LECTURES ON PLEATING COORDINATES FOR THE MASKIT EMBEDDING OF THE PUNCTURED TORUS

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Acknowledgments. These lectures are based on joint work with Linda Keen, City University, New York. The investigation was motivated by computer studies of the boundary of Teichmüller space carried out by David Wright, Oklahoma State University. All the computer drawn pictures of limit sets and the parameter space were made by David Wright. A detailed account of this work is contained in reference [5].

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Lecture 1. Kleinian groups and the Maskit embedding: Constructing the basic example.

Given a transformation T on a space X, we are used to the idea of studying the orbit of point $T^{-2}(x), T^{-1}(x), x, T(x), T^2(x), \dots$ and its limiting behaviour. The behaviour divides into two types:

Dissipative: $\Omega(T) = \{x \in X | x$ has a neighbourhood U for which the sets $T^n(U)$, $n \in \mathbf{Z}$, are disjoint$\}$ (or more generally, $\#\{n \in \mathbf{Z} : T^n(U) \cap U \neq \phi\}$ is finite), and

Recurrent : $\Lambda(T) = \{x \in X |$ for any neighbourhood U of x, $T^n U \cap U \neq \phi$ for infinitely many $n \in \mathbf{Z}\}$.

We also frequently want to study the dependence of T, and hence of $\Omega(T)$ and $\Lambda(T)$, on a parameter μ.

A famous example is when $X = \hat{\mathbf{C}} = \mathbf{C} \cup \infty$, the *Riemann sphere*, and T is the map $T(z) = z^2 + c$, where c is a complex parameter. In this case $\Omega(T)$ is called the *Fatou set* and $\Lambda(T)$ the *Julia set*. For different c values, either $\Lambda(T)$ is a Cantor set (i.e. totally disconnected) or $\Lambda(T)$ is connected. The *Mandelbrot set* is the set of c values *in parameter space* for which $\Lambda(T)$ is connected.

What we shall be studying in these lectures is what happens when we replace the single transformation T by a group of transformations G, and allow G to depend on varying parameters.

In analogy with the above example, we shall study $G \subset PSL(2, \mathbf{C})$ acting on $\hat{\mathbf{C}}$. Thus a typical element $g \in G$ can be written $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbf{C}$, $ad - bc = 1$, with g acting on $z \in \hat{\mathbf{C}}$ by $g : z \mapsto \frac{az+b}{cz+d}$. We take a finitely generated group G and allow the coefficients a, b, c, d of the generators to depend on parameters $\mu_1, \dots, \mu_k \in \mathbf{C}$. A very simple example, which we shall be studying here, is the group G_μ generated by $S_1 : z \mapsto z + 2$ and $T_\mu : z \mapsto \mu + \frac{1}{z}$, represented by the matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -i\mu & -i \\ -i & 0 \end{pmatrix}$. In this case $\Omega(G)$ is called the *regular set* or *ordinary set* and $\Lambda(G)$ is the *limit set*. We shall be investigating how $\Lambda(G_\mu)$ varies with μ.

In general, G is called *Kleinian* if $\Omega(G) \neq \phi$. This implies that G is discrete as a group of matrices; the converse in general is false.

Teichmüller theory.

There is another whole set of motivations for studying this problem.

Recall that a *Riemann surface* is a surface S for which there are charts to open subsets of $\mathbb{C}$, such that the overlap maps are complex analytic. Another way to say this is that a Riemann surface is a surface with a *complex analytic* or *conformal* structure. The *Teichmüller space* of a surface S_0 is the space of possible conformal structures on S_0. More precisely, consider the space of pairs (S, f) such that S is a Riemann surface and $f : S \to S_0$ is a diffeomorphism. Define an equivalence relation by $(S, f) \sim (S', f')$ if there is a conformal homeomorphism $h : S' \to S$ such that $f^{-1}f'$ is isotopic to h. The *Teichmüller space of* S_0, $Teich(S_0)$, is the space of equivalence classes of pairs.

Note that by introducing this equivalence relation we are implicitly taking account of a *marking* on the surface: if $g : S \to S$ is a conformal diffeomorphism which is not homotopic to the identity then the pairs (S, f) and $(S, f \circ g)$ are different points in Teich(S_0).

There are several different approaches to studying Teich(S_0) in a more concrete fashion. The one which relates to our problem here is based on the *Ahlfors finiteness theorem*:

If $G \subset PSL(2, \mathbb{C})$ is discrete and finitely generated, then $\Omega(G)/G$ is a (possibly disconnected) union of a finite number of Riemann surfaces, each of finite type.

Finite type means that the surface has finite genus and no boundary, except for possibly a finite number of punctures. By definition, in the neighbourhood of a puncture there is a chart which maps to a punctured disc in $\mathbb{C}$.

One method of studying Teich(S_0) is to look for a Kleinian group G for which $\Omega(G)/G = S_0 \cup S_1 \cup \cdots \cup S_k$ and then to make G depend on parameters which vary in such a way that $S_1, \ldots, S_k$ are fixed Riemann surfaces while S_0 varies over Teich(S_0). The *Maskit embedding* of Teich(S_0) refers to the case in which $S_1, \ldots, S_k$ are all thrice-punctured spheres. It is an important fact from complex analysis that Teich (thrice-punctured sphere) consists of one point. Thus any variation of G automatically produces variation in S_0 but not $S_1, \ldots, S_k$. Maskit gave a construction to find such a G corresponding to any S_0 of negative Euler characteristic, for a suitable choice of k. This is called the *Maskit embedding* of Teich(S_0).

Whatever the choice of $S_1, \ldots S_k$, there are two important ingredients to this programme:

(a) the Klein-Maskit combination theorems, and

(b) the simultaneous uniformisation theorem.

The theorems referred to in (a) give recipes for building up complicated groups from simpler ones. One can interpret the different ways of combining groups topologically in terms of glueing on handles and pasting together neighbourhoods of punctures in $\Omega(G)/G$.

Using (a) we can build up $\Omega(G)/G$ to have more or less any prespecified topological type.

The result of (b) is that if G is a group for which $\Omega(G)/G = S_0 \cup \cdots \cup S_k$, and if $S'_0, \ldots, S'_k$ are Riemann surfaces such that S'_i is homeomorphic to S_i for each i, with any given (marked) conformal structures, then there exists a group G' for which $\Omega(G')/G' = S'_0 \cup \cdots \cup S'_k$.

The proof is based on the *Ahlfors-Bers measurable Riemann mapping theorem*, which we shall describe in the next lecture.

We shall see later how both these theorems work in our G_μ example above.

Our aims in studying this example will be the following:

(a) to see that for 'good' μ-values, $\Omega(G_\mu)/G_\mu$ is the union of a punctured torus and a thrice-punctured sphere,

(b) to investigate the space of 'good' μ-values, and

(c) to find a picture of Teich(S_0) as it sits in $\mathbb{C}$, and in particular to study ∂ Teich(S_0).

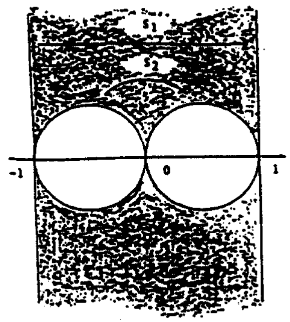

Figure 1.1

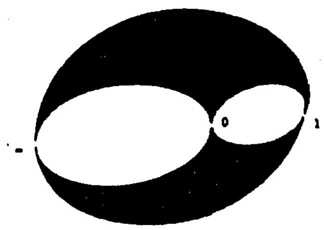

Figure 1.2

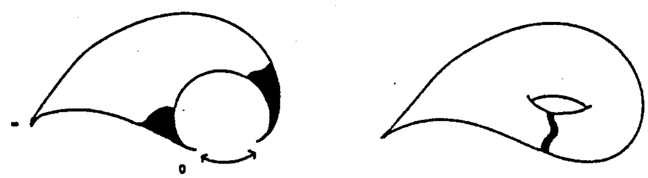

Figure 1.3

Construction of the example

Start with the group $\Gamma =< S_1 : z \mapsto z+2,\ S_2 : z \mapsto \frac{z}{2z+1} >$. This is a Fuchsian group, i.e. it leaves the upper half plane $\mathbb{H}$ and lower half plane $\mathbb{L}$ invariant. It has a fundamental domain as shown shaded in Figure 1.1. Here $\Lambda(G) = \mathbb{R} \cup \infty$, $\Omega(G) = \mathbb{H} \cup \mathbb{L}$.

In Figure 1.2 we see $\Omega(G)/G$ which has 2 connected components, both of which are thrice-punctured spheres.

Now we shall do the group theoretical equivalent of deleting punctured disc neighbourhoods of 0 and ∞ in $\mathbb{H}$ and glueing the boundaries together. This is shown in Figure 1.3.

We do this by adjoining an element T to Γ such that $TS_2T^{-1} = S_1$. The content of Maskit's second combination theorem is, roughly, that if one can draw what looks like a fundamental domain for $G =< T, \Gamma >$, then G *is* discrete and the domain one has drawn actually works. In our case it is easy to compute that if $TS_2T^{-1} = S_1$, then necessarily $T(z) = \mu + \frac{1}{z}$ for some $\mu \in \mathbb{C}$.

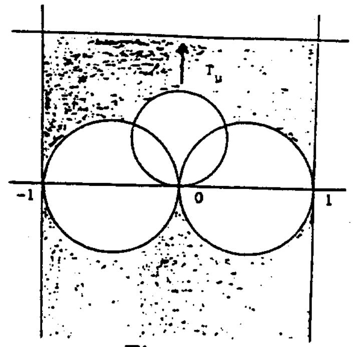

Figure 1.4

If we set $\mu = it$, for $t > 2$, we can draw the picture shown in Figure 1.4

The horocycle neighbourhoods of $0, \infty$ are identified by T_μ and Maskit's theorem asserts that the shaded region is a fundamental domain for $G = <\Gamma, T_\mu>$. We see that $\Omega(G)/G$ is the union of a punctured torus and a thrice-punctured sphere.

(If $0 < t < 2$ the horocycles intersect, so the construction doesn't work; if $t = 2$ they touch and we shall study this case later; if $t < 0$ we get a symmetrical picture in **L**.)

Notice that this construction has exactly implemented the idea suggested by Figure 1.3: delete neighbourhoods of cusps and glue the boundaries, see Figure 1.5.

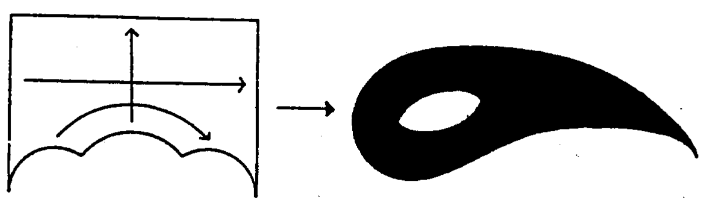

Figure 1.5

Thus the family of Riemann surfaces $\mathbb{H}/G_{it}$, $2 < t < \infty$, are all punctured tori. It is clear that if we specify that the generators of $\pi_1(S_0)$ are S_1 and T_μ, then no two of these (marked) groups are conjugate in $PSL(2, \mathbb{C})$ and hence the conformal structures are distinct. Specifying S_1, T_μ as generators is like specifying the marking on the torus. The trace $-i\mu$ of T_μ is an invariant of conjugacy in $PSL(2, \mathbb{C})$.

In the next lecture we shall study what happens when μ varies off the imaginary axis.

Lecture 2. The rough shape of Teichmüller space.

From now on, let $\mathcal{S}_0$ denote a punctured torus. We begin by discussing the topology of Teich($\mathcal{S}_0$). We first make the important observation that there is a bijection between the spaces Teich($\mathcal{S}_0$) and Teich($\mathcal{T}$), where $\mathcal{T}$ is an unpunctured torus. This correspondence is made as follows. Clearly, given a conformal structure on $\mathcal{T}$ one obtains a structure on $\mathcal{S}_0$ by removing the puncture. Conversely, since a chart in a neighbourhood of the puncture maps to a punctured disc, we can extend to a chart in a neighbourhood of the filled in puncture. Finally we note that if $P, Q \in \mathcal{T}$, then there is a conformal homeomorphism $f : (\mathcal{T}, P) \to (\mathcal{T}, Q)$ isotopic to the identity. Hence up to the equivalence relation on Teich($\mathcal{S}_0$) the position of the puncture P can be chosen arbitrarily. This proves our claim that Teich($\mathcal{S}_0$) $=$ Teich($\mathcal{T}$). (Of course this statement needs to be amplified in order to discuss the topology or analytic structure on Teich($\mathcal{S}_0$). We shall not do this here.)

Description of Teich($\mathcal{T}$).

It is well known that the Teichmüller space of an unpunctured torus can be identified with the upper half plane $\mathbb{H}$. This works as follows.

The universal cover of $\mathcal{T}$ has an induced conformal structure and can be identified with $\mathbb{C}$. The covering transformations are conformal bijections $\mathbb{C} \to \mathbb{C}$, hence of the form $z \mapsto az + b$, $a, b \in \mathbb{C}$. The covering group is $\mathbb{Z}^2$, hence a lattice in $\mathbb{C}$ with marked generators w_1, w_2 say. After applying a similarity and rotation (these do not change the equivalence class in Teich($\mathcal{T}$)), we can set $w_2 = 1$ and $\tau = w_1$, and regard these as our marked generators.

We note that the map $\tau' = \frac{a\tau+b}{c\tau+d}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, is induced by a base change for the lattice

$$w_1' = aw_1 + bw_2, \quad w_2' = cw_1 + dw_2.$$

Such a map leaves the conformal structure invariant but changes the marking on $\mathcal{T}$, hence τ, τ' represent *different* points in Teich($\mathcal{T}$).

From general theory, which we shall not discuss here, it is possible to put a complex analytic structure on Teich($\mathcal{S}_0$) and show that Teich($\mathcal{S}_0$) and Teich($\mathcal{T}$) are equivalent under an analytic bijection. Further, the analytic structure on Teich($\mathcal{S}_0$) is exactly that which it inherits from $\mathbb{C}$ when it sits inside $\hat{\mathbb{C}}$ as a set of suitably normalised "good" μ-values $\mathcal{M}$ for which the group G_μ (see Lecture 1) is such that $\Omega(G_\mu)/G_\mu$ is the union of $\mathcal{S}_0$ and a thrice-punctured sphere. Thus $\mathcal{M}$ is the image of $\mathbb{H}$ under an analytic bijection. We want to find a way of actually determining $\mathcal{M}$.

To do this we need one more ingredient, namely the *Measurable Riemann Mapping Theorem.*

In general a conformal structure on $\hat{\mathbb{C}}$ is given by a metric of the form $ds = |dz+\nu \overline{dz}|$, where $\nu = \nu(z)$ is a bounded measurable function of z, called a *Beltrami differential.* Pictorially we can think of ν as an ellipse field on $\hat{\mathbb{C}}$, where ν describes the eccentricity and orientation of the ellipse. The MRMT states that given any ν with $\|\nu\|_\infty < 1$, there exists a homeomorphism $\phi_\nu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\nu = \frac{\partial \phi}{\partial \bar{z}} / \frac{\partial \phi}{\partial z}$. This condition is equivalent to saying that $\phi_{\nu^*}(ds)$ is the standard metric $|dz|$ on $\hat{\mathbb{C}}$. In other words, up to homeomorphism, there is a *unique* conformal structure on $\hat{\mathbb{C}}$.

This fact explains the *rigidity* of the thrice-punctured sphere referred to in the last lecture. We can always normalise, by conjugating in $PSL(2,\mathbb{C})$), so that the three punctures are at $0, 1, \infty$. Just as for the punctured torus, we can fill in the punctures without changing the conformal structure. Now use the MRMT to map to the standard structure (which we can clearly do fixing the three punctures.).

We are now ready to return to the example G_μ.

The parameterisation of G_μ has been chosen to make the "good" μ-values $\mathcal{M}$ look as much like $\mathbb{H}$ as possible. To take care of the problem of conjugacy in $PSL(2,\mathbb{C})$, we begin by looking at a larger set $\tilde{\mathcal{M}}$.

Precisely, let $\tilde{\mathcal{M}}$ denote those $\mu \in \mathbb{C}$ for which:

(a) G_μ is a free group on the two generators S_1, T_μ, and S_1 is parabolic,

(b) $\Omega(G_\mu)$ has one simply connected invariant component Ω_0 for which Ω_0/G_μ is a punctured torus, and

(c) the other components Ω_i of $\Omega(G_\mu)$ are all conjugate under G_μ and the orbit space is a thrice-punctured sphere.

We saw in Lecture 1 that if $\mu \in i\mathbb{R}$, $|\mu| > 2$, then $\mu \in \tilde{\mathcal{M}}$.

Conjugacy classes of groups in $\tilde{\mathcal{M}}$ can be identified with points in Teich($\mathcal{S}_0$) as follows. Recalling the rigidity of the thrice-punctured sphere, we see that $\Omega(G_\mu)/G_\mu$ can be thought of as a point in Teich($\mathcal{S}_0$). Suppose conversely we are given a conformal structure on $\mathcal{S}_0$. Let $\mu_0 \in i\mathbb{R}$, $|\mu_0| > 2$. The given structure on $\mathcal{S}_0$ lifts to a structure on $\Omega_0(G_{\mu_0})$.

We extend this structure by zero to a G_{μ_0}-invariant conformal structure (i.e. a G_{μ_0}-invariant Beltrami differential) on $\hat{\mathbb{C}}$. We use here the non-trivial fact that $\Lambda(G_{\mu_0})$ has null Lebesgue measure, which follows from the fact that G_{μ_0} is geometrically finite and a result of Ahlfors that for geometrically finite Kleinian groups the measure of $\Lambda(G_{\mu_0})$ is zero. (Recall that for a Kleinian group, necessarily $\Omega(G) \neq \phi$.)

Use the MRMT to find a homeomorphism ϕ mapping the given structure to the standard structure. This will conjugate G_{μ_0} to a group $\tilde{G}$ of conformal automorphisms, i.e. a Kleinian group. If we normalise correctly and use the rigidity of the thrice-punctured sphere we find that $\tilde{G}$ contains the thrice-punctured sphere subgroup Γ (see Lecture 1). Further, T_{μ_0} maps to $\tilde{T}$ such that $\tilde{T}S_1\tilde{T}^{-1} = S_2$. An easy computation shows that necessarily $\tilde{T} = T_\mu$ some $\mu \in \hat{\mathbb{C}}$. Thus $\tilde{G} = G_\mu$ for some μ, and since ϕ is a homeomorphism, $\mu \in \tilde{\mathcal{M}}$. By multiplying the Beltrami differential by t, $0 \leq t \leq 1$, we see that μ_0 and μ can be connected by a path in $\tilde{\mathcal{M}}$.

The fact that $\mu, \mu' \in \tilde{\mathcal{M}}$ represent the same conformal structure *only if* G_μ and $G_{\mu'}$ are conjugate in $SL(2, \mathbb{C})$ needs a harder result. Very briefly, the construction gives us a conformal homeomorphism $\phi : \Omega(G_\mu) \to \Omega(G_{\mu'})$. Maskit's combination theorem implies that G_μ is geometrically finite. By Marden's isomorphism theorem it follows that ϕ extends to a conformal homeomorphism $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which is then necessarily a conjugacy in $SL(2, \mathbb{C})$. We refer to [5] for more details.

We can pick *one* point in each conjugacy class by choosing the fixed points of S_1, S_2 and $S_2^{-1}S_1$ to be at $\infty, 0$ and -1 respectively, and choosing $\operatorname{Im}\mu > 0$. (In fact $\tilde{\mathcal{M}}$ has two connected components separated by $\mathbb{R}$. Note that if $\mu \in \mathbb{R}$ then G_μ is Fuchsian and Ω/G_μ does not have the right shape.)

The Shape of $\mathcal{M}$.

We let $\mathcal{M}$ be the connected component of $\tilde{\mathcal{M}}$ satisfying the conditions above. From the general theory and the discussion above, $\mathcal{M}$ is the image of $\mathbb{H}$ under an analytic bijection.

(a) $\mathcal{M}$ is invariant under translation $\mu \mapsto \mu + 2$.

This follows since $S_1 T_\mu = T_{\mu+2}$ and the observation that if (S_1, T_μ) generate G_μ, so do $(S_1, S_1 T_\mu)$.

(b) $\mathcal{M}$ is symmetrical under $\mu \mapsto -\bar{\mu}$.

This follows since any word in $G_{-\bar{\mu}}$ is obtained from a word in G_μ by replacing μ by $-\bar{\mu}$. Hence $\Lambda(G_\mu) \to \Lambda(G_{-\bar{\mu}})$ under the map $\mu \mapsto -\bar{\mu}$. Clearly this map preserves all relevant properties of G_μ, and we obtain $\mu \in \mathcal{M} \iff -\bar{\mu} \in \mathcal{M}$.

(c) $\mathcal{M} \supset \{\mu \in \mathbb{C} \,|\, Im\,\mu > 2\}$

If $Im\,\mu > 2$ then the region shown in Figure 1.4 is always a fundamental domain for G_μ to which Maskit's theorem applies.

(d) If $\mu = 2i$ then the upper horizontal line in Figure 1.4 becomes tangent to the circle tangent to the real axis at zero. We can then apply Maskit's theorem to the shaded region in Figure 1.4 to see that Ω/G_μ is the union of two thrice-punctured spheres. We find $2i \in \partial\mathcal{M}$.

(e) Notice that if $Im\,\mu = 0$, then G_μ would be Fuchsian, which is not the case. Thus $\mathcal{M} \subset \{\mu \in \mathbb{C} \,|\, Im\,\mu > 0\}$. In fact it is not hard to show that $\mathcal{M} \subset \{\mu \in \mathbb{C} \,|\, Im\,\mu > 1\}$.

From (a)-(e) we get a rough idea of the shape of $\mathcal{M}$.

David Wright's Programme

We wish to study $\partial\mathcal{M}$. The idea is to find certain special points on $\partial\mathcal{M}$ called *cusps*. The point $\mu = 2i$ is called a *cusp*; as $t \searrow 2$, $\operatorname{Trace} T_{it} \to +2$, in other

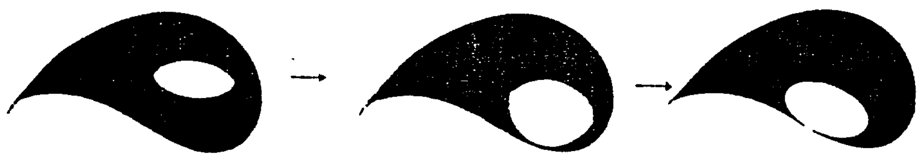

Figure 2.1

words, T_μ becomes parabolic. This means that the curve on S_0 represented by the element T in G_μ shrinks as $\epsilon \searrow 0$ to length zero. Geometrically, we obtain the pictures of Ω_0/G_μ shown in Figure 2.1.

We can make a similar construction starting with other simple closed curves on S_0. David Wright's idea was the following:

(a) Enumerate the simple closed curves on S_0 and find elements of G_μ representing them.

(b) Compute the traces of these group elements and look for μ-values at which those elements are parabolic. At least some of these points should be cusps on $\partial\mathcal{M}$.

As we shall see, simple closed curves on S_0 correspond to the (p,q) curves an unpunctured torus T; in other words, they are enumerated by $\mathbf{Q} \cup \infty$.

The condition that a given trace is ± 2 (ie the corresponding group element is parabolic) has many solutions. By using Newton's method in a systematic way, Wright found a sequence of solutions which appeared to give points on $\partial\mathcal{M}$. The boundary he found in this way is shown in Picture 1. He found that groups corresponding to these solutions had regular sets which were unions of tangent circles. In fact in these cases, Ω/G is a union of two thrice-punctured spheres. The arrangement of the circles depends on the continued fraction expansion of p/q. One such example is shown in Picture 2, where $p/q = 5/16$. There are two families of circles, coloured black and white in the picture, corresponding to the two spheres in Ω/G. In the following lecture we shall discuss the enumeration (a) in detail, and then go on to see how Wright's choice of root is indeed the unique choice of root which is a cusp point on $\partial\mathcal{M}$. This leads us to the introduction of *pleating rays* which foliate the interior of $\mathcal{M}$. One interpretation of the pleating ray is that it describes the visual shape of $\Lambda(G_\mu)$.

Lecture 3. Enumerating the simple closed curves on a punctured torus.

Let us begin by being a bit more precise about the definition of a *cusp* on $\partial(\text{Teich}(S_0))$. (This use of the term cusp is not to be confused with the idea of a cusp or puncture on a hyperbolic surface. We are now talking about a point in parameter space, i.e. a particular Kleinian group in which a certain element is parabolic, causing there to be a cusp in the sense of puncture on Ω/G.)

To make life simple, we will only define cusps for our groups G_μ. In fact of course, the idea is much more general.

First, if G is a Kleinian group, an element $g \in G$ is said to *represent* a curve γ on $\Omega(G)/G$ if γ lifts to a G-invariant curve $\tilde{\gamma} \subset \Omega$ whose endpoints (in $\Lambda(G)$) are the fixed points of g. Both parabolic and loxodromic elements may be represented in this way.

By a *simple closed curve* on a surface we mean a curve with no proper self-intersections.

Notice that if $\mu, \mu' \in \mathcal{M}$, then there is a homeomorphism $\Omega(G_\mu) \to \Omega(G_{\mu'})$ which induces a group isomorphism $\phi : G_\mu \to G_{\mu'}$. One can show that ϕ is *type preserving*, i.e. $\phi(g)$ is loxodromic (respectively parabolic) if and only if the same is true of g. An element g represents a curve on $\Omega(G_\mu)/G_\mu$ if and only if it does on $\Omega(G_{\mu'})/G_{\mu'}$.

Definition. Let $g \in G_\mu$ be a loxodromic element which represents a simple closed curve on $\Omega(G_\mu)/G_\mu$ for some (and hence any) $\mu \in \mathcal{M}$. A point $\mu_0 \in \partial\mathcal{M}$ is a *cusp* relative to g if there is a group isomorphism $\phi : G_\mu \to G_{\mu_0}$ which is type preserving except that $\phi(g)$ is parabolic.

It is a result of Maskit that there is a cusp group corresponding to every simple closed curve on $\Omega(G)/G$ represented by a loxodromic $g \in G$. By a recent result of L. Keen, B. Maskit and C. Series [2], this cusp is *unique* (in fact one does not even need the condition $\mu_0 \in \partial\mathcal{M}$. This follows automatically if G_μ is discrete and a cusp.) By a very important recent result of Curt McMullen, cusps are dense on $\partial\mathcal{M}$ (and more generally on the boundary of any suitable Teichmüller space). These two results justify the programme of David Wright discussed in the last lecture: to get a good picture of $\partial\mathcal{M}$ it suffices to enumerate those simple closed curves on $\Omega_0(G_\mu)/G_\mu$ which are represented by elements of G, (note there are *no* non-trivial simple closed curves on a thrice-punctured torus) and then to find the corresponding cusps.

Enumeration of simple closed curves

This is a technique developed by J. Birman and C. Series [1], which works in a quite general context.

In order to avoid problems with the relation $T_\mu S_1 T_\mu^{-1} = S_2$ in G_μ, we shall use a slight variant on the fundamental domain for G_μ acting in Ω_0 from that used in lecture 1. Since the problem is a topological one it does not matter which μ value we choose in $\mathcal{M}$; so for simplicity we take $\mu = it$, $t > 2$. The fundamental domain R we use is shown shaded in Figure 3.1. Its sides are paired by S_1, T as shown.

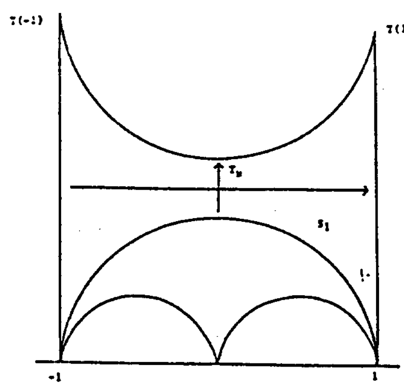

Figure 3.1

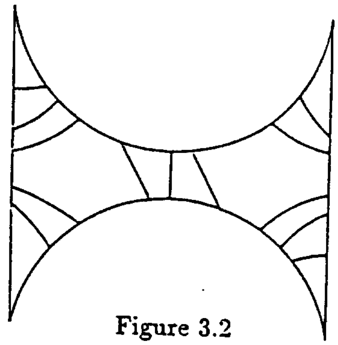

Figure 3.2

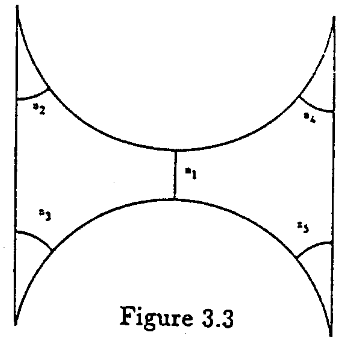

Figure 3.3

Any simple closed curve on Ω_0/G_μ lifts to a number of pairwise disjoint strands joining sides of R; see Figure 3.2. We collapse these strands as in Figure 3.3.

The n_i in Figure 3.3 are integers indicating the number of strands joining the sides in question.

Because the sides are paired, we have the *side pairing relations*

$$n_1 + n_2 + n_4 = n_1 + n_3 + n_5, \quad n_2 + n_3 = n_4 + n_5.$$

Note that if all of n_2, n_3, n_4, n_5 are non-zero, then where the strands are separated out and rejoined to form a curve, we get a loop around the puncture -1. This loop is represented by the element $S_2^{-1}S_1 \in G$, which by computation is parabolic. Thus there is no corresponding cusp on $\partial\mathcal{M}$, and we can omit this loop from the enumeration. In other words, we may as well assume that at least one of $n_2, \ldots, n_5$ is zero. Now we solve the side-pairing relations (using $n_i \geq 0$, $\forall i$). If, for example, $n_3 = 0$, we find that $n_2 = n_5$, $n_3 = n_4 = 0$. All the other solutions are symmetrical. (There may be of course a horizontal n_1-strand in place of the vertical one.) It is also easy to see that after rejoining the strands there will be one connected loop if and only if n_2 and n_1 are relatively prime. Henceforth we focus on the situation $n_1 = r$, $n_2 = s$. Observe that we can read off the group element corresponding to this curve by following along the strands in the order they must link up. This order is completely determined by the order in which the strands meet the sides of the fundamental domain. Up [illegible] permutation

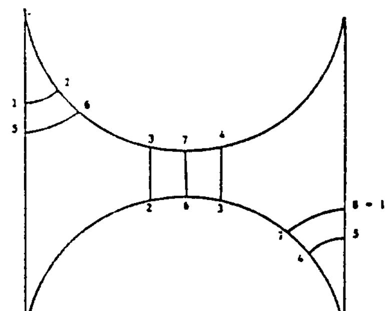

Figure 3.4

(conjugacy in G_μ) this order is fixed by r and s. Thus for example, if $r = 3,\ s = 2$, by following the arrows in Figure 3.1, we obtain the word S_1TTTS_1TT.

(The numbers in Figure 3.4 indicate the order in which the strands are joined.) There is an obvious correspondence here with the homotopy classes of simple closed curves on an unpunctured torus. It is well known that there is one such curve for each line of rational slope p/q in the plane. The vertical line of slope ∞ is represented by $p = 1, q = 0$. (Warning: the p/q and r/s we have above do not quite correspond as you would think. In the example above, $r = 2,\ s = 3$ but $p = 5\ (= 2 + 3)$ and $q = 2$.)

Using the above construction one proves:

Proposition 3.1. *For each $p/q \in \mathbb{Q} \cup \infty$, there is a unique conjugacy class in G_μ which represents a simple closed curve on the punctured torus. This class corresponds to the (p, q) curve on the unpunctured torus.*

One can make an explicit choice, see [5] for more details, of a word in the conjugacy class, denoted $W_{p/q}$, in such a way that if $\frac{p}{q} < \frac{r}{s}$ and $ps - rq = -1$, then $W_{\frac{p+r}{q+s}} = W_{r/s}W_{p/q}$. The conventions are chosen so that when $W_{p/q}$ is abelianised we get the word $S_1^{-p}T^q$. To start the recursion we define $W_{0/1} = T$ and $W_{1/1} = S_1^{-1}T$. For example, we find $W_{1/4} = S_1^{-1}T^4$ and $W_{2/5} = S_1^{-1}T^2S_1^{-1}T^3$. This recursion derives from the idea that a line segment of slope r/s, juxtaposed to one of slope p/q, "pulls tight"into one of slope $\frac{p+r}{q+s}$.

This construction of words $W_{p/q}$ is closely related to the way in which the rationals are built up in the *Farey tree*. Starting from $\frac{0}{1}$ and $\frac{1}{1}$ we perform the operation $\frac{p}{q},\ \frac{r}{s} \mapsto \frac{p}{q},\ \frac{p+q}{r+s},\ \frac{r}{s}$ whenever $\frac{p}{q} < \frac{r}{s}$, $ps - rq = -1$.

(Note that in this case automatically $\frac{p}{q} < \frac{p+r}{q+s} < \frac{r}{s}$.)

For example, after three steps we reach the sequence

$$\frac{0}{1}\ \frac{1}{4}\ \frac{1}{3}\ \frac{2}{5}\ \frac{1}{2}\ \frac{3}{5}\ \frac{2}{3}\ \frac{3}{4}\ \frac{1}{1}.$$

It is a well known fact that all the rationals in $[0,1]$ will appear exactly once, in lowest terms. Notice that $\frac{p}{q}$, $\frac{r}{s}$ are neighbours if and only if $ps - rs = \pm 1$.

We shall need to be able to compute the traces of the words $W_{p/q}$. Recall the easily verified *trace relation* $\operatorname{Tr} AB = \operatorname{Tr} A \operatorname{Tr} B - \operatorname{Tr} AB^{-1}$ which holds for elements of $SL(2,\mathbb{C})$. Applying this to the recurrence formula for $W_{p/q}$, we find that $\operatorname{Tr} W_{p/q}$ is a polynomial in μ with leading terms $(-i)^q(\mu^q - 2p\mu^{q-1} + \cdots)$.

David Wright's scheme now becomes more explicit. From the symmetries of $\mathcal{M}$ discussed in lecture 2 we only need look in the region $0 \le \operatorname{Re}\mu \le 2$. We already have cusps at $2i$ and $2+2i$ which are the places where the words $W_{0/1} = T$ and $W_{1/1} = S_1^{-1}T$ are parabolic. We then look for a root of the equation $\operatorname{Tr} W_{\frac{1}{2}} = 2$, using $\operatorname{Tr} W_{0/1} = -i\mu$, $\operatorname{Tr} W_{1/1} = -i(\mu - 2)$ and $\operatorname{Tr} W_{1/2} = \operatorname{Tr} W_{0/1} \operatorname{Tr} W_{1/1} - \operatorname{Tr} W_{0/1} W_{1/1}^{-1} = i\mu(i(\mu-2)) - 2 = -\mu^2 + 2\mu - 2$. There is a root at $\mu = 1 + \sqrt{3}i$ which 'looks right'.

Wright continued to find roots iteratively in this way, using as a starting point for the iteration for $W_{p+r/q+s}$ a point midway between the roots for $W_{p/q}$ and $W_{r/s}$. The word $W_{\frac{p}{q}} W_{\frac{r}{s}}^{-1}$ which enters the recursion is always a word which occurs earlier in the sequence and hence is a polynomial of lower degree in μ.

The picture built up in this way, shown in Picture 1, looked very convincingly as if it must be $\partial\mathcal{M}$. We shall be seeing a proof that this is indeed the case in lectures 4 and 5.

Circle packings.

Wright also obtained computer pictures of the groups G_{μ_0} at the "boundary" points μ_0 he had found. The case of $p/q = 5/16$ is shown in Picture 2. These limit sets are always unions of tangent circle, i.e. circle packings. In the picture some circles are white and some are black. The action of G_{μ_0} permutes the white circles among themselves, and the black circles likewise. There is always a black circle Δ in the upper half plane, tangent to $\mathbb{R}$ at -1, which is fixed by both the commutator $K = S_2^{-1} S_1$ and the parabolic word $W_{p/q}$. All the white circles are equivalent under G to the lower half-plane $\mathbb{L}$ and all the black ones to Δ. In particular, the half-plane above $T(\Delta)$, which is $T(\mathbb{L})$, is a white circle. In the example of the picture where $p/q = 5/16$, we have $W_{5/16} = (S_1^{-1} T^3)^5 T$, and the circles Δ and $T(\Delta)$ are tangent at the fixed point of $W_{5/16}$.

These *circle chains* can be explained as follows. At a cusp μ_0, Ω/G_{μ_0} consists of two thrice-punctured spheres. The punctured torus Ω_0/G_{μ_0} has degenerated to a thrice-punctured sphere by 'pinching' a simple closed curve to length zero. The rigidity of the thrice-punctured sphere means that up to conjugacy in $PSL(2, \mathbb{C})$, the group Γ (see lecture 1) is the only group corresponding to this surface. A group conjugate to Γ leaves a disc in $\hat{\mathbb{C}}$ invariant. The two families of circles we see are thus the lifts to $\hat{\mathbb{C}}$ of the two components of Ω/G_{μ_0} at the cusp.

The cusps on Ω/G_{μ_0} are 'paired' as in Figure 3.5.

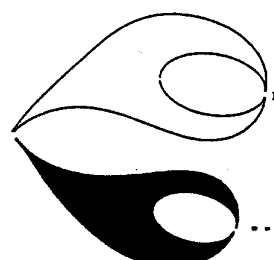

Figure 3.5

The idea in what follows will be to *open up the cusp*, i.e., to move from the cusp to the interior of $\mathcal{M}$ in such a way that the pattern of white and black circles is preserved, but the black circles cease to be tangent. Instead, they overlap, intersecting at the fixed points of $W_{p/q}$ and its conjugates. It turns out that this happens when we move in such a way that $\operatorname{Tr} W_{p/q}$ remains real and $|\operatorname{Tr} W_{p/q}| > 2$, i.e. $W_{p/q}$ becomes purely hyperbolic, and the subgroup generated by K and $W_{p/q}$ continues to preserve the circle Δ, i.e. remains Fuchsian. The quotient

$\Delta/\langle K, W_{p/q}\rangle$ is a cylinder with boundary, represented by $W_{p/q}$, and one puncture K. The element T matches the ends of the cylinder to form a punctured torus.

We define the *hyperbolic locus* $\mathcal{H}_{p/q}$ of $W_{p/q}$ by $\mathcal{H}_{p/q} = \{\mu \in \mathcal{M} \mid \operatorname{Tr} W_{p/q}(\mu) \in \mathbb{R},\ |\operatorname{Tr} W_{p/q}(\mu)| > 2\}$.

We shall prove:

Theorem 3.2. *Let $\mathcal{P}_{p/q}$ be the connected component of $\mathcal{H}_{p/q}$ which is asymptotic to $\operatorname{Re}\mu = 2p/q$ as $|\mu| \to \infty$. Then $\mathcal{P}_{p/q}$ contains no critical points. It limits in a unique point on $\partial\mathcal{M}$ at which $\operatorname{Tr} W_{p/q} = 2$. The curves $\mathcal{P}_{p/q}$ are pairwise disjoint and the closure of their union is $\mathcal{M}$. Along $\mathcal{P}_{p/q}$, $\Omega_0(G_\mu)$ is a union of overlapping circles arranged in the same combinatorial pattern as at the p/q cusp.*

The locus $\mathcal{P}_{p/q}$ is called the *p/q-pleating ray*. These rays act as 'internal rays' for $\mathcal{M}$, and are analogous to the famous *external rays* to the Mandelbrot set.

To understand why all this is true (and to see the reason for the terminology *pleating ray*) we need to study the hyperbolic convex hull of $\Lambda(G_\mu)$ in $\mathbb{H}^3$. This we shall do in the next lecture.

Lecture 4. The Convex Hull Boundary of the Limit Set.

In order to understand the interior of Teich(S_0), and to prove the facts stated at the end of the last lecture, we need to study some concepts introduced by Thurston.

The Convex Hull Boundary

Think of $\hat{\mathbb{C}}$ as the boundary of the ball model of hyperbolic 3-space $\mathbb{H}^3$. All one needs to know about $\mathbb{H}^3$ is that geodesic lines are arcs of circles in $\mathbb{H}^3$ which meet $\hat{\mathbb{C}}$ orthogonally, and that planes are portions of spheres which meet $\hat{\mathbb{C}}$ in the same way. The action of $PSL(2,\mathbb{C})$ on $\hat{\mathbb{C}}$ by linear fractional transformations extends to an action on $\mathbb{H}^3$ by using *inversions*. In fact any linear fractional transformation on $\hat{\mathbb{C}}$ can be expressed as a product of inversions in circles, and inversion in a circle C extends naturally to inversion in the sphere (i.e. the hyperbolic plane P) which meets C orthogonally. This inversion maps $\mathbb{H}^3$ to itself and should be thought of as hyperbolic reflection in P. (In the above description we are thinking of $\hat{\mathbb{C}}$ as a sphere whose interior is a ball. You may prefer to think of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ as a plane together with the point at infinity bounding the upper half space model of $\mathbb{H}^3$. In this case P is the hemisphere which meets $\mathbb{C}$ in C. The two pictures are equivalent under a conformal map which is nothing other than stereographic projection.)

Now let G be a Kleinian group acting on $\hat{\mathbb{C}}$ with limit set $\Lambda = \Lambda(G)$. Form the convex hull of Λ, denoted, $\mathrm{Co}(\Lambda)$ in $\mathbb{H}^3$. This means we have to join all possible pairs of points in Λ by geodesic lines in $\mathbb{H}^3$. This produces a very complicated object ! To get some insight into what it looks like, consider the very special case in which Λ is a circle. Then $\mathrm{Co}(\Lambda)$ is a hemisphere sitting above this circle. If there are parts of two overlapping circles C_1 and C_2 in Λ, as in Figure 4.1, then $\mathrm{Co}(\Lambda)$ cuts the two part-hemispheres H_1 and H_2 which meet along the geodesic in $\mathbb{H}^3$ which joins the intersection points of C_1 and C_2.

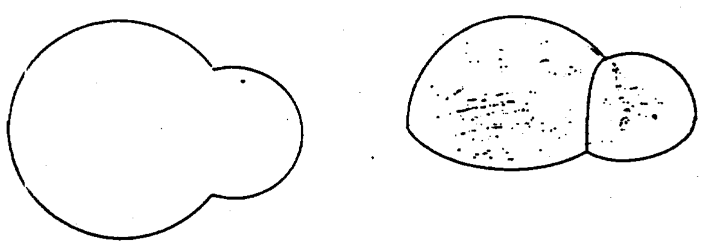

Figure 4.1

If there are no points of Λ in the region R 'inside' C_1 and C_2, then H_1 and H_2 form part of the boundary $\partial(\mathrm{Co}(\Lambda)$ in $\mathbb{H}^3$. (In this case $\mathrm{Co}(\Lambda)$, of course, contains many other points besides H_1 and H_2.)

Think now about the case of the cusp groups in the last lecture, in which Λ is a union of tangent circles. We can see that $\partial\mathrm{Co}(\Lambda)$ will include all the hemispheres over these circles, in fact one can show that the closure of this set is exactly $\partial\mathrm{Co}(\Lambda)$. One should think as each hemisphere as a 'roof' over the corresponding disc.

Now suppose that we move inside Teich($\mathcal{S}_0$) in the manner indicated at the end of the last lecture, in such a way that the pattern of black and white circles remains but some of the points of tangency open up and black circles begin to overlap. Then over each connected chain of overlapping circles, there will be a component of $\partial\mathrm{Co}(\Lambda)$ which will be a 'roof' over the chain. In fact it is not hard to show that, quite generally, there is a bijection between the connected components of Ω and of $\partial\mathrm{Co}(\Lambda)$, the components of $\partial\mathrm{Co}(\Lambda)$ forming 'roofs' over the corresponding components of Ω. Each component of Ω is homeomorphic with its 'roof', by a map which commutes with the action of G. Thus in the case of our groups G_μ, the convex hull boundary consists of one simply connected component whose quotient is a punctured torus, and a countable number of hemispheres whose quotients are thrice-punctured spheres, which are permuted among themselves by the action of G_μ.

What is the geometry of $\partial\mathrm{Co}(\Lambda)$? It is an example of what Thurston calls *pleated surface.* This means that it is a surface in $\mathbb{H}^3$, made up of portions of hyperbolic planes with geodesic boundaries, glued together along geodesics called *bending lines.* The bending lines are pairwise disjoint complete geodesics, meaning that they extend all the way out to $\hat{\mathbb{C}}$.

Another way to get a picture of this is to consider a Euclidean pleated surface in $\mathbb{R}^3$. To make sure it is convex, we need to be sure that all the bending angles are in the same direction. Of course whether or not this surface is embedded in $\mathbb{R}^3$ with depend on the bending angles and the distances between the bending lines, see Figure 4.2.

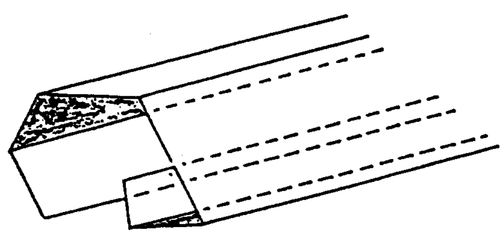

Figure 4.2

Notice that we can reverse the bending process and flatten the surface by 'unrolling' it along the plane $\mathbb{R}^2$. In this way we transfer the Euclidean metric on $\mathbb{R}^2$ in a natural way to the pleated surface in $\mathbb{R}^3$.

It is a theorem of Thurston (proved in detail by D. Epstein and A. Marden) that any hyperbolic pleated surface "rolls flat" in a similar way, in other words, that it carries a natural hyperbolic metric with respect to which it is isometric to a hyperbolic surface. In particular this theorem applies to $\partial\mathrm{Co}(\Lambda)$, where Λ is the limit set of any Kleinian group G. In fact, since the construction is G-invariant, $\partial\mathrm{Co}(\Lambda)/G$ will also be a disjoint union of pleated surfaces, one for each component of $\Omega(G)/G$. Topologically the components of $\partial\mathrm{Co}(\Lambda)/G$ and $\Omega(G)/G$ are homeomorphic in pairs, however the conformal structures (coming from the hyperbolic metric on $\partial\mathrm{Co}(\Lambda)/G$ and the natural complex structure on $\Omega(G)/G$)) do not coincide.

What are the possibilities for the bending lines of $\Omega(G)/G$? The collection of bending lines is always a closed family of pairwise disjoint simple geodesics on the hyperbolic surface $\partial\mathrm{Co}(\Lambda)/G$, in other words, a *geodesic lamination.* One can also show that provided G is geometrically finite, no bending line can end in a cusp. One knows how to enumerate all possible geodesic laminations on a hyperbolic surface. In particular the thrice-punctured sphere does not support *any* non-trivial laminations (assuming no leaves end at the cusps) and hence cannot be bent. In other words the components of $\partial\mathrm{Co}(\Lambda)/G$ corresponding to thrice-punctured sphere components of Ω/G are just quotients of hyperbolic planes in $\mathbb{H}^3$. In our example, these planes are, of course, just the planes you see sitting above the circles corresponding to the conjugates of the subgroup Γ of G.

If S is a hyperbolic punctured torus, then the possible geodesic laminations correspond exactly to the possible foliations of a Euclidean flat torus by parallel lines. We have already seen that a line of rational slope gives a homotopy class of simple closed curves on the punctured torus, and it is a basic fact of hyperbolic geometry that this homotopy class contains a unique geodesic, relative to a given hyperbolic metric. If we take a foliation by lines of irrational slope, puncture the torus and then give it a hyperbolic metric, then each leaf of the foliation 'pulls tight' to a unique geodesic, which is however no longer closed. The collection of all these geodesics no longer covers the whole torus. In particular no leaf of the lamination goes too near the puncture. (In fact in total the lamination only occupies a set of Hausdorff dimension one.) In cross-section the leaves of the lamination cut transversals in Cantor sets. They behave like the attractors of certain flows on the torus.

Now return to our example. Given a group G_μ, $\mu \in \mathcal{M}$, we know that $\partial \mathrm{Co}(\Lambda)/G_\mu$ has a component which is a torus, bent along a lamination λ. By the above discussion, we can identify λ with a point in $\mathbf{R} \cup \infty$, indicating the 'slope' of the lamination. (The point ∞ represents the $(1,0)$ curve S_1.) Thus we have a map $pl : \mathcal{M} \to \mathbf{R} \cup \infty$, $pl(\mu) = \lambda$.

Theorem 4.1. *The map pl is continuous.*

This is proved in [4].

We also remark that if $p/q \in \mathrm{Im}(pl(\mu))$, then since the (p,q) curve is a bending line for $\partial \mathrm{Co}(\Lambda)/G_\mu$, it is represented by a geodesic curve in $\mathbb{H}^3$, in other words the word $W_{p/q}$ in G must have an axis in $\mathbb{H}^3$. Now in the special case $(p,q) = (1,0)$, we have $W_{1/0} = S_1$ and $\mathrm{Tr}\, S_1 = 2$. Thus S_1 is parabolic and has no axis in $\mathbb{H}^3$; in other words, $\infty \notin \mathrm{Im}(pl(\mu))$.

We can now explain the geometrical significance of the special branches $\mathcal{P}_{p/q}$ of the hyperbolic locus $\mathcal{H}_{p/q}$ discussed in the last lecture. Recall that $\mathcal{H}_{p/q} = \{\mu \in \mathcal{M} \mid \mathrm{Tr}\, W_{p/q}(\mu) \in \mathbf{R},\ |\mathrm{Tr}\, W_{p/q}(\mu)| > 2\}$ and that $\mathcal{P}_{p/q}$ is the connected component of $\mathcal{H}_{p/q}$ which is asymptotic to $\mathrm{Re}\,\mu = 2_{p/q}$ as $|\mu| \to \infty$.

Theorem 4.2. *For $p/q \in \mathbf{Q}$, $\mathcal{P}_{p/q} = pl^{-1}(p/q)$.*

This is proved in [5].

In fact $\mathcal{P}_{p/q}$ contains no singularities (critical points of $\mathrm{Tr}\, W_{p/q}$) and $\bar{\mathcal{P}}_{p/q} - \mathcal{P}_{p/q}$ consists of a unique point on $\partial \mathrm{Teich}(S_0)$ which represents the p/q-cusp group. The lines $\mathcal{P}_{p/q}$ are pairwise disjoint and fill $\mathcal{M}$ densely.

The lines $\mathcal{P}_{p/q}$ are the 'vertical' lines in Picture 3.

Most of the proof of Theorem 4.2 we leave till the next lecture. However part of it is not hard:

Lemma 4.3. $pl^{-1}(p/q) \subset \mathcal{H}_{p/q}$.

Proof. The lemma says that if the bending line of $\partial \mathrm{Co}(\Lambda)/G$ is the axis of $W_{p/q}$, then $\mathrm{Tr}\, W_{p/q} \in \mathbb{R}$. In general a loxodromic $g \in SL(2,\mathbb{C})$ is a translation plus a rotation along its axis, i.e. the geodesic in $\mathbb{H}^3$ joining its two fixed points. The argument of $\mathrm{Trace}\, g$ measures the rotation about the axis. Since the axis is a bending line, we can measure the rotation relative to one of the planes in $\partial \mathrm{Co}(\Lambda)/G$ which meet in the bending line, say the left hand plane.

This plane is invariant under the action of $W_{p/q}$; hence the rotation maps the plane back to itself. Thus the rotation must be through an angle $2k\pi$, and hence $\mathrm{Trace}\, g \in \mathbb{R}$ as required.

It is easy also to see from our characterisation that the sets $pl^{-1}(p/q)$ are pairwise disjoint. It remains to explain why these sets are identified with the special components $\mathcal{P}_{p/q}$ of $\mathcal{H}_{p/q}$, why they contain no critical points and why they fill $\mathcal{M}$ densely. This we do in the next lecture.

Chapter 5. Pleating co-ordinates.

Recall from lecture 3 that we indicated that the pleating rays $\mathcal{P}_{p/q}$ are lines along which the chains of tangent circles which you see in the limit sets of the cusp groups have "opened up" into chains of overlapping circles. In order to understand this characterisation of $\mathcal{P}_{p/q}$, let us consider what such a chain of overlapping circles means in terms of the bending lamination of $\partial \mathrm{Co}(\Lambda)$.

Suppose the torus component of $\partial \mathrm{Co}(\Lambda)/G$ is bent along the geodesic $\gamma(p/q)$ corresponding to the word $W_{p/q}$, i.e. the (p,q) curve on the torus. If we cut along this curve then the remaining part of the torus is made up of part of a hyperbolic plane. It follows that when we lift to $\mathbb{H}^3$ we shall see a chain of portions of planes which intersect in lines conjugate to the lifts of $\gamma(p/q)$, i.e., the axes of the conjugates of $W_{p/q}$. Each of these planes forms part of the 'roof' over $\Omega_0(G)$, so we see that $\Omega_0(G)$ will be made up of a chain of overlapping circles which intersect at the fixed points of $W_{p/q}$ and its conjugates. The way in which these circles are permuted among themselves by elements of G depends on the word $W_{p/q}$. From the way in which the group is normalised, there is always one circle Δ in the chain tangent to $\mathbb{R}$ at -1 and containing the two fixed points of $W_{p/q}$. This circle is fixed by the subgroup generated by $K = S_2^{-1}S_1$ (which is parabolic and fixes -1) and $W_{p/q}$. Thus $\langle K, W_{p/q}\rangle$ is *Fuchsian*. Furthermore, no limit points of G appear inside this circle. We call a Fuchsian subgroup with this property, F-peripheral.

Now it is not hard to show that conversely, whenever $\operatorname{Tr} W_{p/q} \in \mathbb{R}$, then $\langle K, W_{p/q}\rangle$ is Fuchsian. It leaves invariant the circle containing the fixed points of $W_{p/q}$ and -1. This follows since in this case the traces of K, $W_{p/q}$ and $KW_{p/q}$ are all real. However the condition $\operatorname{Tr} W_{p/q} \in \mathbb{R}$ does not ensure that $\langle K, W_{p/q}\rangle$ is F-peripheral. In other words, there may be limit points both inside and outside the invariant circle. If this is not the case, i.e. if the invariant circle it is F-peripheral, then there is a circle chain in $\Omega(G)$ made up of overlapping discs and it is not hard to see that in this situation $\partial\mathrm{Co}(\Lambda)/G$ will be bent exactly along the axis of $\gamma(p/q)$. In other words:

Proposition 5.1. (a) *If $\mu \in \mathcal{H}_{p/q}$ then $\langle K, W_{p/q}\rangle$ is Fuchsian with invariant circle containing the fixed points of $W_{p/q}$ and tangent to $\mathbb{R}$ at -1:*

(b) *If $\mu \in \mathcal{M}$, then $pl(\mu) = p/q$ if and only if $\langle K, W_{p/q}/rangle$ is Fuchsian and F-peripheral.*

There does not seem to be a nice algebraic condition to determine when $\langle K, W_{p/q}\rangle$ is F-peripheral. However we have:

Proposition 5.2. $\{\mu \in \mathcal{H}_{p/q} | \langle K, W_{p/q}\rangle$ *is F-peripheral$\}$ is open in $\mathcal{H}_{p/q}$.*

Proof. Suppose μ_0 is such that $\langle K, W_{p/q}\rangle$ is F-peripheral. We can use the circle chain to construct a fundamental domain for G of the type suitable for applying Maskit's second combination theorem, as described in lecture 1. This is illustrated in Picture 4, for which $p/q = 3/8$. The fundamental domain is shown in black. Let us call the circles in the chain involved in the construction, the ***basic chain***. Now for any $\mu \in \mathcal{H}_{p/q}$ the subgroup $\langle K, W_{p/q}\rangle$ is still Fuchsian and hence it still defines a chain of invariant circles. What we have to see is that for μ close to μ_0, there are no limit points of G inside the chain. But for μ sufficiently close to μ_0, the basic chain of circles still has the same pattern of overlaps, and we can copy the shape of the fundamental domain for G into this perturbed region, in such a way that its sides are paired by the perturbed generators in the same pattern. Thus we can apply Maskit's theorem to see that G_μ is still a discrete group. Further, we can see that the images of this fundamental domain completely cover the basic circle chain, which therefore must be contained in $\Omega(G_\mu)$. This means, in particular, that $\langle K, W_{p/q}\rangle$ is F-peripheral.

This result has an important corollary.

Corollary 5.3. *The set $pl^{-1}(p/q)$ is a connected component of $\mathcal{H}_{p/q}$.*

Proof. We have just proved that $pl^{-1}(p/q)$ is open in $\mathcal{H}_{p/q}$. It is closed because the function $pl : \mathcal{M} \to \mathbb{R}$ is continuous.

Remark: To prove Theorem 4.2 we really need to show that $pl^{-1}(p/q)$ is closed in $\mathbb{C}$. This is also not hard, see [5] for details.

Proof of Theorem 4.2. We now have all the ingredients necessary for the proof of Theorem 4.2. We shall proceed by *induction on the Farey tree.* This means that we prove results for $p/q = 0/1$ or $1/1$, and then, assuming them true for p/q, r/w, where $ps - rq = -1$, prove them for $\frac{p+r}{q+s}$. In this way we reach all of $\mathbb{Q}$.

So start with $p/q = 0/1$. In this case the basic chain is shown in Figure 5.1. The word $W_{0/1}$ is just T_μ, and Trace$W_{0/1} = -i\mu$. Thus $\mathcal{H}_{0/1} = \{it | t > 2\}$. For $\mu \in \mathcal{H}_{0/1}$, the fixed points of T_μ are on the imaginary axis and the black chain is made up of two circles tangent to $\mathbb{R}$ at -1 and $+1$, which intersect in these fixed points. The shaded region in Figure 5.1 is a fundamental domain for the action of G on $\Omega_0(G)$. The vertical sides are paired by S_1 and the curving 'horizontal' sides by T_μ. Now it is not hard to see that the images of this fundamental domain under G_μ entirely cover the region inside the two circles of the basic chain, thus

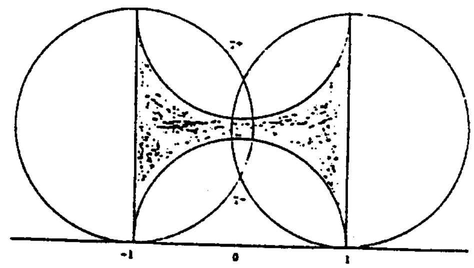

Figure 5.1

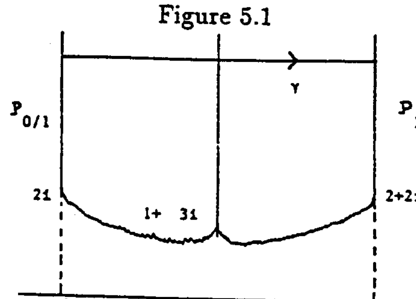

Figure 5.2

this region contains no limit points of G_μ. In other words $\langle K, W_{0/1}\rangle$ is F-peripheral and so we have $\mathcal{H}_{0/1} = \mathcal{P}_{0/1} = pl^{-1}(0/1)$.

Similar pictures work for $p/q = 1/1$.

This gives us the picture for the lines $\mathcal{P}_{0/1}$ and $\mathcal{P}_{1/1}$ shown in Figure 5.2.

Now let us look for $\mathcal{P}_{1/2}$. First, $\mathcal{P}_{1/2} \neq \phi$. For consider a horizontal path γ in $\mathcal{M}$ joining two points high up on $\mathcal{P}_{0/1}$ and $\mathcal{P}_{1/1}$, as shown in Figure 5.2. The function pl is continuous along γ and $pl(\gamma(0)) = 0$, $pl(\gamma(1)) = 1$; hence there is some point t_0, $0 < t_0 < 1$, with $pl(\gamma(t_0)) = 1/2$. By Corollary 5.3, the connected component of $\mathcal{H}_{1/2}$ containing $\gamma(t_0)$ is contained in $\mathcal{P}_{1/2}$. (Of course we can easily calculate $\mathcal{H}_{1/2}$ by hand since $W_{1/2} = S_1^{-1}T_\mu^2$ and $\operatorname{Tr} W_{1/2}(\mu) = -\mu^2 + 2\mu - 2$.)

Let B be the connected component of $\mathcal{H}_{1/2}$ containing $\gamma(t_0)$. We claim that B contains no singularities and is asymptotic to $\operatorname{Re}\mu = 1$ as $|\mu| \to \infty$. For if this is not the case, it would be possible to follow B in the direction of increasing trace (possibly passing through critical points) in such a way that $|\mu| \to \infty$, but so that the approach to ∞ is not asymptotic to $\operatorname{Re}\mu = 1$. This means that B would have to cross one of the vertical lines $\mathcal{P}_{0/1}$ or $\mathcal{P}_{1/1}$. But by Corollary 5.3, $B \subset \mathcal{P}_{1/2}$, and pleating rays corresponding to distinct rationals are disjoint, since the torus in $\partial\mathrm{Co}(\Lambda)/G$ can be pleated along at most one of the geodesics represented by $W_{0/1}$ and $W_{1/2}$ at once. Thus such a crossing is impossible.

By similar reasoning we argue that there are no other connected components of $\mathcal{P}_{1/2}$ in $\mathcal{H}_{1/2}$. Finally, we argue that as we approach $\partial\mathcal{M}$ along B, then $\mathrm{Tr}\, W_{1/2} \to 2$; for at any point where $\mathrm{Tr}\, W_{1/2}(\mu_0) > 2$ we can construct a chain of overlapping circles and hence, as in the proof of Proposition 5.2, a fundamental domain showing that $\mu_0 \in \mathcal{M}$.

We thus can add the line $\mathcal{P}_{1/2}$ to Figure 5.2, and by calculation find that the $\frac{1}{2}$ cusp is at the point $1 + \sqrt{3}\, i$.

To find $\mathcal{P}_{1/3}$ we repeat the above argument, applied to the strip between $\mathcal{P}_{0/1}$ and $\mathcal{P}_{1/2}$. In general, we find , since the leading terms of $\mathrm{Tr}\, W_{p/q}(\mu)$ are $(-1)^q(\mu^q - 2p\mu^{q-1})$, that $\mathcal{H}_{p/q}$ has a unique branch asymptotic to $\mathrm{Re}\,\mu = 2p/q$ as $|\mu| \to \infty$. Similar reasoning to the above shows that this branch is the unique component of $\mathcal{P}_{p/q}$ in $\mathcal{H}_{p/q}$, and that it contains no critical points.

Since when we enumerate $\mathbb{Q}$ using the Farey tree, the rationals appear in the correct order on $\mathbb{R}$ (note that if $p/q < r/s$ then $p/q < \frac{p+q}{r+s} < r/s$), we have proved Theorem 4.2 except for the claim that the union of the $\mathcal{P}_{p/q}$ pleating rays is dense in $\mathcal{M}$.

To do this needs a bit more work.

Irrational Pleating Rays.

Let us outline briefly what is involved in interpolating the rational $\mathcal{P}_{p/q}$ rays with "irrational pleating rays." Recall from lectures 3 and 4 that the bending lamination of $\partial\mathrm{Co}(\Lambda)/G$ may be any real number λ, corresponding to the foliation of a Euclidean torus by parallel lines of slope λ. We define $\mathcal{P}_\lambda = pl^{-1}(\lambda)$. The continuity argument used above, shows that $\mathcal{P}_\lambda \neq \phi$ for $\lambda \in \mathbb{R}$. One can also show that $\mathcal{P}_\lambda$ is contained in the real locus of some complex analytic function. If $\mathrm{Tr}\, W_{p/q}(\mu) \in \mathbb{R}$, then we have seen that the fixed points of $W_{p/q}$ lie on a circle tangent to $\mathbb{R}$ at -1. This condition can be expressed by saying that a certain cross-ratio is real-valued. There is a similar condition which expresses the fact that a leaf of the bending lamination lies in the foliation λ. We deduce that $\mathcal{P}_\lambda$ has empty interior in $\mathcal{M}$. This fact, combined with the ordering of the pleating rays, shows that $\cup\{\mathcal{P}_{p/q}|p/q \in \mathbb{Q}\}$ is dense in $\mathcal{M}$. (Notice that because of the periodicity of $\mathcal{M}$ under the translation $\mu \mapsto \mu+2$, we only need consider the range $0 \leq p/q \leq 1$.)

It would be nice to have an explicit function which replaces $\operatorname{Tr} W_{p/q}(\mu)$ for irrational λ. However there is no group element corresponding to a lamination whose leaves are not closed. Moreover the family of functions $\{\operatorname{Tr} W_{p/q}\}_{p/q\in\mathbb{Q}}$ is not a normal family: as $p/q \to \lambda \notin \mathbb{Q}$, then $q \to \infty$ so the trace polynomials, being polynomials of degree q in μ, are very badly behaved. We might hope that the functions $(\operatorname{Tr} W_{p/q}(\mu))^{\frac{1}{q}}$ would do better. In fact, something very close to this has a geometrical meaning. Recall that $\operatorname{Tr} W_{p/q}(\mu)$ is related by a simple formula to the length $\rho_{p/q}$ of the axis of $W_{p/q}$ in $\mathbb{H}^3$, in fact $\operatorname{Trace} W_{p/q} = 2\cosh(\rho_{p/q}/2)$. Thus $(\operatorname{Tr} W_{p/q})^{\frac{1}{q}}$ behaves like $\frac{1}{q}\ell_{p/q}$. Now the (p,q)-curve $\gamma_{p/q}$ on $\partial\text{Co}(\Lambda)/G$ cuts the $(1,0)$-curve $\gamma_{1/0}$ represented by S_1 exactly q times. Thus we can regard $\frac{1}{q}\ell_{p/q}$ as the average return time of $\gamma_{p/q}$ to $\gamma_{1/0}$. With this definition, we can extend the function $\frac{1}{q}\ell_{p/q}$ continuously to irrational laminations. Correspondingly one can show that the functions $\frac{1}{q}\operatorname{arccosh}(\operatorname{Tr} W_{p/q}/2)$ form a normal family which have a (complex analytic) limit whenever $p/q \to \lambda \in \mathbb{R}$. This limit is in the sense of uniform convergence on compact subsets of $\mathcal{M}$. Details are in [5]. The limiting functions L_λ are real valued exactly when λ is the bending lamination of $\partial\text{Co}(\Lambda)/G$. One can identify L_λ as the *complex length* of the bending lamination relative to a certain choice of transverse measure. It is these functions which are the analogue of the trace functions we seek.

The 'horizontal' lines in Picture 3 are the level curves $\{\mu \in \mathcal{M}|L_{pl(\mu)}(\mu) = \text{constant}\}$. These level curves can be shown to be continuous on μ.

Thus the rays $\mathcal{P}_{p/q}$ and the level curves together form a co-ordinate grid for μ.

Theorem 5.4. *The map $\pi : \mathcal{M} \to \mathbb{R} \times \mathbb{R}^+$, $\pi : \mu \mapsto (pl(\mu), L_{pl(\mu)}(\mu))$, is a homeomorphism.*

We call the co-ordinate system given by Theorem 5.4, pleating co-ordinates.

We have sketched in these lectures all of the proof of Theorem 5.4, except for the surjectivity of π. A method for the surjectivity, involving a different kind of argument in $\mathbb{H}^3$, was suggested by J.-P. Otal. Details of the remainder are to be found in [5]. The proofs of continuity of pl and related results, in a more general context are in [4]. The results of [2], which also use different techniques in $\mathbb{H}^3$, show that the cusp groups at the end of the pleating rays are the *unique* p/q cusps.

Generalisations and comments

1. We have seen that the lines $\mathcal{P}_{p/q}$ limit in a unique point on $\partial\mathcal{M}$ which is a p/q-cusp group. It is a much harder problem to study the limiting behavior of the irrational pleating rays as $\mu \to \partial\mathcal{M}$. The question is related to the rigidity of groups with specified ending lamination; and is equivalent to the question of whether $\partial\mathcal{M}$ is a Jordan curve. This result has been announced by McMullen and will have important implications.

2. Pleating co-ordinates can be set up for more general Teichmüller spaces and more generally for deformation spaces of Kleinian groups. One other one dimensional case has been looked at in detail and appears in [6]. Since usually the parameter space has more than one complex dimension the situation gets much more complicated. Together with J. Parker we are investigating in detail the case of a torus with two punctures, where the parameter space has complex dimension two. It appears that we have we have all the main ingredients for a general theory.

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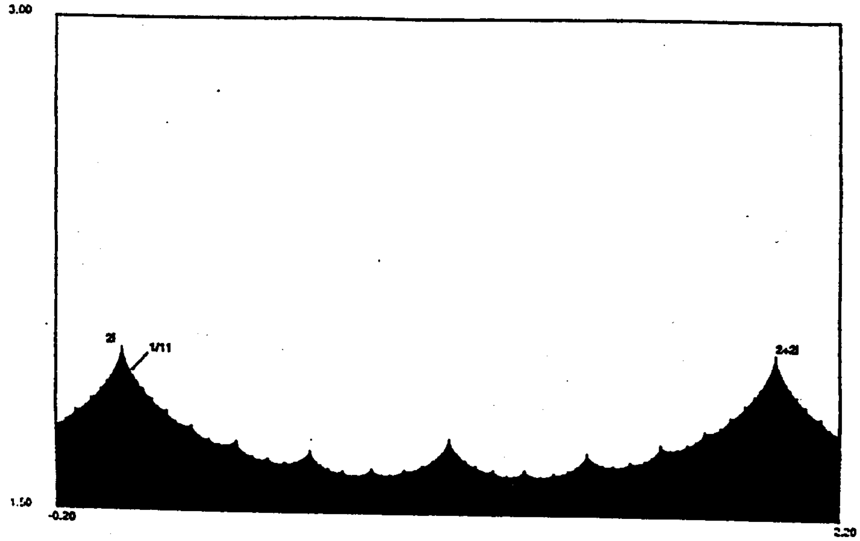
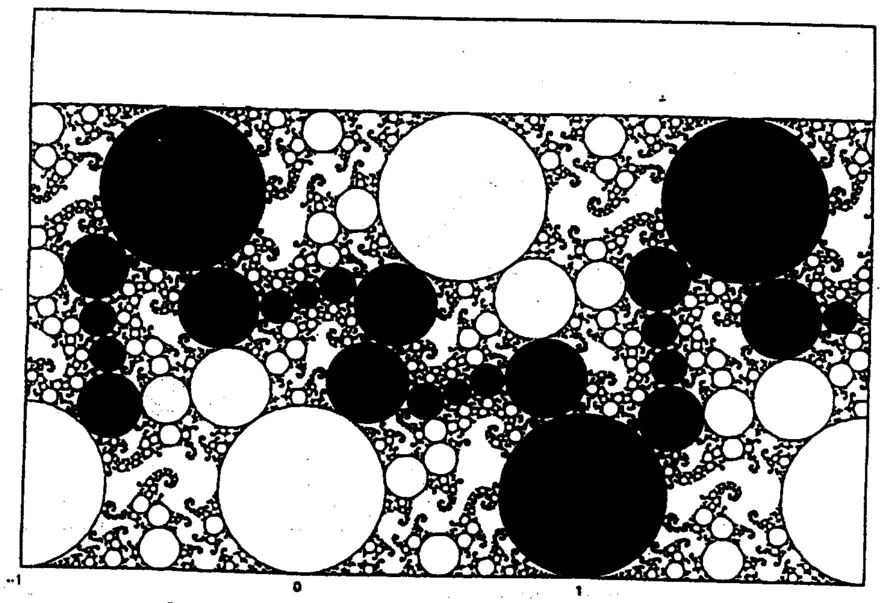
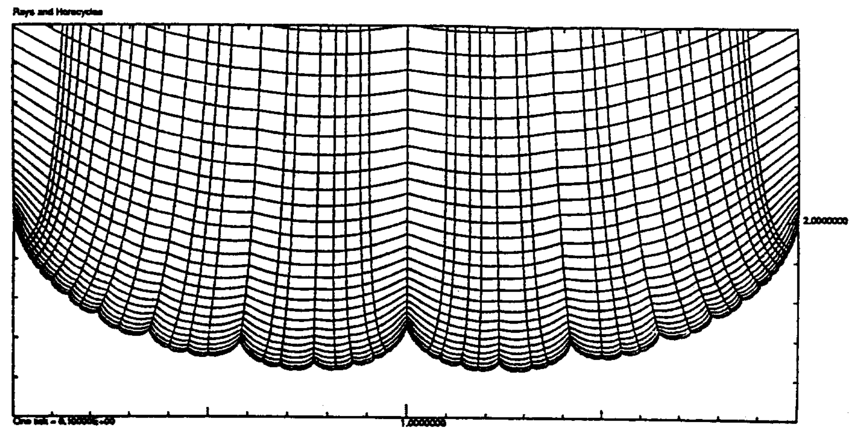


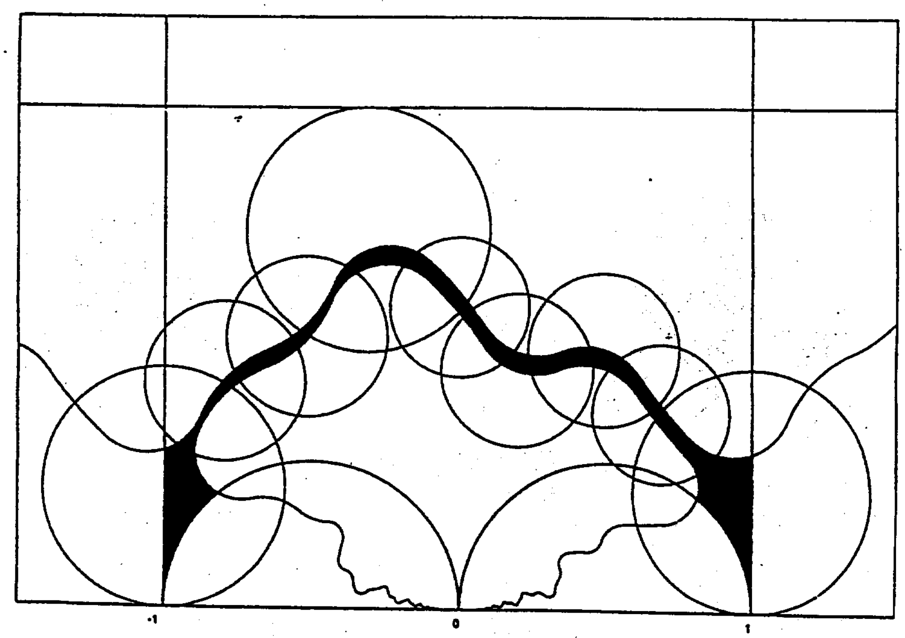
Picture 1. Boundary of $\mathcal{T}$.



Picture 2. Limit set for 5/16 cusp group.



Picture 3. Pleating coordinates for $\mathcal{M}$.



Picture 4. Circle chains and fundamental domain for p/q = 3/8.