closed curves tranverse to α which effectively define the marking of S. This allows one to define not only t_α modulo the length l_α, but $t_\alpha \in \mathbb{R}$. We omit a more detailed disussion of the conventions whose exact form will not be important here. Good references are [1, 8]. It is important to notice however that we define t_α as an actual hyperbolic length and not, as the above authors, a fraction of l_α.

Since t_α is a *signed* distance, it defines a real analytic function on $\mathcal{F}$. The Fenchel Nielsen theorem is that, if $\mathcal{A} = \alpha_i, i = 1, \dots, d$ is a pants decomposition of S, the map $p \mapsto (l_{\alpha_i}(p), t_{\alpha_i}(p)),\ i = 1, \dots, d$ defines a real analytic bijection between $\mathcal{F}$ and $(\mathbb{R}^+)^d \times (\mathbb{R})^d$. We denote the group with coordinates $(l_{\alpha_i}, t_{\alpha_i})$ by $G(l_{\alpha_i}, t_{\alpha_i})$.

Complex Fenchel Nielsen Coordinates

It is shown in [16] and [24] that the Fenchel Nielsen construction extends to quasi-Fuchsian groups. The idea is to replace the length coordinate l_α with the complex translation length λ_α of the element $g_\alpha \in \pi_1(S)$ representing the curve α. Complex length is defined by the formula $\operatorname{Tr} g_\alpha = 2\cosh \lambda_\alpha/2$ and is chosen so that $\Re\lambda_\alpha > 0$. There is a problem in this definition: there is an ambiguity of sign which depends on whether one chooses the half length as $\lambda_\alpha/2$ or $\lambda_\alpha/2 + i\pi$. To determine a quasi-Fuchsian representation $\pi_1(S) \to PSL(2, \mathbb{C})$ uniquely, it turns out one needs to specify the *half* lengths $\lambda_\alpha/2$. The data $\{\lambda_\alpha/2, \alpha \in \mathcal{A}, \Re\lambda_\alpha > 0\}$ determines the lifts of boundary geodesics of the pants Π, uniquely up to isometries of $\mathbb{H}^3$. Conversely, given a point $q \in \mathcal{QF}$, it is possible to define the half lengths $\lambda_\alpha/2$ in a canonical (and holomorphic) way. The complex twist parameter τ_α describes the gluing of neighbouring pants; with suitable conventions, it is the *signed complex distance* between the oriented common perpendiculars to lifts of appropriate boundary curves in the two pants, measured along their oriented common axis. Notice that even though the pants are skewed, the lifts of axes and their common perpendiculars are well defined, so that it still makes sense to speak about orientations inherited from the orientation of S. Once again, the precise details of the construction will not be relevant to us here. A more detailed summary appears in [22], and careful discussions of signed complex distance are to be found in [7, 22]. As discussed at length in [22], it is important to take τ_α to be *signed* complex distance so as to obtain a holomorphic function on $\mathcal{QF}$.

The data $\{\lambda_\alpha/2 \in \mathbb{C}^+, \tau_\alpha \in \mathbb{C}/2\pi i;\ \alpha \in \mathcal{A}\}$ determine a unique point in $\mathcal{R}(S)$, where $\mathbb{C}^+ = \{z \in \mathbb{C}: \Re z > 0\}$. In contrast to the Fuchsian case, the resulting group is in

general neither discrete nor quasi-Fuchsian. One does however get a holomorphic embedding of $\mathcal{QF}$ into $(\mathbb{C}^+)^d \times (\mathbb{C}/2\pi i)^d$ which restricts to the Fenchel Nielsen embedding on $\mathcal{F}$, see [24] theorem 1. Some examples of specific computations of traces given the complex Fenchel Nielsen coordinates of the corresponding representation are given in [4].

2.3. GEODESIC LAMINATIONS

For details on this section see for example [3] or [18]. Let S be a surface as in Section 2.1. We denote by $\mathcal{S} = \mathcal{S}(S)$ the set of all homotopy classes of simple closed nonboundary parallel curves on S. For a fixed hyperbolic structure on S, there is exactly one geodesic in each such homotopy class, and it is well known that $\mathcal{S}$ is independent of the hyperbolic structure on S.

A geodesic lamination on S is a closed set that is a union of pairwise disjoint simple complete geodesics called its *leaves*. A geodesic lamination is *measured* if it carries a tranverse invariant measure, that is, an assignment of a finite Borel measure to each transversal which is invariant under the push forward map along leaves. We always assume that the support of the measure μ is the entire underlying lamination $|\mu|$, and that $|\mu|$ contains no leaves which end in a puncture. Notice that every puncture is contained in a definite horocyle neighbourhood which is not intersected by any leaf of any lamination of this kind. We denote the set of such measured laminations by $\mathcal{ML}(S)$, topologised with the topology of weak convergence: $\mu_n \to \mu$ if $\mu_n(T) \to \mu(T)$ for any transversal T. For $\mu \in \mathcal{ML}$, one can define the length l_μ as the total mass of the measure which is the product of hyperbolic distance along the leaves of μ with the transverse measure μ.

A measured geodesic lamination $\mu \in \mathcal{ML}(S)$ is *rational* if its support $|\mu|$ is a disjoint union of closed geodesics $\alpha_i \in \mathcal{S}$. We write such laminations $\sum_i a_i \delta_{\alpha_i}$, usually abbreviated to $\sum_i a_i \alpha_i$, where $a_i \in \mathbb{R}^+$ and δ_{α_i} is the lamination with support α_i which assigns unit mass to each intersection with α_i. We denote the set of all rational measured laminations by $\mathcal{ML}_Q(S)$; the set $\mathcal{ML}_Q$ is dense in $\mathcal{ML}$.

The length of the rational lamination $\sum_i a_i \alpha_i$ is just $\sum_i a_i l_{\alpha_i}$, where l_{α_i} means of course the function which assigns to $p \in \mathcal{F}$ the hyperbolic length of the geodesic α_i on the surface $S(p)$. Kerckhoff [13, 14] has shown that if $\mu_n \in \mathcal{ML}_Q$ converge to μ in $\mathcal{ML}$, then (l_{μ_n}) converges to l_μ uniformly on compact subsets of $\mathcal{F}$. In fact this gives an alternative way of defining the length function l_μ. In a similar way, the geometric intersection number $i(\alpha, \alpha')$ of two geodesics $\alpha, \alpha' \in \mathcal{S}$ extends by linearity and continuity to a continuous function $i(\mu, \nu)$ on $\mathcal{ML}(S)$, (see, for example, [13, 18]). A *projective measured lamination* is an equivalence class of measured laminations, with the relation that $\mu, \mu' \in \mathcal{ML}$ are equivalent if they have the same underlying support and if there exists $k > 0$ such that for any transversal T, $\mu'(T) = k\mu(T)$. We write $[\mu]$ for the projective class of μ and denote the set of all *nonzero* projective measured laminations on S by $\mathrm{P}\mathcal{ML}(S)$. If S has genus g and b punctures, as usual we set $d = 3g - 3 + b$. Thurston showed that $\mathrm{P}\mathcal{ML}(S)$ is a

compactification of $\mathcal{F}(S)$, and hence that $\mathrm{P}\mathcal{ML}(S)$ is homeomorphic to the sphere S^{2d-1} (see [6, 18]).

3. The Convex Hull Boundary and Pleating Varieties

3.1. THE CONVEX HULL BOUNDARY

For details on this section, see [5, 10]. Let G be a Kleinian group. The *convex hull* or *convex core* $\mathcal{C}/G$ of the 3-manifold $\mathbb{H}^3/G$ is the smallest closed convex set containing all closed geodesics of $\mathbb{H}^3/G$. Alternatively, $\mathcal{C}$ can be defined in the universal cover $\mathbb{H}^3$, as the hyperbolic convex hull of the limit set Λ. If G is quasi-Fuchsian then $\partial\mathcal{C}$ has exactly two components $\partial\mathcal{C}^{\pm}$ which 'face' the components $\Omega^{\pm}$ of Ω. The quotients $\partial\mathcal{C}^{\pm}/G$ are homeomorphic to $\Omega^{\pm}/G$ and, hence, to S, see [10], proposition 3.1.

In the special case in which G is Fuchsian, $\mathcal{C}$ is contained in a single flat plane. We regard this as a degenerate case in which $\partial\mathcal{C}$ is two sided, each side facing one component of $\Omega(G)$.

The convex hull boundary $\partial\mathcal{C}$ is made up of convex pieces of flat hyperbolic planes which meet along a disjoint set of complete geodesics called *pleating* or *bending lines*. They project to geodesic laminations, called the *bending laminations*, on $\partial\mathcal{C}^{\pm}/G$. As explained in detail in [5], these laminations carry transverse measures, the *bending measures*, denoted $pl^{\pm}(G)$. We proved in [10] that the map $\mathcal{QF} \to \mathcal{ML}, q \mapsto pl^{\pm}(q)$ is continuous.

We shall be especially interested in the case when the bending lamination is rational. In this case, the bending measure is determined by the angles θ_i between the planes which meet along the axes $Ax\,\alpha_i$ of the geodesics γ_i in the support of $pl^{\pm}$. Precisely, if $|pl^{+}| = U_i\alpha_i$ where $\alpha_i \in \mathcal{S}$, then $pl^{+}(T) = \sum_i i(T, \alpha_i)|\theta_i|$, where θ_i is taken to be 0 when the oriented support planes coincide. In particular, if G is Fuchsian, then $pl^{\pm} = 0$.

3.2. CONSTRAINTS ON THE BENDING LINES

In this section we describe two relatively easy conditions constraining the possible bending laminations $pl^{\pm}$ which can appear. The first concerns the topology of their support. We use the following result proved in [11], proposition 3.3.

LEMMA 3.1. *Suppose that G is a Kleinian group which is not Fuchsian, and that α is a geodesic axis. Then $\alpha \subset \partial\mathcal{C}_i/G$ for at most one component $\partial\mathcal{C}_i/G$ of the convex hull boundary $\partial\mathcal{C}/G$.*

Proof. The idea is to study the inverse image of α under the retraction map $r\colon \mathbb{H}^3 \cup \hat{\mathbb{C}} \to \partial\mathcal{C}$. On the one hand, if $\alpha \subset \partial\mathcal{C}^{+} \cap \partial\mathcal{C}^{-}$, then $r^{-1}(\alpha) \cap \hat{\mathbb{C}}$ intersects both components $\Omega^{\pm}(G)$ of $\Omega(G)$; while on the other, it is not hard to show that $r^{-1}(\alpha) \cap \hat{\mathbb{C}}$ is connected unless G is Fuchsian. $\square$

DEFINITION. Laminations $\mu, \nu \in \mathcal{ML}(S)$ *fill up* S if $i(\mu, \eta) + i(\nu, \eta) > 0$ for all $\eta \in \mathcal{ML}$. An equivalent condition is that every component of $S - |\mu| \cup |\nu|$ contain at most one puncture, which, after filling in the puncture if needed, is compact and simply connected.

For the equivalence of these two conditions we refer to [15], proposition 1.1 and lemma 4.4, or to [20], proposition 2. These authors treat the case of closed surfaces but there is no essential difference for surfaces with punctures provided we allow complemetary components which are punctured disks. We shall usually be dealing with rational laminations, in which case it is enough to require that the components of the complement be disks or punctured disks. For general laminations this is not enough, since the complement of a generic lamination is a union of ideal polygons. This is why one needs in addition that the complementary components be compact. The following result has also been proved in [2].

PROPOSITION 3.2. *Let G be quasi-Fuchsian,* $G \in \mathcal{QF}(S)$. *Let* $|pl^{\pm}|$ *be the supports of the pleating laminations of* $\partial\mathcal{C}^{\pm}$. *Then* $|pl^+|$ *and* $|pl^-|$ *fill up S.*

Proof. Suppose not, then there is a simple non-boundary peripheral loop α on S disjoint from both $|pl^+|$ and $|pl^-|$. Cut $\partial\mathcal{C}^+$ along $|pl^+|$. The remaining surface is geodesically convex and contained in a single support plane. It contains a geodesic representative of α which must be its axis in $\mathbb{H}^3$. The same is true of $\partial\mathcal{C}^-$. This contradicts Lemma 3.1. □

The second constraint relates geometry to analysis and is the key to understanding the arrangement of the pleating laminations in $\mathcal{QF}$. It is a simple consequence of the observation that the two planes in $\partial\mathcal{C}^{\pm}$ which meet along a geodesic axis $Ax(g)$ are invariant under translation by g.

PROPOSITION 3.3 ([9], lemma 4.6). *Suppose that the axis of* $g \in G$ *is a bending line of* $\partial\mathcal{C}^{\pm}(G)$. *Then* $\lambda(g)$, *the complex length of g, is real.*

We remark that following [12], one can define the complex length of a lamination and extend the above result to the case of general $\mu \in \mathcal{ML}$. This more delicate result is not needed here.

COROLLARY 3.4. *Suppose that* $q \in \mathcal{QF}$, *and that the support of* pl^+ *is contained in a pants decomposition* $\mathcal{A} = \{\alpha_i\}$ *of S. Then the complex lengths* $\lambda_{\alpha_i}(q) \in \mathbb{R}$ *for all i.*

Proof. By hypothesis $pl^+ = \sum_i a_i\alpha_i$ where $a_i \geqslant 0$. If $a_i > 0$ then α_i is a bending line of $\partial\mathcal{C}^{\pm}$ and we can apply Proposition 3.3. Otherwise, $i(\alpha_i, \mu) = 0$ and so α_i lies in a flat part of $\partial\mathcal{C}^{\pm}$. Translation along the axis of α_i leaves this flat part invariant and the result follows. □

In what follows we shall be concerned with the converse of this result, that is, conditions under which the lengths $\lambda_{\alpha_i} \in \mathbb{R}$ imply that the support of $pl^{\pm}$ is contained in $\mathcal{A}$.

3.3. PLEATED SURFACES AND THE COMPLEX FENCHEL NIELSEN CONSTRUCTION

Let $G \in \mathcal{R}(S)$. Slightly varying the definitions in [3, 26] to allow for non-discrete groups G, we define a *G-pleated surface map* to be a hyperbolic structure $p \in \mathcal{F}$ on S, together with a map $\sigma: \mathbb{H}^2 \to \mathbb{H}^3$ such that

(1) σ conjugates the action of $\Gamma(p)$ on $\mathbb{H}^2$ with the action of G on $\mathbb{H}^3$.
(2) σ is an isometry onto its image with the path metric induced from $\mathbb{H}^3$.
(3) For each $x \in \mathbb{H}^2$, there exists at least one geodesic γ containing x such that $\sigma|_\gamma$ is an isometry.

Usually we shall refer to the image of σ as a *pleated surface* in $\mathbb{H}^3$. The set of points in $\mathbb{H}^2$ satisfying (3) for exactly one geodesic Γ project to a geodesic lamination $B(\sigma)$ on S. We call a pleated surface *rational* if $B(\sigma)$ is a union of closed curves. Let $p \in \mathcal{F}$ and let $\mathcal{A}, \mathcal{B}$ be pants decompositions of S. There is an intimate connection between rational pleated surfaces and the complex Fenchel Nielsen construction when the complex lengths $\lambda_\alpha, \alpha \in \mathcal{A}$ are real. In the real Fenchel Nielsen construction, the hyperbolic surface S with coordinates $\{(l_\alpha, t_\alpha); \alpha \in \mathcal{A}\}$ is made by gluing together planar pairs of pants whose boundary curves have lengths l_α with a shift of t_α along the axis of α. Since each pair of pants is made by gluing two planar right angled hexagons, the glued up pants lift to the hyperbolic plane $\mathbb{H}^2$, forming the universal cover of the surface S. In the complex construction, the planar hexagons are replaced by skew right angled hyperbolic hexagons of (complex) side lengths $\lambda_\alpha/2$. In the special case in which all the prescribed lengths $\lambda_\alpha/2$ are real, this part of the construction proceeds exactly as in the real case. The skew hexagons are planar and the lift of each pair of pants is a flat surface contained in one hyperbolic plane. The only difference between the two constructions is that in the first case the planar pants are glued together with a shift t_α along the axis of α, while in the second the same pants are glued with the same shift $t_\alpha = \Re\tau_\alpha$ and an additional bend $\theta_\alpha = \Im\tau_\alpha$, which means that the glued up pants now form a pleated surface in $\mathbb{H}^3/G$. More precisely, there is a pleated surface map $dev: \mathbb{H}^2 \to \mathbb{H}^3$ conjugating the actions of $G(\lambda_\alpha, t_\alpha)$ and $G(\lambda_\alpha, t_\alpha + i\theta_\alpha)$. The image is a rational pleated surface $\mathrm{Dev}(\mathcal{A})$ in $\mathbb{H}^3$ whose bending lamination is exactly $\{\alpha \in \mathcal{A}: \theta_\alpha \neq 0\}$.

3.4. RATIONAL PLEATED SURFACES AND THE CONVEX HULL BOUNDARY

If $G \in \mathcal{QF}$, then the two components $\partial\mathcal{C}^\pm$ of $\partial\mathcal{C}$ are pleated surfaces [5]. If $\mathcal{A} = \{\alpha_i\}$ is a pants decomposition, let

$$\mathcal{P}^\pm(\mathcal{A}) = \{q \in \mathcal{QF} - \mathcal{F}: |pl^\pm| \subset \mathcal{A}\}.$$

Notice that if $q \in \mathcal{P}^\pm(\mathcal{A})$ then $\mu = \sum_i a_i\alpha_i$ with $a_i \geqslant 0$. If $a_i = 0$ then the convex hull is not bent along α_i. (We cannot have $a_i = 0$ for all i since by definition $q \notin \mathcal{F}$.)

By Corollary 3.4, if $q \in \mathcal{P}(\mathcal{A})$ then all the complex lengths $\lambda_{\alpha_i}(q)$ are real. If for example $q \in \mathcal{P}^+(\mathcal{A})$, it is clear that $\mathrm{Dev}(\mathcal{A}) = \partial\mathcal{C}^+$ and that the bending measure

is exactly $\sum_{\mathcal{A}} \Im\tau_\alpha \delta_\alpha$. Since $\partial\mathcal{C}^+$ bounds a convex set, all the angles $\Im\tau_\alpha$, $\alpha \in \mathcal{A}$ have the same sign. Conversely, suppose that the group $G \in \mathcal{R}(S)$ has complex Fenchel Nielsen coordinates $\{\lambda_\alpha, \tau_\alpha\}_{\alpha\in\mathcal{A}}$ with $\lambda_\alpha \in \mathbb{R}$, and such that all the bending angles $\theta_\alpha = \Im\tau_\alpha$ have the same sign. The following gives conditions on the τ_α which ensure that $G \in \mathcal{QF}$ and that $\mathrm{Dev}(\mathcal{A})$ equals $\partial\mathcal{C}^\pm$.

THEOREM 3.5. *Let $\mathcal{A} = \{\alpha_i\}$ be a pants decomposition of S and suppose that $G \in \mathcal{R}(S)$ has complex Fenchel Nielsen coordinates $\{\lambda_\alpha, \tau_\alpha\}_{\alpha\in\mathcal{A}}$ with $\lambda_\alpha \in \mathbb{R}$. Let* $\mathrm{Dev}(\mathcal{A})$ *be the associated pleated surface in $\mathbb{H}^3$. Then there exists $\varepsilon > 0$ such that if $|\Im\tau_\alpha| < \varepsilon$ for all $\alpha \in \mathcal{A}$, then the group G is quasi-Fuchsian. If in addition* $\Im\tau_\alpha \geqslant 0$ *for all $\alpha \in \mathcal{A}$, then* $\mathrm{Dev}(\mathcal{A})$ *is a component of $\partial\mathcal{C}(G)$.*

This was proved in [11], theorems 6.1 and 7.2 for the once punctured torus. The proof extends with only trivial changes to the general case. Notice that the proof of 6.1 is still valid whether or not all the bending angles have the same sign. Lemma 7.1 holds because the condition $\Im\tau_\alpha \geqslant 0, \alpha \in \mathcal{A}$ implies that the half space cut out by $\mathrm{Dev}\,\mathcal{A}$ is convex. This result is closely related to Thurston's *bending* construction described in detail in [5]. In fact, if $G(l_{\alpha_i}, t_i) \in \mathcal{F}$, then Theorem 3.5 shows that for small $\theta_i \geqslant 0$, $G(l_{\alpha_i}, t_i + i\theta_i)$ is exactly the group obtained by bending $G(l_{\alpha_i}, t_i)$ along the rational lamination $\sum_i \theta_i\alpha_i$. In other words, for any given pants decomposition $\mathcal{A}$, we can always construct groups with $q \in \mathcal{P}^+(\mathcal{A})$, or if we prefer with $q \in \mathcal{P}^-(\mathcal{A})$. In the remainder of the paper, we want to address the problem of when we can construct groups which are simultaneously in $\mathcal{P}^+(\mathcal{A}) \cap \mathcal{P}^-(\mathcal{B})$ for an arbitrary pair of pants decompsitions $\mathcal{A}, \mathcal{B}$ which fill up S.

4. Lines of Minima

In [11], we showed that in the case of the once punctured torus, the problem just described could be solved using Kerckhoff's results about minima of length functions along of earthquake paths in $\mathcal{F}$. For higher genus, the situation becomes considerably more complicated, but the same general idea works. We therefore begin by collecting the results from Kerckhoff [15] which we need. Recall that the time *t left earthquake* [13, 25] along a lamination $\mu \in \mathcal{ML}$ is a real analytic map $\mathcal{E}_\mu(t)\colon \mathcal{F} \to \mathcal{F}$ which generalises the classical Fenchel–Nielsen twist. The earthquake shifts complementary components of the lamination $|\mu|$ on the surface $S(p)$ a distance $t\mu(T)$ relative to one another, where $\mu(T)$ is the μ-measure of a transversal T joining the two components. In particular, if $\mathcal{A}$ is a pants decomposition of S and if $\mu \in \mathcal{ML}_Q, \mu = \sum_i a_i\alpha_i$, then the earthquake is expressed in terms of Fenchel Nielsen coordinates relative to $\mathcal{A}$ by $\mathcal{E}_\mu(t)\colon (l_{\alpha_i}, t_{\alpha_i}) \mapsto (l_{\alpha_i}, t_{\alpha_i} + a_i t), i = 1, \ldots, d$.

If $p \in \mathcal{F}$, we denote by $\mathcal{E}_\mu = \mathcal{E}_\mu(p)$ the earthquake path $\mathcal{E}_\mu(t)(p), t \in \mathbb{R}$. This flow induces a tangent vector field $\partial/\partial t_\mu$ on $\mathcal{F}$. In [13], Kerckhoff showed that if $\nu \in \mathcal{ML}$, then the length l_ν is a real analytic function of t along $\mathcal{E}_\mu(p)$, strictly convex

if $i(\mu, \nu) > 0$ and constant otherwise. Wolpert [27] proved the antisymmetry formula

$$\partial l_\nu / \partial t_\mu = -\partial l_\mu / \partial t_\nu, \tag{1}$$

from which one deduces easily that the minimum points for l_ν along $\mathcal{E}_\mu$ and l_μ along $\mathcal{E}_\nu$ coincide. He also showed that if $i(\mu, \nu) > 0$, then at the unique minimum, $\partial^2 l_\nu / \partial t_\mu^2 > 0$. The following is the fundamental theorem of [15].

THEOREM 4.1 ([15], theorem 1.2). *Suppose μ and ν fill up S. Then for every $k \in (0, \infty)$ the function $f = l_\mu + k l_\nu$ has a unique critical point on $\mathcal{F}$ which is a global minimum for f.*

The uniqueness part of this result uses that length is convex along earthquake paths, and Thurston's earthquake theorem [25] and [13], corollary 5.4, that there is a left earthquake path between any two points in $\mathcal{F}$. In fact if there were two critical points, they would be joined by an earthquake path $\mathcal{E}_\eta$ along which $l_\mu + k l_\nu$ is strictly convex, since by hypothesis μ, ν fill up S. This is clearly impossible. (The proof of the earthquake theorem requires a bit of attention when S has punctures. The main point is that in the approximation arguments, one has to use the observation noted earlier that every leaf in the support of a lamination $\mu \in \mathcal{ML}$ avoids a definite horocycle neighbourhood of each cusp.) The next result shows that we really do need the assumption that μ, ν fill up S.

PROPOSITION 4.2. *Suppose that if $\mu, \nu \in \mathcal{ML}$ do not fill up S, and that $k \in (0, \infty)$. Then the function $l_\nu + k l_\mu$ has no critical point on $\mathcal{F}$.*

Proof. It suffices to show that if μ and ν fail to fill up S, then $l_\nu + k l_\mu$ can always be decreased. Precisely this statement is proved by Kerckhoff in part II of his theorem 2.1, see pp 194–195 in [15]. Once again the argument needs minor changes if S has cusps to allow for the possibility that the complementary components of various laminations may be punctured disks. $\square$

Suppose that $\mu \in \mathcal{ML}$. Following Kerckhoff, we denote $\mathcal{ML}_\mu$ the set of $\nu \in \mathcal{ML}$ such that μ and ν fill up S. This set is described in [15], theorem 4.7:

THEOREM 4.3. *Let $\mu \in \mathcal{ML}$. Then $\mathcal{ML}_\mu$ is homeomorphic with its induced topology to an open disk of (real) dimension $2d = 6g + 2b - 6$.*

We remark that we shall mainly use this result in the case $\mu \in \mathcal{ML}_Q$, in which case it is particularly easy to see using the Dehn Thurston coordinates for $\mathcal{ML}$, see Section 7. Fix once and for all a continuous section $j\colon \mathrm{P}\mathcal{ML} \to \mathcal{ML}$, for example by choosing a fixed surface S_0 and defining $j([\nu]) = \nu / l_\nu(S_0)$. The following is theorem 2.1 of [15].

THEOREM 4.4. *Let $\mu \in \mathcal{ML}$. The map $\Phi_\mu\colon \mathrm{P}\mathcal{ML}_\mu \times (0, \infty) \to \mathcal{F}(S)$, sending $([\nu], k)$ to the point where $l_\mu + k l_{j([\nu])}$ attains its minimum, is a homeomorphism.*

For fixed μ, the line $k \to \Phi(\nu, k): k \in (0, \infty)$ is called the *line of minima* of μ, ν, denoted $\mathcal{L}_{\mu,\nu}$. Notice that if we change μ in $[\mu]$, the map Φ_μ changes but the line $\mathcal{L}_{\mu,\nu}$ remains the same.

There is a useful alternative characterisation of lines of minima in terms of tangent vectors along earthquake paths. As remarked above, the earthquake flow $\mathcal{E}_\mu(t)$ generates a field of tangent vectors $\partial/\partial t_\mu$ on $\mathcal{F}$. Likewise, the length function l_μ defines a cotangent vector field dl_μ.

PROPOSITION 4.5 ([15], theorem 3.5). *For all $p \in \mathcal{F}$, tangent vectors $\partial/\partial t_\mu$, $\mu \in \mathcal{ML}$, span the tangent space $T_p(\mathcal{F})$ and the cotangent vectors dl_μ, $\mu \in \mathcal{ML}$ span the cotangent space $T_p^*(\mathcal{F})$. The maps $\mu \to \partial/\partial t_\mu(p)$ and $\mu \to dl_\mu(p)$ are homeomorphisms onto $T_p(\mathcal{F})$ and $T_p^*(\mathcal{F})$ respectively.*

This gives the following alternative characterisation of critical points. Notice that for this result, we do not need the assumption that $k > 0$.

PROPOSITION 4.6. *Let $\mu, \nu \in \mathcal{ML}$ and $k \in \mathbb{R}$. The point $p \in \mathcal{F}$ is a critical point of $l_\mu + kl_\nu$ if and only if $\partial/\partial t_\mu = -k(\partial/\partial t_\nu)$ at p.*

Proof. At a critical point, we have $\partial/\partial t_\eta(l_\mu + kl_\nu) = 0$ for every $\eta \in \mathcal{ML}$. Using the antisymmetry formula (1) this gives $\partial l_\eta/\partial t_\mu = -k(\partial l_\eta/\partial t_\nu)$. Since the one forms dl_η span the cotangent space at p, we conclude that $\partial/\partial t_\mu = -k(\partial/\partial t_\nu)$. The converse follows similarly. □

The following simple consequence of Kerckhoff [15], theorem 4.8, shows that in fact the sign of k is automatic.

PROPOSITION 4.7. *Suppose that $\mu, \nu \in \mathcal{ML}$ and that $\mu \notin [\nu]$. Suppose also that $\partial/\partial t_\mu = -k(\partial/\partial t_\nu)$ for some $k \in \mathbb{R}$. Then μ and ν fill up S and $k > 0$.*

Proof. First suppose $i(\mu, \nu) = 0$. If $k \leqslant 0$ then we get an immediate contradiction to Proposition 4.5. If $k > 0$ then $\mu' = \mu + k\nu \in \mathcal{ML}$ and $\partial/\partial t_{\mu'} = 0$, again contradicting 4.5. If $i(\mu, \nu) > 0$, then by [15], theorem 4.8, for any point $p \in \mathcal{F}$, there exists a lamination $\eta \in \mathcal{ML}$ such that

$$\frac{\partial l_\eta}{\partial t_\nu}(p) > \frac{\partial l_\mu}{\partial t_\nu}(p) \quad \text{and} \quad \frac{\partial l_\nu}{\partial t_\mu}(p) > \frac{\partial l_\eta}{\partial t_\mu}(p).$$

This clearly forces $k \neq 0$. From $\partial/\partial t_\mu = -k(\partial/\partial t_\nu)$ we deduce $-k(\partial l_\mu/\partial t_\nu) = \partial l_\mu/\partial t_\mu(p) = 0$ and hence

$$\frac{\partial l_\eta}{\partial t_\nu}(p) > 0 > \frac{\partial l_\eta}{\partial t_\mu}(p) = -k\frac{\partial l_\eta}{\partial t_\nu}(p)$$

which forces $k > 0$. From Proposition 4.2, we conclude that μ, ν fill up S. □

COROLLARY 4.8. *Suppose that* $\mu, \nu \in \mathcal{ML}$ *with* $[\mu] \neq [\nu]$. *Suppose also that at* $p \in \mathcal{F}$ *there exists* $k \in \mathbb{R}$ *such that* $\partial/\partial t_\mu = -k(\partial/\partial t_\nu)$. *Then* $k > 0$, *the laminations* μ *and* ν *fill up* S, *and* p *is the unique minimum of* $l_\mu + kl_\nu$.

5. Necessary Conditions for Bending Away

We are finally ready to address the main object of this paper: the arrangement of possible bending laminations for quasi-Fuchsian groups near Fuchsian space $\mathcal{F}$. For $\mu, \nu \in \mathrm{P}\mathcal{ML}$, we set

$$\mathcal{P}_{\mu,\nu} = \{q \in \mathcal{QF}: pl^{\pm}(q) \in [\mu], pl^{\pm}(q) \in [\nu]\}.$$

Notice that this definition depends only on the projective equivalence classes of μ and ν. As in Section 3.4, if $\mathcal{A}$ and $\mathcal{B}$ are pants decompositions, we let

$$\mathcal{P}^{\pm}(\mathcal{A}) = \{q \in \mathcal{QF} - \mathcal{F}: |pl^{\pm}| \subset \mathcal{A}\}$$

and set

$$\mathcal{P}(\mathcal{A}, \mathcal{B}) = \mathcal{P}^{+}(\mathcal{A}) \cap \mathcal{P}^{-}(\mathcal{B}).$$

We call sets $\mathcal{P}^{+}(\mathcal{A})$, $\mathcal{P}_{\mu,\nu}$ and so on, *pleating varieties*. By Theorem 3.5, $q \in \mathcal{P}(\mathcal{A}, \mathcal{B})$ provided $\theta_{\alpha_i} \geqslant 0$ and $\theta_{\beta_i} \leqslant 0$ for $i = 1, \dots, d$, and in addition $0 < \sum_i \theta_{\alpha_i}$, $-\sum_i \theta_{\beta_i} < \varepsilon$, for sufficiently small $\varepsilon > 0$, where $\theta_\gamma = \Im\tau_\gamma$ is the bending angle on the curve γ.

Set

$$\mathcal{Q} := \left\{(s_1, \dots, s_d): s_i \geqslant 0, \sum_i s_i = 1\right\} \subset \mathbb{R}^d,$$

and for a pants decomposition $\mathcal{A} = \{\alpha_i\}$, let

$$\mathcal{Q}(\mathcal{A}) = \left\{\sum_i s_i\alpha_i \in \mathcal{ML}_Q: (s_1, \dots, s_d) \in \mathcal{Q}\right\}.$$

The following is our first main result.

THEOREM 5.1. *Suppose that pants decompositions* $\mathcal{A}, \mathcal{B}$ *fill up* S *and that* $p \in \mathcal{F}$ *is in the closure of* $\mathcal{P}(\mathcal{A}, \mathcal{B})$. *Then* $p \in \mathcal{L}_{\xi,\eta}$ *for some* $\xi \in \mathcal{Q}(\mathcal{A})$, $\eta \in \mathcal{Q}(\mathcal{B})$ *which fill up* S. *If* $p \in \mathcal{F}$ *is in the closure of* $\mathcal{P}_{\mu,\nu}$ *for some* $\mu \in \mathcal{Q}(\mathcal{A})$, $\nu \in \mathcal{Q}(\mathcal{B})$, *then* $\xi = \mu$ *and* $\eta = \nu$.

Proof. By hypothesis, we can find $q_n \in \mathcal{P}(\mathcal{A}, \mathcal{B})$ such that $q_n \to p$. Thus $q_n \in \mathcal{P}_{\mu_n,\nu_n}$ for some $\mu_n \in \mathcal{Q}(\mathcal{A})$, $\nu_n \in \mathcal{Q}(\mathcal{B})$. Passing to a subsequence, we may assume $\mu_n \to \mu \in \mathcal{Q}(\mathcal{A})$, $\nu_n \to \nu \in \mathcal{Q}(\mathcal{B})$. We need to show that there exists $k > 0$ such that μ, ν fill up S and such that p is the global minimum of $l_\mu + kl_\nu$. By Corollary 4.8, it will be enough to show that there exists $k \in \mathbb{R}$ such that $\partial/\partial t_\mu = -k(\partial/\partial t_\nu)$ at p.

Since $\lambda_{\beta_i}, \tau_{\beta_i}$ are coordinates for $\mathcal{QF}$, we have

$$\frac{\partial}{\partial\tau_\mu} = \sum_i \frac{\partial\lambda_{\beta_i}}{\partial\tau_\mu}\frac{\partial}{\partial\lambda_{\beta_i}} + \sum_i \frac{\partial\tau_{\beta_i}}{\partial\tau_\mu}\frac{\partial}{\partial\tau_{\beta_i}}.$$

Suppose that $\mu = \sum_i a_i\alpha_i$ and $\nu = \sum_i b_i\beta_i$. We claim that

(1) $\partial\lambda_{\beta_i}/\partial\tau_\mu(p) = 0,\ i = 1, \dots, d$.
(2) There exists $k \in \mathbb{R}$ such that $\partial\tau_{\beta_i}/\partial\tau_\mu(p) = -kb_i,\ i = 1, \dots, d$.

Combining these results gives $\partial/\partial\tau_\mu = -k(\partial/\partial\tau_\nu)$ at p as required. To prove the first claim, expand λ_{β_i} as a Taylor series about p, using complex Fenchel Nielsen coordinates with respect to $\mathcal{A}$. For any function $f\colon \mathcal{QF} \to \mathbb{C}$, we write $\Delta f(q) = f(q) - f(p)$. Since $q_n \in \mathcal{P}(\mathcal{A}, \mathcal{B})$, we have $\Delta\lambda_{\alpha_i}(q_n), \Delta\lambda_{\beta_i}(q_n) \in \mathbb{R}$. We also note that the complex derivatives $\partial\lambda_{\beta_i}/\partial\lambda_{\alpha_j}(p), \partial\lambda_{\beta_i}/\partial\tau_{\alpha_j}(p)$ are in fact real at $p \in \mathcal{F}$.

We have

$$\Delta\lambda_{\beta_i}(q_n) = \sum_j \frac{\partial\lambda_{\beta_i}}{\partial\lambda_{\alpha_j}}(p)\Delta\lambda_{\alpha_j}(q_n) + \sum_j \frac{\partial\lambda_{\beta_i}}{\partial\tau_{\alpha_j}}(p)\Delta\tau_{\alpha_j}(q_n) + R$$

where R denotes terms at least quadratic in $\Delta\lambda_{\alpha_i}(q_n)$ and $\Delta\tau_{\alpha_i}(q_n)$. Thus taking imaginary parts we find

$$0 = \sum_j \frac{\partial\lambda_{\beta_i}}{\partial\tau_{\alpha_j}}(p)\Delta\theta_{\alpha_j}(q_n) + ||\theta_{\mathcal{A}}(q_n)||E(q_n),$$

where $||\theta_{\mathcal{A}}(q)|| = \sum_i |\Delta\theta_{\alpha_i}(q)|$ and $E(q_n) \to 0$ as $q_n \to p$. Notice also that $||\theta_{\mathcal{A}}(q_n)|| \neq 0$ since $q_n \notin \mathcal{F}$. Now by hypothesis,

$$\frac{\Delta\theta_{\alpha_i}(q_n)}{||\theta_{\mathcal{A}}(q_n)||} \to a_i,$$

so taking limits proves the first claim.

The proof of the second claim is similar. Expanding τ_{β_i} about p and taking imaginary parts we find:

$$\Delta\theta_{\beta_i}(q_n) = \sum_j \frac{\partial\tau_{\beta_i}}{\partial\tau_{\alpha_j}}(p)\Delta\theta_{\alpha_j}(q_n) + ||\theta_{\mathcal{A}}(q_n)||E(q_n)$$

and thus

$$\Delta\theta_{\beta_i}(q_n)/||\theta_{\mathcal{A}}(q_n)|| \to \sum_j a_j\frac{\partial\tau_{\beta_i}}{\partial\tau_{\alpha_j}}(p) = \frac{\partial\tau_{\beta_i}}{\partial\tau_\mu}(p).$$

On the other hand, by our hypothesis $\Delta\theta_{\beta_i}(q_n)/||\theta_{\mathcal{B}}(q_n)|| \to b_i$ for each i. Choose i_0 such that $b_{i_0} \neq 0$. Then

$$\frac{||\theta_{\mathcal{B}}(q_n)||}{||\theta_{\mathcal{A}}(q_n)||} = \frac{||\theta_{\mathcal{B}}(q_n)||}{\Delta\theta_{\beta_{i_0}}(q_n)}\frac{\Delta\theta_{\beta_{i_0}}(q_n)}{||\theta_{\mathcal{A}}(q_n)||} \to \frac{1}{b_{i_0}}\frac{\partial\tau_{\beta_{i_0}}}{\partial\tau_\mu}(p) = -k$$

say. Thus

$$\frac{\partial \tau_{\beta_i}}{\partial \tau_\mu}(p) = \lim \frac{\Delta\theta_{\beta_i}(q_n)}{||\theta_{\mathcal{A}}(q_n)||} = -kb_i$$

for each i, as claimed. □

6. Sufficient Conditions for Bending Away

In this section we prove a partial converse to the result in the last section. Suppose as usual that $\mathcal{A}$ and $\mathcal{B}$ are pants decompositions which fill up S, and that $p \in \mathcal{F}$ is lies on the line of minima $\mathcal{L}_{\mu,\nu}$ for some $\mu \in \mathcal{Q}(\mathcal{A})$, $\nu \in \mathcal{Q}(\mathcal{B})$. We give conditions under which p is in the closure of $\mathcal{P}_{\mu,\nu}$.

For $q \in \mathcal{QF}$, let $M = M(\mathcal{A}, \mathcal{B}, q)$ be the $d \times d$ matrix whose entry in the ith row and jth column is $\partial\lambda_{\beta_i}/\partial\tau_{\alpha_j}(q)$. The following lemma, trivially checked by multiplying out the equations in question, gives a relationship between critical points of length functions and null vectors of the matrix M.

LEMMA 6.1. *Let $\mathcal{A}, \mathcal{B}$ be any two pants decompositions of S and let $M = M(q)$ as above. Let $q \in \mathcal{QF}$ and let $(a_1, \dots, a_d), (b_1, \dots, b_d) \in \mathbb{R}^d$. Then:*

(1) $(b_1, \dots, b_d)M = 0$ *if and only if* $\sum_i b_i \partial\lambda_{\beta_i}/\partial\tau_{\alpha_j}(q) = 0, j = 1, \dots, d.$
(2) $M(a_1, \dots, a_d)^T = 0$ *if and only if* $\sum_i a_i \partial\lambda_{\beta_j}/\partial\tau_{\alpha_i}(q) = 0, j = 1, \dots, d.$

Remark. Notice that if $a_i, b_i \geqslant 0$ for all $i = 1, \dots, d$ then setting $\mu = \sum_i a_i\alpha_i$ and $\nu = \sum_i b_i\beta_i$ the above two conditions may be written more concisely as

(1) $(b_1, \dots, b_d)M = 0$ if and only if $\partial\lambda_\nu/\partial\tau_{\alpha_j} = 0,\ j = 1, \dots, d,$
(2) $M(a_1, \dots, a_d)^T = 0$ if and only if $\partial\lambda_{\beta_j}/\partial\tau_\mu = 0,\ j = 1, \dots, d.$

We shall also need the following result which describes the relationship between right and left null vectors of M.

LEMMA 6.2. *Suppose that $(a_1, \dots, a_d)^T$ is a right null vector of $M(\mathcal{A}, \mathcal{B}, q)$. Then there exist $(b_1, \dots, b_d)$ such that*

$$\sum_i b_i \frac{\partial}{\partial \tau_{\beta_i}} = \sum_i a_i \frac{\partial}{\partial \tau_{\alpha_i}}, \quad j = 1, \dots, d$$

at q. Moreover, this condition implies that that $(b_1, \dots, b_d)$ is a left null vector of M.

Proof. Write X for the tangent vector $\sum_i a_i(\partial/\partial\tau_{\alpha_i})$ at q. By the change of variable formula we have

$$X = \sum_i X(\lambda_{\beta_i}) \frac{\partial}{\partial \lambda_{\beta_i}} + \sum_i X(\tau_{\beta_i}) \frac{\partial}{\partial \tau_{\beta_i}}.$$

On the other hand, by Lemma 6.1 we have $X(\lambda_{\beta_i})(q) = 0$ for $i = 1, \dots, d$. This proves

the first assertion with $b_i = X(\tau_{\beta_i})$. Now let $Y = \sum_i b_i(\partial/\partial\tau_{\beta_i})(q)$. By Lemma 6.1, we just have to check that $Y(\lambda_{\beta_i})(q) = 0$ for all i. By construction, $Y(q) = X(q)$ and so $Y(\lambda_{\beta_i})(q) = X(\lambda_{\beta_i})(q) = 0$. □

COROLLARY 6.3. *If $p \in \mathcal{F}$ is a minimum of $l_\mu + kl_\nu$ for some $\mu \in \mathcal{Q}(\mathcal{A})$, $\nu \in \mathcal{Q}(\mathcal{B})$, $k > 0$, then* $\operatorname{rank} M(\mathcal{A}, \mathcal{B}, p) < d$.

Proof. At p, we have

$$\frac{\partial(\lambda_\mu + k\lambda_\nu)}{\partial\tau_{\alpha_j}} = 0, \quad j = 1, \ldots, d$$

and, hence,

$$\frac{\partial\lambda_\nu}{\partial\tau_{\alpha_j}} = 0, \quad j = 1, \ldots, d. \qquad \square$$

Now we state our second main theorem.

THEOREM 6.4. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are pants decompositions which fill up S and that $p \in \mathcal{F}$ is a global minimum of $l_\mu + kl_\nu$ for some $\mu \in \operatorname{Int}\mathcal{Q}(\mathcal{A})$, $\nu \in \operatorname{Int}\mathcal{Q}(\mathcal{B})$ and $k > 0$. Suppose also that* $\operatorname{rank} M(\mathcal{A}, \mathcal{B}, p) = d - 1$. *Then p is in the closure of $\mathcal{P}_{\mu,\nu}$.*

Proof. Let $\mu = \sum_i a_i\alpha_i$, $\nu = \sum_i b_i\beta_i$. By reordering the curves in $\mathcal{A}$ and $\mathcal{B}$ if necessary, we may assume that the $d - 1 \times d - 1$ matrix

$$J = \left(\frac{\partial\lambda_{\beta_i}}{\partial\tau_{\alpha_j}}(p)\right)_{i,j=1}^{d-1}$$

is nonsingular. It follows that $b_d \neq 0$, since otherwise $(b_1, \ldots, b_{d-1})$ is a left null vector of J. We claim the map $F \colon \mathcal{QF} \to \mathcal{QF}$,

$$F(\lambda_{\alpha_1}, \ldots, \lambda_{\alpha_d}, \tau_{\alpha_1}, \ldots, \tau_{\alpha_d}) \longmapsto (\lambda_{\alpha_1}, \ldots, \lambda_{\alpha_d}, \lambda_{\beta_1}, \ldots, \lambda_{\beta_{d-1}}, \tau_{\alpha_d})$$

is nonsingular at p. Indeed its Jacobian matrix is of the form

$$\begin{pmatrix} I & 0 & 0 \\ * & J & * \\ 0 & 0 & 1 \end{pmatrix}$$

which is clearly nonsingular. Thus

$$(\lambda_{\alpha_1}, \ldots, \lambda_{\alpha_d}, \lambda_{\beta_1}, \ldots, \lambda_{\beta_{d-1}}, \tau_{\alpha_d})$$

are local coordinates for $\mathcal{QF}$ near p. In particular, we can certainly move away from p keeping all of $\lambda_{\alpha_1}, \ldots, \lambda_{\alpha_d}, \lambda_{\beta_1}, \ldots, \lambda_{\beta_{d-1}}$ constant. Now $\lambda_\nu = \sum_i b_i\lambda_{\beta_i}$ and so, since $b_d \neq 0$, we have in addition $\lambda_{\beta_d} \in \mathbb{R}$ if and only if $\lambda_\nu \in \mathbb{R}$.

Write $\zeta = \tau_{\alpha_d} - \tau_{\alpha_d}(p)$ in the new coordinate system, so that $\zeta(p) = 0$. Let H denote the ζ-axis

$$\lambda_{\alpha_1} = \lambda_{\alpha_1}(p), \ldots, \lambda_{\alpha_d} = \lambda_{\alpha_d}(p), \lambda_{\beta_1} = \lambda_{\beta_1}(p), \ldots, \lambda_{\beta_{d-1}} = \lambda_{\beta_{d-1}}(p).$$

Thus on H, we can write $\lambda_\nu = \lambda_\nu(\zeta)$ and the restriction of $\lambda_\nu(\zeta)$ to H has a minimum at p.

Since ζ is real valued on $\mathcal{F}$, by Proposition 4.5 we have $\partial/\partial\zeta_{|_p} = \partial/\partial\tau_{\xi_{|_p}}$ for some $\xi \in \mathcal{ML}$. Since p is a global minimum for $\lambda_\mu + k\lambda_\nu$ on $\mathcal{F}$ and l_μ is constant on H, we have $\partial\lambda_\nu/\partial\zeta(p) = 0$, so that p is the minimum point for l_ν along the earthquake path along ξ through p. By Wolpert's theorem, the second derivative does not vanish, hence on H we have $\lambda_\nu = \lambda_\nu(\zeta) = h(\zeta)\zeta^2$ for some analytic function h with $h(0) > 0$. It is now an easy exercise in one variable complex analysis to find a path $t\colon [0, \varepsilon) \longmapsto \sigma(t) \in \mathcal{QF}$ such that $\sigma(0) = p$, $\sigma(t) \notin \mathcal{F}$ for $t > 0$, and along which $\lambda_\nu(\sigma(t)) \in \mathbb{R}$. Using Theorem 3.5, it remains only to show that all the angles $\theta_{\alpha_i} = \Im\tau_{\alpha_i}$ have the same sign along σ, and likewise for the angles θ_{β_i}. Now

$$\lim_{t\to 0}\left[\frac{\theta_{\alpha_1}}{||\theta_{\mathcal{A}}||}, \ldots, \frac{\theta_{\alpha_d}}{||\theta_{\mathcal{A}}||}\right] = \left[\frac{\mathrm{d}\theta_{\alpha_1}}{\mathrm{d}\zeta}(p), \ldots, \frac{\mathrm{d}\theta_{\alpha_d}}{\mathrm{d}\zeta}(p)\right].$$

Since clearly

$$0 = \sum_j \frac{\partial\lambda_{\beta_i}}{\partial\tau_{\alpha_j}}(p)\frac{\mathrm{d}\lambda_{\alpha_i}}{\mathrm{d}\zeta}(p),$$

we conclude that $(\mathrm{d}\theta_{\alpha_i}/\mathrm{d}\zeta(p))^T$ is a right null vector of $M(p)$. Since $\operatorname{rank} M(p) = d - 1$, we see that $(\mathrm{d}\theta_{\alpha_i}/\mathrm{d}\zeta(p))^T$ must be a multiple of $(a_1, \ldots, a_d)^T$. Since $\mu \in \operatorname{Int} \mathcal{Q}(\mathcal{A})$, all the a_i are strictly positive and the result follows. □

Based on Theorems 5.1 and 6.4, we make the following conjecture:

CONJECTURE 6.5. *Let $\mu, \nu \in \mathcal{ML}$ be laminations which fill up S. Then the closure of $\mathcal{P}_{\mu,\nu}$ meets $\mathcal{F}$ precisely in the Kerckhoff critical line $\mathcal{L}_{\mu,\nu}$.*

Remarks. (1) It is not hard to show that $\operatorname{rank} M(p) = d - 1$ provided that p lies on a line $\mathcal{L}_{\mu,\nu}$ for a *unique* pair μ, ν in the interior of $\mathcal{Q}(\mathcal{A}) \times \mathcal{Q}(\mathcal{B})$. On the other hand, in [4] we show by computation that the degenerate case $\operatorname{rank} M(p) < d - 1$ definitely occurs; in fact we have an example with $d = 2$ and a point p at which all coefficients in $M(p)$ vanish.

(2) The difficulty in extending 6.4 to the general case is exemplified by the above example. We have length functions $l_1, l_2\colon \mathbb{C}^2 \to \mathbb{C}$ which are real valued on $\mathbb{R}^2$, such that $\partial l_i/\partial z_j(p) = 0$ for all i, j. One needs to show that for any $a_i \geqslant 0$, there is a path starting from p into $\mathbb{C}^2$ along which both l_1 and l_2 stay real valued and such that the imaginary parts of z_1, z_2 are in the ratio $[a_1, a_2]$ as $(z_1, z_2) \to 0$. This statement

is false for arbitrary holomorphic functions l_j: consider for example

$$p = 0,\ l_1(z_1, z_2) = z_1^2 + z_2^2,\ l_2(z_1, z_2) = z_1^2 + z_2^2 + (z_1 + z_2)^3.$$

We believe it to be true in the present case and hope to give a proof elsewhere.

(3) Bonahon and Otal [2] have shown that $\mathcal{P}_{\mu,\nu}$ is non-empty for any pair μ, ν which fill up S. Given this result, one might well be able to prove Conjecture 6.5. However their methods are based on the Hodgson Kerckhoff deformation theory of cone manifolds. It would be nice to have a direct proof based on the more elementary techniques developed here.

7. Bending Away From Curves of Fixed Length

In this section we discuss the implications of Kerckhoff's picture of lines of minima for the arrangement of pleating varieties which meet $\mathcal{F}$ along a locus at which one or several curves have prescribed fixed lengths. Our discussion is motivated by the following result, proved in [11]:

THEOREM 7.1. *Suppose S is a once punctured torus and that laminations $\mu, \nu \in \mathcal{ML}$ fill up S. Then for any $c > 0$, the closure of the pleating variety $\mathcal{P}_{\mu,\nu}$ meets $\mathcal{F}$ at a unique point on the locus $l_\mu(p) = c$.*

For a general surface S, as usual we let $\mathcal{A} = \{\alpha_i\}$ be a pants decomposition and let $\gamma \in \mathcal{S}(S)$. For $c > 0$, define the *horocycle plane* $\mathcal{H}_{\gamma;c} = \{p \in \mathcal{F}: l_\gamma(p) = c\}$ and for $\mathbf{c} = (c_1, \ldots, c_d)$, $c_i > 0$, define the *shearing plane* $\mathcal{E} = \mathcal{E}_{\mathcal{A},\mathbf{c}} = \{p \in \mathcal{F}: l_{\alpha_i}(p) = c_i,\ i = 1, \ldots, d\}$. As usual, we let $\mathcal{ML}_\gamma = \{\nu \in \mathcal{ML}: \gamma, \nu \text{ fill up } S\}$. We shall prove the following:

THEOREM 7.2. *For each $\nu \in \mathcal{ML}_\gamma$, the function l_ν has a unique minimum on $\mathcal{H}_{\gamma;c}$. This point p_ν is also the unique point in which the Kerckhoff line of minima for the pair δ_γ, ν meets $\mathcal{H}_{\gamma;c}$. The map $\mathcal{PML}_\gamma \to \mathcal{H}_{\gamma,c}$ which sends to $[\nu]$ to $p_{j([\nu])}$ is a homeomorphism.*

Assuming Conjecture 6.5, this means that the closure of the pleating variety $\mathcal{P}_{\gamma,\nu}$ meets $\mathcal{H}_{\gamma,c}$ in a unique point, generalising Theorem 7.1 to the special case in which the support of μ is a single curve γ. By contrast, the following result shows that if we prescribe the lengths of all the curves in a pants decomposition, then we cannot hope to obtain all possible pleating varieties $\mathcal{P}_{\mu,\nu}$ for which $\mu \in \mathcal{Q}(\mathcal{A})$ and μ, ν fill up S. Choose μ and let $\mathcal{ML}_\mathcal{A}$ denote the set of laminations ν such that μ, ν fill up S. Clearly, $\mathcal{ML}_\mathcal{A}$ is independent of the choice of μ.

THEOREM 7.3. *Let $\mathcal{A}$ be a pants decomposition of S and let $\mathbf{c} = (c_1, \ldots, c_d)$, $c_i > 0$. Then there is an open set $U \subset \mathcal{ML}_\mathcal{A}$ such that for $\nu \in U$, there are no laminations $\mu \in \mathcal{Q}(\mathcal{A})$ for which $\mathcal{L}_{\mu,\nu}$ meets $\mathcal{E}_{\mathcal{A},\mathbf{c}}$.*

COROLLARY 7.4. *There exist laminations $\mu \in \mathcal{Q}(\mathcal{A})$ and $\nu \in \mathcal{ML}_{\mathcal{A}}$ for which the closure of $\mathcal{P}_{\mu,\nu}$ does not meet $\mathcal{E}$.*

Proof. This follows immediately from the above result in combination with Theorem 5.1. □

In [4] we give an example of a pants decomposition $\mathcal{A}$ and a curve $\gamma \in \mathcal{ML}_{\mathcal{A}}$ for which the closure of $\mathcal{P}_{\mathcal{A},\gamma}$ meets $\mathcal{E}_{\mathcal{A},\mathbf{c}}$ for *any* choice of $\mathbf{c}$.

We begin with the proof of Theorem 7.2.

PROPOSITION 7.5. *Let $\gamma \in \mathcal{S}(S)$ and let $c > 0$. Then any vector X in the tangent space $T_p(\mathcal{H})$ to $\mathcal{H} = \mathcal{H}_{\gamma,c}$ at p can be written in the form*

$$a\frac{\partial}{\partial t_\gamma} + b\frac{\partial}{\partial t_\eta}$$

where $a, b \in \mathbb{R}$ and $\eta \in \mathcal{ML}(S_\gamma)$.

Proof. Choose a pants decomposition $\mathcal{A} = \{\alpha_i\}_{i=1}^d$ for S in which $\alpha_1 = \gamma$. For convenience write

$$x_{2i-1} = l_{\alpha_i}, x_{2i} = t_{\alpha_i}, \quad i = 2, \dots, d$$

so that $\partial/\partial t_\gamma, \partial/\partial x_j, j = 3, \dots, 2d$ are a basis for $T_p(\mathcal{H})$.

Let $\mathcal{ML}(S_\gamma)$ be the set of measured laminations on S whose support does not intersect γ. Using Dehn Thurston coordinates for $\mathcal{ML}$ (see, for example, [21] or the discussion preceding Lemma 7.11 below), one sees easily that $\mathcal{ML}(S_\gamma)$ is homeomorphic to $\mathbb{R}^{2(d-1)}$. If $\eta \in \mathcal{ML}(S_\gamma)$, then the time t earthquake $\mathcal{E}_\eta(t)$ clearly leaves l_γ invariant and thus $\partial/\partial t_\eta \in T_p(\mathcal{H})$. (Notice however that *a priori* $\mathcal{E}_\eta(t)$ may change the twist coordinate t_γ. This is the reason for the complication of this proof.) Thus at p we have

$$\frac{\partial}{\partial t_\eta} = a(\eta)\frac{\partial}{\partial t_\gamma} + \sum_{i=3}^{2d} a_i(\eta)\frac{\partial}{\partial x_i}.$$

We claim that the map

$$\phi: \mathcal{ML}(S_\gamma) \to \mathbb{R}^{2d-2}, \qquad \phi(\eta) = (a_3(\eta), \dots, a_{2d}(\eta))$$

is injective. Recall that by Proposition 4.5, the map

$$\psi: \mathcal{ML}(S) \to T_p(\mathcal{F}), \qquad \psi(\zeta) = \frac{\partial}{\partial t_\zeta}$$

is a homeomorphism.

Suppose that $\phi(\eta) = \phi(\eta')$, $\eta, \eta' \in \mathcal{ML}(S_\gamma)$. If $a(\eta) = a(\eta')$ there is nothing to prove, so suppose that $a(\eta) > a(\eta')$. Then we can write

$$\frac{\partial}{\partial t_\eta} = (a(\eta) - a(\eta'))\frac{\partial}{\partial t_\gamma} + \frac{\partial}{\partial t_{\eta'}}.$$

Since by hypothesis the supports of η and γ are disjoint, the right-hand side is of the form $\partial/\partial t_\zeta$ for some $\zeta \in \mathcal{ML}$. Since ψ is injective we conclude that $\eta = \zeta$. This forces $a(\eta) = a(\eta')$, since by hypothesis the support of η is disjoint from γ. Now ϕ is clearly continuous, and so by invariance of domain it must be open. Finally $\phi(s\eta) = s\phi(\eta)$ for $s > 0$, from which we conclude that ϕ is a homeomorphism onto $\mathbb{R}^{2d-2}$.

Now if $X \in T_p(\mathcal{H})$, write

$$X = x_\gamma \frac{\partial}{\partial t_\gamma} + \sum_{i=3}^{2d} x_i \frac{\partial}{\partial x_i}, \quad x_\gamma, x_i \in \mathbb{R}.$$

Use the surjectivity of ϕ to find $\eta \in \mathcal{ML}_\gamma$ with

$$\frac{\partial}{\partial t_\eta} = a(\eta)\frac{\partial}{\partial t_\gamma} + \sum_{i=3}^{2d} x_i \frac{\partial}{\partial x_i}.$$

Combining these two equations gives the result. □

PROPOSITION 7.6. *Let $\gamma \in \mathcal{S}$ and let $c > 0$. For each $\nu \in \mathcal{ML}_\gamma$, the function l_ν has a unique minimum on $\mathcal{H} = \mathcal{H}_{\gamma;c}$. This point is also the unique intersection of the Kerckhoff line of minima for ν, γ with $\mathcal{H}$.*

Proof. To show a minimum exists, it is enough to show that the restriction of l_ν to $\mathcal{H}$ is proper. The proof of Kerckhoff [15], theorem 1.2, shows that $l_\nu + l_\gamma$ is proper on $\mathcal{F}$, but l_γ is constant on $\mathcal{H}$. The result follows. Now we show that every minimum point of l_ν lies on $\mathcal{L}_{\gamma,\nu}$. Denote such a minimum point by p_ν. Choose disjoint curves $\alpha_2, \dots, \alpha_d \in \mathcal{S}$ such that $\gamma = \alpha_1, \alpha_2, \dots, \alpha_d$ are a pants decomposition for S. We have

$$\frac{\partial}{\partial t_\nu} = \sum_i \frac{\partial l_{\alpha_i}}{\partial t_\nu}\frac{\partial}{\partial l_{\alpha_i}} + \sum_i \frac{\partial t_{\alpha_i}}{\partial t_\nu}\frac{\partial}{\partial t_{\alpha_i}}.$$

Now the tangent vectors $\partial/\partial t_{\alpha_i}$, $i = 1, \dots, d$ all lie in the tangent space to $\mathcal{H}$ and so since the restriction of l_ν to $\mathcal{H}$ is minimum at p_ν we have

$$\frac{\partial l_{\alpha_i}}{\partial t_\nu}(p_\nu) = -\frac{\partial l_\nu}{\partial t_{a_i}}(p_\nu) = 0, \quad i = 1, \dots, d.$$

Therefore at p_ν,

$$\frac{\partial}{\partial t_\nu} = \sum_i \frac{\partial t_{\alpha_i}}{\partial t_\nu}\frac{\partial}{\partial t_{\alpha_i}}.$$

Since $\partial/\partial t_{\alpha_i}$ is in the tangent space $T_p(\mathcal{H})$ for all $i > 1$, so is $\partial/\partial t_\nu$. Thus by

Proposition 7.5, there exist $a, b \in \mathbb{R}$ such that at p_ν,

$$\frac{\partial}{\partial t_\nu} = a\frac{\partial}{\partial t_\gamma} + b\frac{\partial}{\partial t_\eta} \tag{2}$$

for some $\eta \in \mathcal{ML}(S_\gamma)$. We claim that $b = 0$. Choose any $\xi \in \mathcal{ML}(S_\gamma)$. Since $\partial/\partial t_\xi \in T_p(\mathcal{H}_\gamma)$ and since p_ν is the minimum of l_ν, we have $\partial l_\xi/\partial t_\nu = -(\partial l_\nu/\partial t_\xi) = 0$. On the other hand, from Equation (2) we find

$$\frac{\partial l_\xi}{\partial t_\nu} = a\frac{\partial l_\xi}{\partial t_\gamma} + b\frac{\partial l_\xi}{\partial t_\eta}$$

at p_ν. Since $\partial l_\xi/\partial t_\gamma = 0$ we conclude that $b(\partial l_\xi/\partial t_\eta) = 0$ for all $\xi \in \mathcal{ML}(S_\gamma)$. Without loss of generality, we may suppose that $b = 1$.

As in the proof of Proposition 7.5, write

$$\frac{\partial}{\partial t_\xi} = a(\xi)\frac{\partial}{\partial t_\gamma} + \sum_{i=3}^{2d} a_i(\xi)\frac{\partial}{\partial x_i}.$$

Since $\eta \in \mathcal{ML}_\gamma$, we have $\partial l_\eta/\partial t_\gamma = 0$ and so

$$\sum_{i=3}^{2d} a_i(\xi)\frac{\partial l_\eta}{\partial x_i} = 0.$$

From the proof of Proposition 7.5, this forces $\partial l_\eta/\partial x_i = 0$, $i = 3, \ldots, 2d$ which gives

$$\mathrm{d}l_\eta = \frac{\partial l_\eta}{\partial l_\gamma}\mathrm{d}l_\gamma + \frac{\partial l_\eta}{\partial t_\gamma}\mathrm{d}t_\gamma + \sum_i \frac{\partial l_\eta}{\partial x_i}\mathrm{d}x_i = \frac{\partial l_\eta}{\partial l_\gamma}\mathrm{d}l_\gamma$$

at p, in other words, $\mathrm{d}l_\eta = k\mathrm{d}l_\gamma$ for some $k \in \mathbb{R}$. Now by Proposition 4.5 again, the map $\mathcal{ML} \to T_p^*(\mathcal{F})$, $\zeta \mapsto \mathrm{d}l_\zeta$ is a homeomorphism which clearly commutes with multiplication by positive scalars. Thus if $k > 0$ we have $\eta = k\gamma$ which is certainly not the case. If $k < 0$ then $\eta - k\gamma \in \mathcal{ML}$ and $d(\eta - k\gamma) = 0$, which forces $\eta = k\gamma = 0$. Thus $\eta = 0$ as claimed.

It follows immediately from Corollary 4.8 that $a < 0$ and that $l_\nu - al_\gamma$ has its global minimum at p_ν. To conclude the proof, note that it is obvious that any intersection of the Kerckhoff line of minima $\mathcal{L}_{\gamma,\nu}$ with $\mathcal{H}$ is also a global minimum for l_ν on $\mathcal{H}$. Suppose that l_ν had minima at points $p, p' \in \mathcal{H}$. By the above argument, there exist $k, k' > 0$ so that $l_\nu + kl_\gamma, l_\nu + k'l_\gamma$ have their unique global minimum at p, p' respectively. Since $l_\gamma(p) = l_\gamma(p')$ we deduce immediately that $l_\nu(p) < l_\nu(p')$ and vice versa, which is impossible. □

Proof of Theorem 7.2. To complete the proof, we have only to show that the map $\mathrm{P}\mathcal{ML}_\gamma \to \mathcal{H}, [\nu] \to p_{j([\nu])}$ is continuous and surjective. Surjectivity is clear from Theorem 4.4. Define the map $[\nu] \to k([\nu])$ by the condition that $l_\gamma + k([\nu])l_{j([\nu])}$ has its unique minimum at $p_\nu \in \mathcal{H}$. This map is continuous because its graph is the closed

set $\Phi_\gamma^{-1}(\mathcal{H})$, where $\Phi_\gamma\colon \mathrm{P}\mathcal{ML}_\gamma \times (0,\infty) \to \mathcal{QF}$ is Kerckhoff's map as in Theorem 4.4. Continuity of $[v] \to \Phi_\gamma([v], k([v])) = p_v$ follows. $\square$

At the other extreme, one is interested in which pleating planes can meet the shearing plane $\mathcal{E} = \mathcal{E}_{\mathcal{A},\mathbf{c}}$. Now $\mathcal{E}$ is a d-dimensional submanifold of $\mathcal{F}$, but there is in addition a $d-1$-dimensional choice of weighting for the bending angles along α_i. As we shall see, this allows us to map injectively onto a domain in $\mathrm{P}\mathcal{ML}$. As usual, let $\mathcal{A}$ be a fixed pants decomposition of S and as in Section 5, let $\mathcal{Q}$ denote the positive cone $\{(s_1,\dots,s_d)\colon s_i \geqslant 0, i=1,\dots,d, \sum_i s_i = 1\}$ in $\mathbb{R}^d$. For $\mathbf{s}\in\mathcal{Q}$, denote by $\mu_\mathbf{s}$ the lamination $\sum_i s_i\alpha_i$ and let $\mathcal{Q}(\mathcal{A}) = \{\mu_\mathbf{s}\colon \mathbf{s}\in\mathcal{Q}\}$. As above, let $\mathcal{ML}_\mathcal{A}$ denote the set of laminations $v\in\mathcal{ML}$ such that v and μ fill up S for some, and hence any, choice of μ in the interior of $\mathcal{Q}(\mathcal{A})$. Let $p\in\mathcal{E}$ and let $\mathbf{s}\in\mathcal{Q}$. By Kerckhoff's Theorem 4.4, there exist a unique $k\in(0,\infty)$ and $[v]\in\mathrm{P}\mathcal{ML}$ such that $\mu_\mathbf{s}, v$ fill up S and such that $l_{\mu_\mathbf{s}} + kl_{j(v)}$ has its global minimum at p. Define a map $\Psi\colon \mathcal{E}\times\mathcal{Q} \to \mathrm{P}\mathcal{ML}$ by $(p,\mathbf{s}) \mapsto [v]$.

The point of studying the map Ψ is the following. By Theorem 5.1, the closure of the pleating variety $\mathcal{P}_{v,\mathcal{A}}$ meets $\mathcal{E}$ at p only if p is on the line of minima $\mathcal{L}_{v,\mu_\mathbf{s}}$ for some $\mathbf{s}\in\mathcal{Q}$. In other words, the possible pleating laminations which can occur as we bend along geodesics α_i with fixed lengths c_i but varying ratios of the bending angles must be contained in the range of the map Ψ. Assuming our Conjecture 6.5, the range of the map Ψ exactly describes the projective classes of pleating laminations which can occur.

THEOREM 7.7. *The map* $\Psi\colon\mathcal{E}\times\mathcal{Q}\to\mathrm{P}\mathcal{ML}$ *is a homeomorphism onto its image.*

We begin the proof with:

PROPOSITION 7.8. *The map* $\Psi\colon\mathcal{E}\times\mathcal{Q}\to\mathrm{P}\mathcal{ML}$ *is a continuous injection.*

Proof. We first show that Ψ is injective. So suppose that $\Psi(p,\mathbf{s}) = \Psi(p',\mathbf{s}') = [v]$. Thus there exists $k\in(0,\infty)$ such that $l_{\mu_\mathbf{s}} + kl_{j(v)}$ has its unique global minimum on $\mathcal{F}$ at $p\in\mathcal{E}$ and so $l_{\mu_\mathbf{s}} + kl_{j(v)}(p) < l_{\mu_\mathbf{s}} + kl_{j(v)}(p')$. Observing that $l_{\mu_\mathbf{s}}$ is constant on $\mathcal{E}$, we deduce immediately that $l_{j(v)}(p) < l_{j(v)}(p')$. Interchanging p' with p gives an impossibility unless $p'=p$. Now suppose that both $l_{\mu_\mathbf{s}} + kl_{j(v)}$ and $l_{\mu_{\mathbf{s}'}} + k'l_{j(v)}$ have their global minima at p. From Proposition 4.6, we have

$$\frac{\partial}{\partial t_v} = -\frac{1}{k}\frac{\partial}{\partial t_{\mu_\mathbf{s}}} = -\frac{1}{k'}\frac{\partial}{\partial t_{\mu_{\mathbf{s}'}}}.$$

Since the vectors $\partial/\partial\tau_{\alpha_i}, i=1,\dots,d$ are independent, this forces $k=k'$ and $\mathbf{s}=\mathbf{s}'$.

To show continuity we use a variant of the argument in Kerckhoff [15], theorem 2.1, part II. Assume that $(p_n,\mathbf{s}_n)\to(p_0,\mathbf{s}_0)$ in $\mathcal{E}\times\mathcal{Q}$ and let $[v_n] = \Psi(p_n,\mathbf{s}_n)$. By definition, there exist $u_n = k_n/1+k_n \in (0,1)$ such that $(1-u_n)l_{\mu_{\mathbf{s}_n}} + u_n l_{j([v_n])}$ has its global minimum at p_n. At the minimum, $(1-u_n)\mathrm{d}l_{\mu_{\mathbf{s}_n}} + u_n\mathrm{d}l_{j([v_n])} = 0$.

Passing to a subsequence, we may assume $u_n\to u_0$ in $[0,1]$ and $[v_n]\to[v_0]$ in $\mathrm{P}\mathcal{ML}$. As proved by Kerckhoff, [14], corollary 2.2, the function which takes $v\in\mathcal{ML}$ to l_v is

continuous with respect to the C^∞ topology on compact subsets of $\mathcal{F}$. Hence, $(1-u_0)\mathrm{d}l_{\mu_{s_0}} + u_0\mathrm{d}l_{j([v_0])} = 0$ at p_0. Following Kerckhoff, since $dl_{j([v_0])} \neq 0$ and $\mathrm{d}l_{\mu_{s_0}} \neq 0$, we must have that $u_0 \neq 0, 1$. We conclude that $(1-u_0)l_{\mu_{s_0}} + u_0 l_{j([v_0])}$ has a critical point at p_0. By Theorem 4.1, p_0 must be a global minimum, which shows that $\Psi(p_0, s_0) = [v_0]$ as required. □

We deduce from the above proposition that the restriction of Ψ to the interior $\mathcal{E} \times \mathrm{Int}\, \mathcal{Q}$ is a continuous injection between spaces of the same dimension, and hence a homeomorphism by invariance of domain. However to prove that Ψ extends to a homeomorphism on the boundary $\mathcal{E} \times \partial\mathcal{Q}$ requires some more work. The proof uses the action on $\mathcal{E}$ of the group Γ generated by Dehn twists about the curves α_i, which clearly map $\mathcal{E}$ to itself. Let tw_i be the left Dehn twist about α_i. Clearly, Γ is the free abelian group on generators $\{tw_i\}_{i=1}^d$. The shearing plane $\mathcal{E}$ is parametrised by the twist coordinates $t_{\alpha_i}, i = 1, \dots, d$. Clearly $t_j(tw_i(p)) = t_j(p)$ if $i \neq j$ and $t_i(tw_i(p)) = t_i(p) - c_i$, where $c_i = l_{\alpha_i}$ which by definition is constant on $\mathcal{E}$. From this it is clear that $\Delta = \{p \in \mathcal{E}: 0 \leqslant t_i(p) \leqslant c_i, i = 1, \dots, d\}$ is a fundamental domain for the action of Γ on $\mathcal{E}$.

Since Γ is contained in the mapping class group of S, it induces an action both on $\pi_1(S)$ and on $\mathcal{F}$. More generally, any quasiconformal homeomorphism f of S induces a homeomorphism f^* of $\mathcal{F}$ and an automorphism f_* of $\pi_1(S)$, (see for example [18]). By linearity and continuity, the action of f_* extends to all laminations $v \in \mathcal{ML}$. In particular, it is clear that Γ fixes every point $\mu_{\mathbf{s}} \in \mathcal{ML}(\mathcal{A})$.

Recall from Section 2.1 that a point $p \in \mathcal{F}$ is a pair (ρ, ϕ) where ρ is a representation $\rho: \pi_1(S) \to PSL(2, \mathbb{R})$ of $\pi_1(S)$ as a Fuchsian group and ϕ is a quasi-conformal homeomorphism ϕ of $\hat{\mathbb{C}}$ which conjugates the representations $\rho(p)$ and $\rho(p_0)$ for some fixed base point p_0. The maps f^* and f_* are connected, see [18], page 15, by the relation that $f^*(\rho, \phi)$ is conjugate in $PSL(2, \mathbb{R})$ to the representation $\gamma \mapsto \rho \circ (f_*)^{-1}(\gamma)$.

LEMMA 7.9. *The map Ψ is equivariant with respect to the action of Γ on $\mathcal{E}$; more precisely,*

$$\Psi(g^*(p), \mathbf{s}) = g_*(\Psi(p, \mathbf{s}))$$

for all $g \in \Gamma, p \in \mathcal{E}, \mathbf{s} \in \mathcal{Q}$.

Proof. Let f be a quasiconformal homeomorphism of S. Using the above relation between f^* and f_* we see that $\mathrm{Tr}(\gamma)(f^*(\rho, \phi)) = \mathrm{Tr}(\rho \circ (f_*)^{-1}(\gamma)) = \mathrm{Tr}((f_*)^{-1}(\gamma))(\rho, \phi)$ and, hence, that $l_\gamma(p) = l_{(f_*)^{-1}(\gamma)}((f^*)^{-1}(p))$ for all $\gamma \in \mathcal{S}$. By linearity and continuity, the same relation extends to all laminations $v \in \mathcal{ML}$.

Now let $g \in \Gamma, p \in \mathcal{E}, \mathbf{s} \in \mathcal{Q}$ and set $\Psi(g^*(p), \mathbf{s}) = v$. By definition, there exists a unique $k = k(p, \mathbf{s}) \in (0, \infty)$ such that $l_{\mu_{\mathbf{s}}} + kl_v$ has its global minimum at p. By the above, $l_v(g^*(p)) = l_{g_*(v)}(p)$ while $l_{\mu_{\mathbf{s}}}(g^*p) = l_{g_*(\mu_{\mathbf{s}})}(p) = l_{\mu_{\mathbf{s}}}(p)$. Thus $l_{\mu_{\mathbf{s}}} + kl_{g_*(v)}$ has its global minimum at $g^*(p)$. It follows that $\Psi(g^*(p), \mathbf{s}) = g_*(v)$ as claimed. □

There is also a simple expression for the action of Γ on $\mathcal{ML}$, in terms of the Dehn Thurston twist coordinates for $\mathcal{ML}$. These coordinates are the analogue for curves of the Fenchel Nielsen coordinates for $\mathcal{F}$ and are described for example in [21] section 1.2. As usual, they are defined relative to a fixed pants decomposition $\mathcal{A}$ of S. Suppose that $\gamma \in \mathcal{S}$. The coordinates $(p_i(\gamma), q_i(\gamma))$ take values in $\mathbb{R} \times (\mathbb{R}^+ \cup 0)$ for $i = 1, \dots, d$. For each i, $q_i(\gamma)$ is just the intersection number $i(\gamma, \alpha_i)$. We shall not need the precise details of the definition of the twist coordinate $p_i(\gamma)$. Roughly, let A_i be a closed annulus with core curve α_i and mark A_i with a fixed simple arc η_i between the two boundaries. The surface $S - \cup A_i$ is a disjoint union of pants Π_j. Homotop γ in such a way that γ cuts the closure of each pair of pants Π_j in the standard way as shown in [21], figure 1.2.2 or [6], exposé 6, Section III. (Essentially this means so that there is no twisting around the boundaries $\partial\Pi_j$.) Then $|p_i(\gamma)|$ is the minimum intersection number of the arcs $\gamma \cap A_i$ with η_i up to homotopy *rel* ∂A_i; and the sign of $p_i(\gamma)$ is taken positive if the twist is in the direction of a positive Dehn twist around A_i and negative otherwise.

The Dehn Thurston theorem, see [21], theorems 1.2.1 and 3.1.1, is that $\gamma \mapsto (p_1(\gamma), q_1(\gamma), \dots, p_d(\gamma), q_d(\gamma))$ extends by linearity and continuity to a homeomorphism $\mathrm{P}\mathcal{ML} \to (\mathbb{R} \times \mathbb{R}^+ \cup \{0\})^d$.

The action of Γ has a nice form in these coordinates. It is easy to see that

$$q_i(tw_i(\gamma)) = q_i(\gamma) \quad \text{and} \quad p_i(tw_i(\gamma)) = p_i(\gamma) + q_i(\gamma), \tag{3}$$

see [21] addendum. As usual, this formula extends by linearity and continuity to $\mathrm{P}\mathcal{ML}$.

We also note:

LEMMA 7.10. *Let $\mathcal{A} = \{\alpha_1, \dots, \alpha_d\}$ be a pants decomposition of S. Then $\nu \in \mathcal{ML}_{\mathcal{A}}$ if and only if $q_i(\nu) = i(\nu, \alpha_i) > 0$, $i = 1, \dots, d$. If $q_i(\nu) = 0$ for all i then ν is equivalent to a lamination in $\mathcal{Q}(\mathcal{A})$.*

Proof. If $\nu \in \mathcal{ML}_{\mathcal{A}}$ then by definition $\nu, \mu_{\mathbf{s}}$ fill up S for some $\mathbf{s} \in \operatorname{Int} \mathcal{Q}$. Thus $i(\nu, \alpha_i) + i(\mu_{\mathbf{s}}, \alpha_i) > 0$ for all i, and $i(\mu_{\mathbf{s}}, \alpha_i) = 0$. Conversely, the condition $i(\nu, \alpha_i) > 0$ implies that ν assigns nonzero weight to each boundary curve of each pair of pants in the decomposition obtained by cutting S along each of the curves α_i. Using the analysis of [6], exposé 6, we see that in all cases the curves α_i together with arcs in the support of ν decompose S into connected pieces of the required kind. The second assertion follows because there are no laminations all of whose leaves are disjoint from $\mathcal{A}$. (Notice that from our definition of $\mathcal{ML}$, if $i(\nu, \alpha_i) = 0$ then $|\nu| \cap \alpha_i = \emptyset$ or $\alpha_i \subset |\nu|$.) $\square$

Now we can prove the following lemma which is needed to complete the proof of Theorem 7.7.

LEMMA 7.11. *Suppose that $(p_n, \mathbf{s}_n) \in \mathcal{E} \times \mathcal{Q}$ and that $[\nu_n] = \Psi(p_n, \mathbf{s}_n) \to [\nu_0]$ in $\mathrm{P}\mathcal{ML}$. Then either the sequence p_n stays in a compact subset of $\mathcal{E}$, or $\nu_0 \in \mathcal{Q}(\mathcal{A})$.*

Proof. If $g = \Pi t w_i^{m^i} \in \Gamma$, write $|g| = \sum_i |m^i|$. Suppose that p_n exits every compact set in $\mathcal{E}$. Then there exist $g_n \in \Gamma$ such that $|g_n| \to \infty$ and such that $g_n(p_n) \in \Delta$, where Δ is the fundamental domain for the action of Γ defined above. Passing to a subsequence, we may assume that $g_n(p_n) \to p_*$ and $\mathbf{s}_n \to \mathbf{s}_*$. Thus $\Psi(g_n(p_n), \mathbf{s}_n) \to \Psi(p_*, \mathbf{s}_*) = [v_*]$ say. For $v \in \mathcal{ML}$, let

$$(p_1(v), q_1(v), \ldots, p_d(v), q_d(v))$$

denote the Dehn Thurston coordinates of v, and let $|v| = \sum_i |p_i(v)| + q_i(v)$. Let $[v_n] = \Psi(p_n, \mathbf{s}_n)$. Choose $\omega_n \in g_n([v_n])$ and $v_* \in [v_*]$ with $|\omega_n| = |v_*| = 1$. By equivariance, we have $\Psi(g_n(p_n), \mathbf{s}_n) = g_n(\Psi(p_n, \mathbf{s}_n)) = g_n([v_n]) = [\omega_n]$ and so $\omega_n \to v_*$. Since $[v_*]$ is in the range of Ψ, we know that $v_* \in \mathcal{ML}_{\mathcal{A}}$ and so by Lemma 7.10, $q_i(v_*) > 0$. Thus $q_i(\omega_n)$ is bounded away from 0 for all sufficiently large n.

Now let $v_n = g_n^{-1}\omega_n$ and let $g_n = \Pi t w_i^{m_n^i}$. Using the formula (3) for the action of Γ on the Dehn Thurston coordinates, $p_i(v_n) = p_i(\omega_n) + m_n^i q_i(\omega_n)$ while $q_i(v_n) = q_i(\omega_n)$. Since $|g_n| = \sum_i |m_n^i| \to \infty$, we have $|v_n| \to \infty$ and $q_i(v_n)/|v_n| \to 0$.

Choose $v_0 \in [v_0]$ with $|v_0| = 1$. By our hypothesis $v_n/|v_n| \to v_0/|v_0|$ so that $q_i(v_0/|v_0|) = 0$. The result follows from Lemma 7.10. □

Finally we can complete the proof of Theorem 7.7. It remains only to check that if $[v_n] = \Psi(p_n, \mathbf{s}_n), [v_0] = \Psi(p_0, \mathbf{s}_0)$ and if $[v_n] \to [v_0]$ as $n \to \infty$, then $p_n \to p_0$ and $\mathbf{s}_n \to \mathbf{s}_0$. Since $[v_0] = \Psi(p_0, \mathbf{s}_0)$, we know that $v_0 \in \mathcal{ML}_{\mathcal{A}}$. It follows from Lemma 7.11 that the sequence p_n is contained in a compact subset of $\mathcal{E}$. Passing to a subsequence, we may suppose that $p_n \to p_*$ say, and also that $\mathbf{s}_n \to \mathbf{s}_*$. Thus $\Psi(p_0, s_0) = [v_0] = \lim_n \Psi(p_n, \mathbf{s}_n) = \Psi(p_*, \mathbf{s}_*)$ so that $p_* = p_0$ and $\mathbf{s}_* = \mathbf{s}_0$ by injectivity of Ψ. The result follows. □

THEOREM 7.12. *The map Ψ is not surjective, more precisely, $\Psi(\mathcal{E} \times \mathcal{Q})$ is not dense in $\mathcal{ML}_{\mathcal{A}}$.*

Proof. Consider the continuous map $h: \mathcal{E} \times \mathcal{Q} \to P\mathbb{R}^d$ given by $h(p, \mathbf{s}) = [q_1(\Psi(p, \mathbf{s})), \ldots, q_d(\Psi(p, \mathbf{s}))]$. Clearly, $h(g(p), \mathbf{s}) = h(p, \mathbf{s})$ for all $g \in \Gamma$. In brief, h induces a map from the compact space $\mathcal{E} \times K/\Gamma$ to the noncompact space $P\mathcal{ML}_{\mathcal{A}}/\Gamma$ and, hence, cannot be surjective.

In more detail, because of the Γ-invariance of h, the image of h equals the image of its restriction to $\Delta \times \mathcal{Q}$ which is compact. By Lemma 7.10, we know that $q_i(\Psi(p, \mathbf{s})) > 0$ for all i and we deduce that there exists $b > 0$ so that $q_i(\Psi(p, \mathbf{s}))/|\Psi(p, \mathbf{s})| > b$ for all i and for all $(p, \mathbf{s}) \in \mathcal{E} \times \mathcal{Q}$.

On the other hand, again by Lemma 7.10, any $v \in \mathcal{ML}$ with $q_i(v) > 0$ for all i is in $\mathcal{ML}_{\mathcal{A}}$. We can certainly find v with $q_i(v) > 0$ for all i but with $q_i(v)/|v| \leqslant b$ for some i, which proves the result. □

Theorem 7.3 is an immediate corollary of Theorem 7.12.

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