

A Garside type structure on the Torelli group

Daan Krammer
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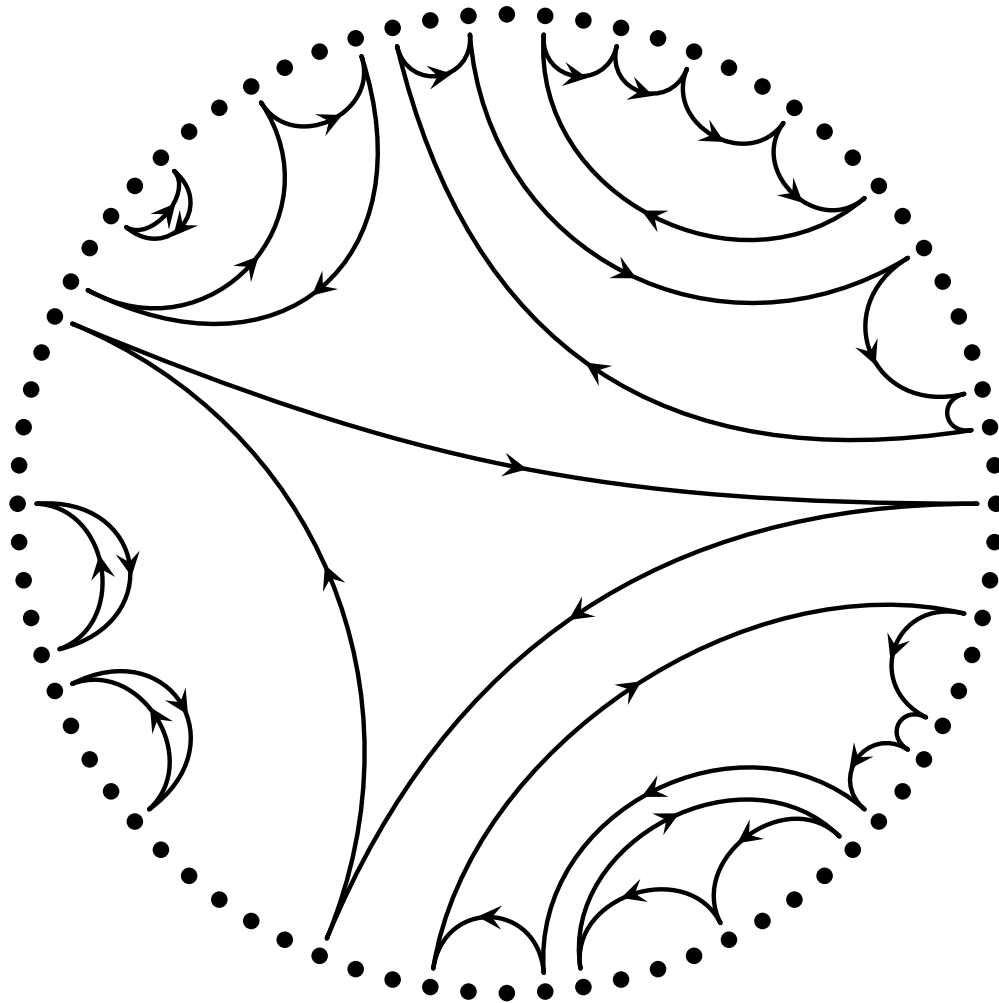
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- Recently, weak versions of “Garside groups” were considered. We shall weaken much further.

Plan of the talk:

- BKL Garside structure.
- Simple example.
- Crash course on Torelli groups.
- General set-up.
- Why does it generalise BKL?

Definition. A *(BKL) simple braid* is a braid like the following example:



Definition.

1. $\Omega := \{\text{simple braids}\}$.
2. $B_n^+ := \text{submonoid } \langle \Omega \rangle \text{ of } B_n$.
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Definition. A *lattice* is an ordered set such that for all $x, y \in L$ there is a least common upper bound or **join** $x \vee y$ and a greatest common lower bound or **meet** $x \wedge y$.

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Theorem. (B_n, \leq) is a lattice.

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2. The greatest simple braid $\leq x_i x_{i+1}$ (which always exists) equals x_i . Same for y_i, y_{i+1} .
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- B_n is automatic (Thurston, 80's).
- Better than automaticity: Grid property to be explained next.

Definition. Let $a, b \in B_n$. The *distinguished path* from a to b is $\{a_0, \dots, a_r\} \subset B_n$ defined by:

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Definition. The *Cayley graph* is the graph with vertex set B_n and edges $\{a, ax\}$ whenever $a \in B_n$ and x is a simple braid.

Theorem (Grid property, K-Dehornoy). *The convex hull of three points in B_n is a planar graph. (This can and should be made more precise).*

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We say that BKL discovered a ***Garside structure*** on B_n . All of the above properties are true for all Garside groups.

A simple example

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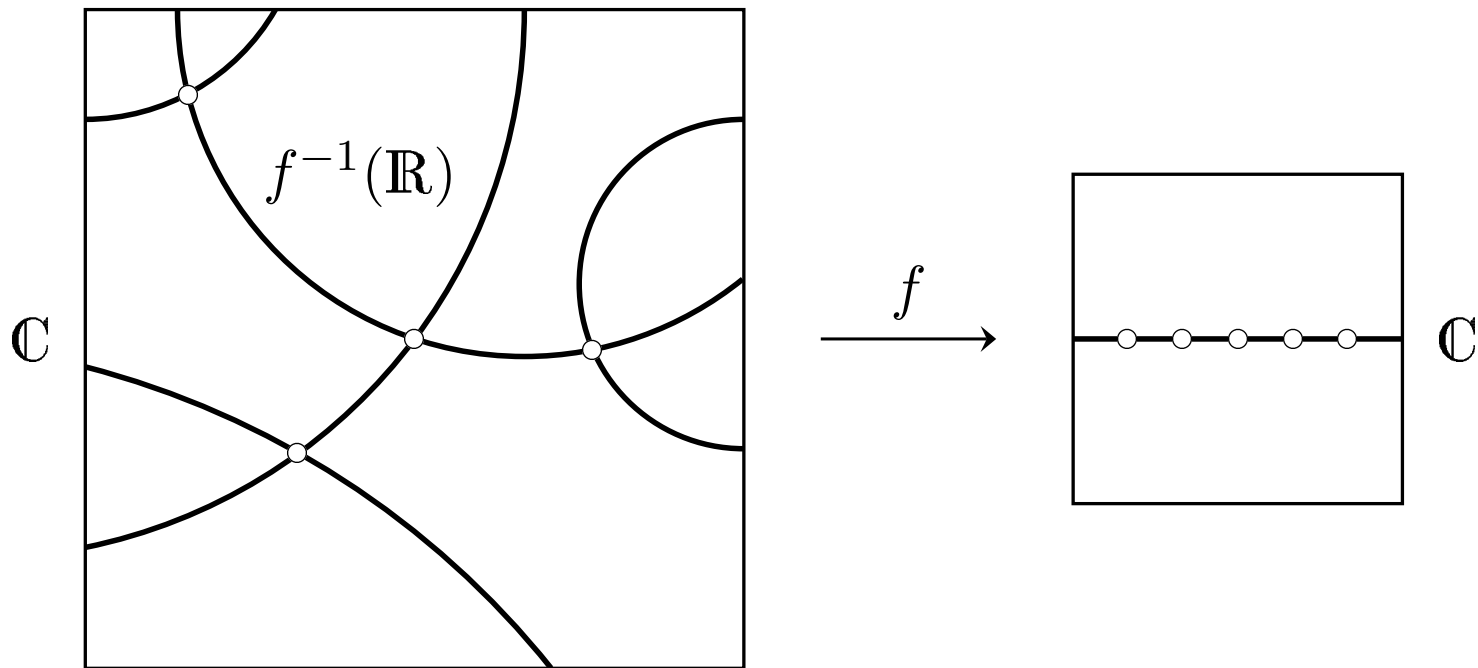
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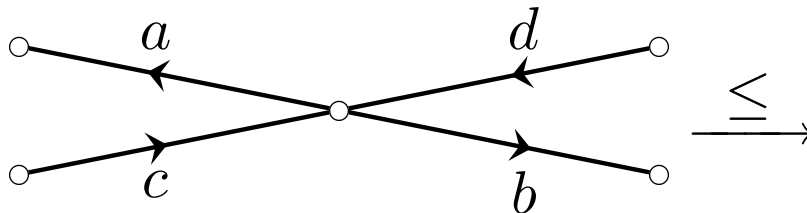
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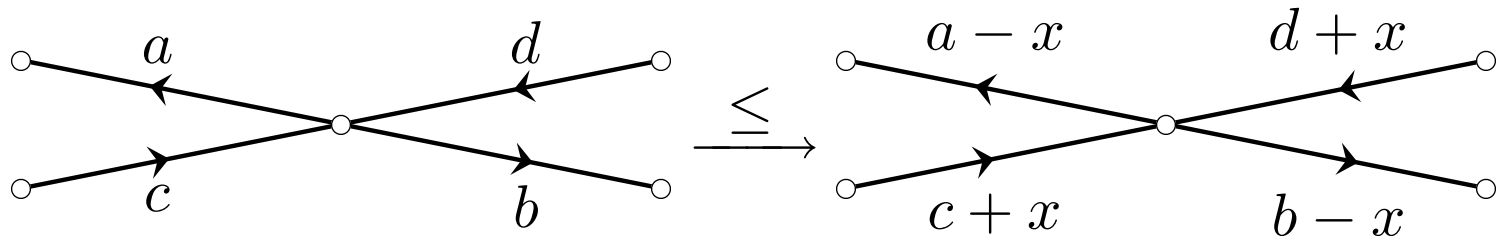
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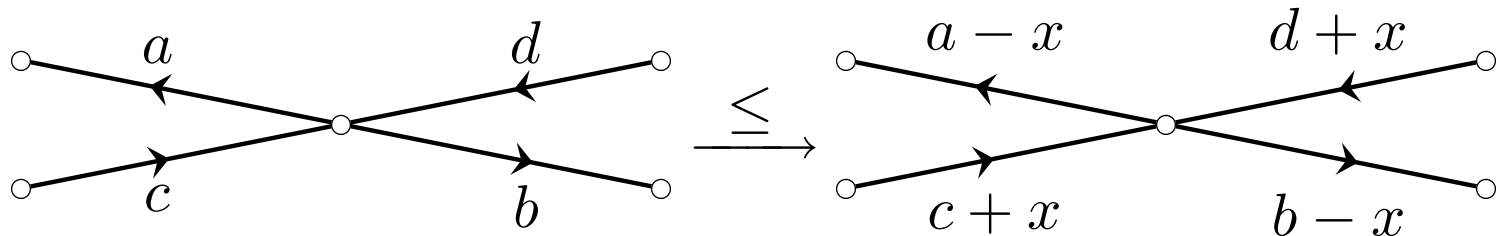
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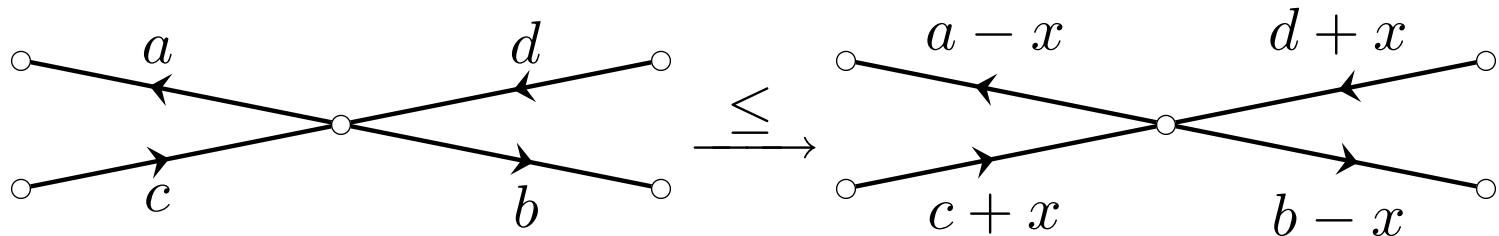
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We will soon see examples of critical points bumping into each other or splitting.

A simple example

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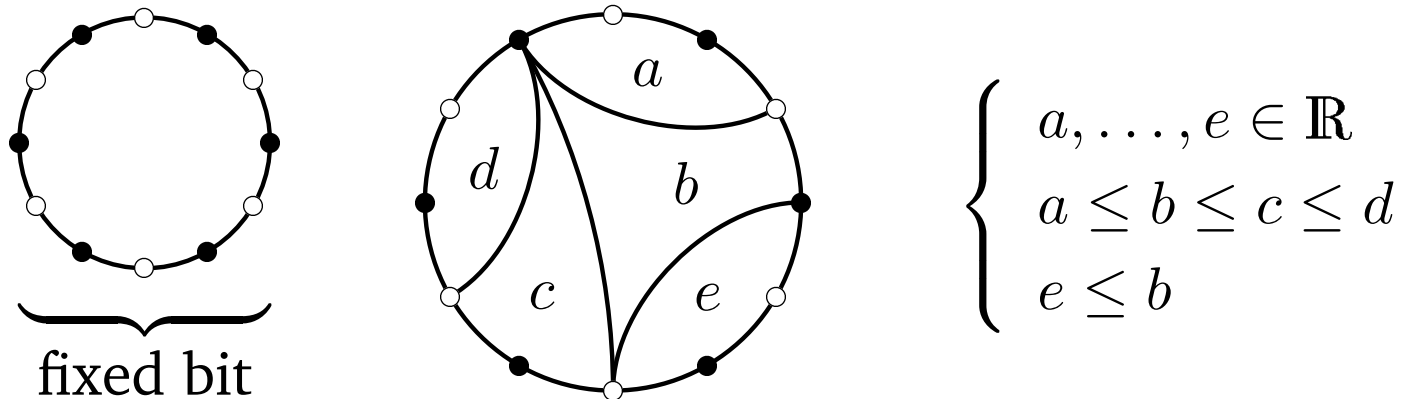
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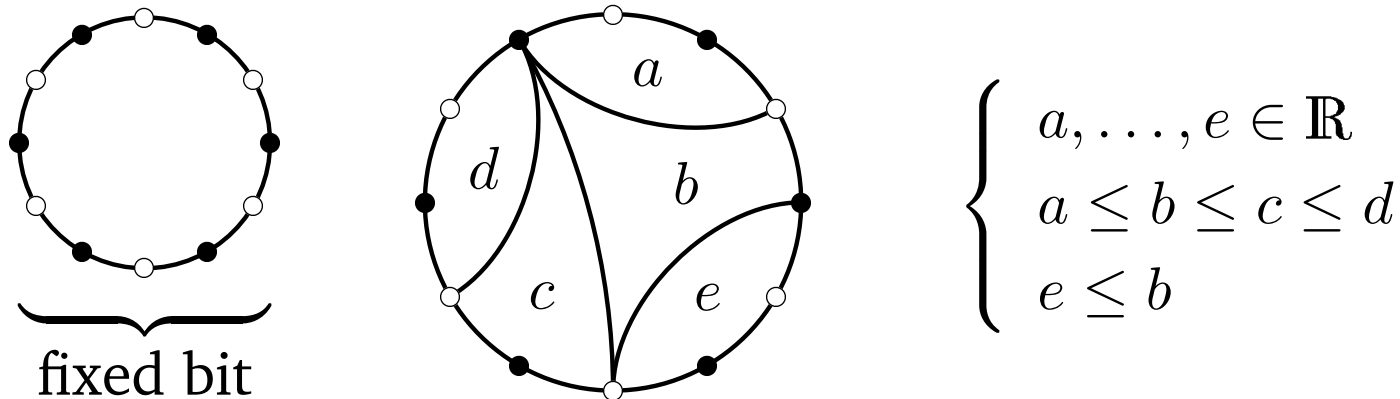
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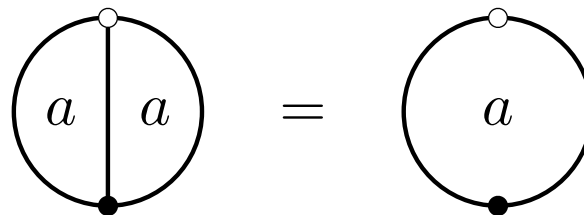
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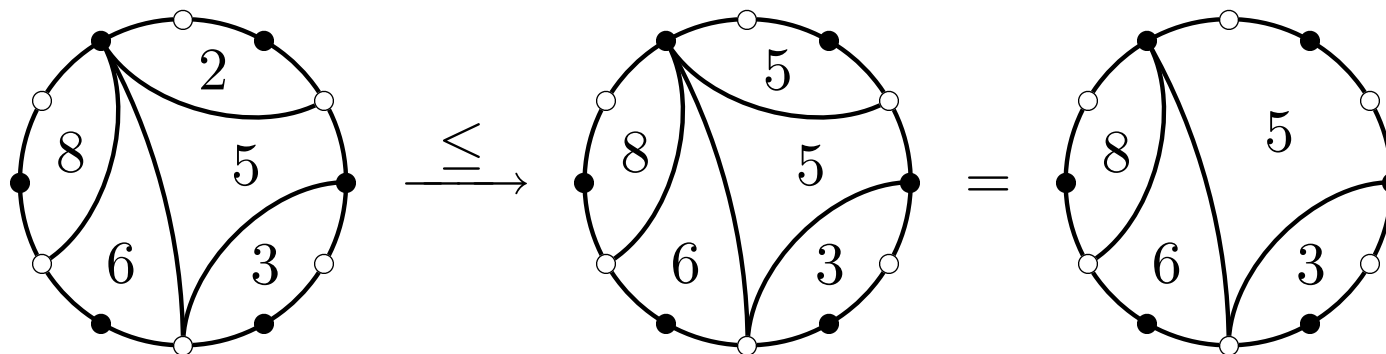


Always modulo removing arcs between regions of equal heights:



A simple example

Example of a generator of the ordering:



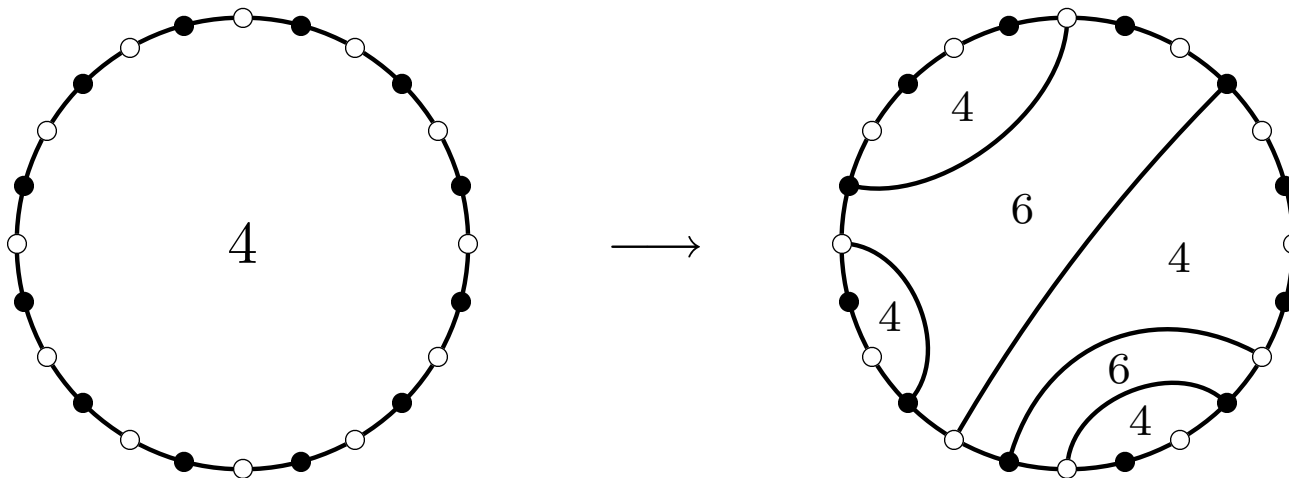
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A simple example

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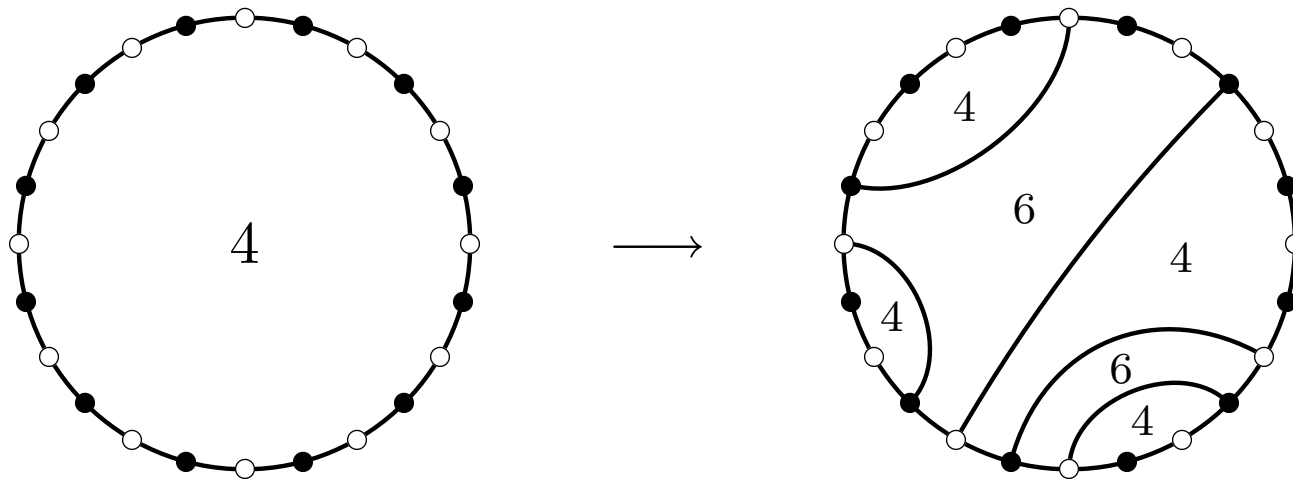
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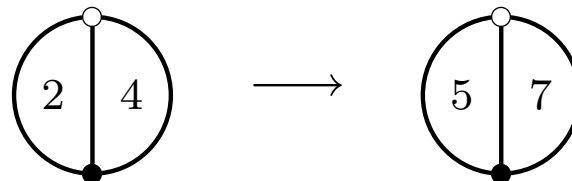
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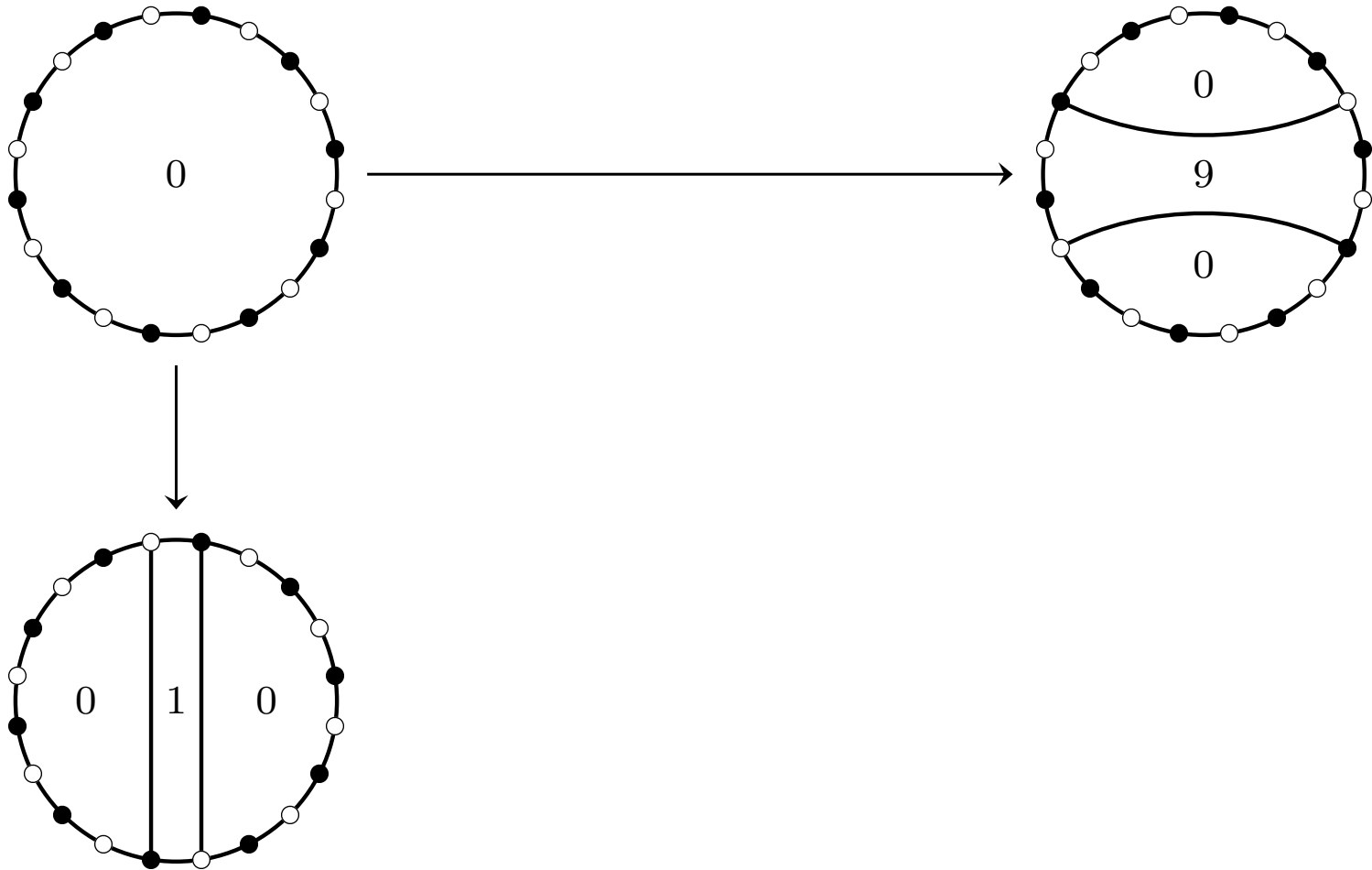


Example. Another example of a semi-simple pair $(x, y) = (x \rightarrow y)$, which is allowed even though $5 > 4$:



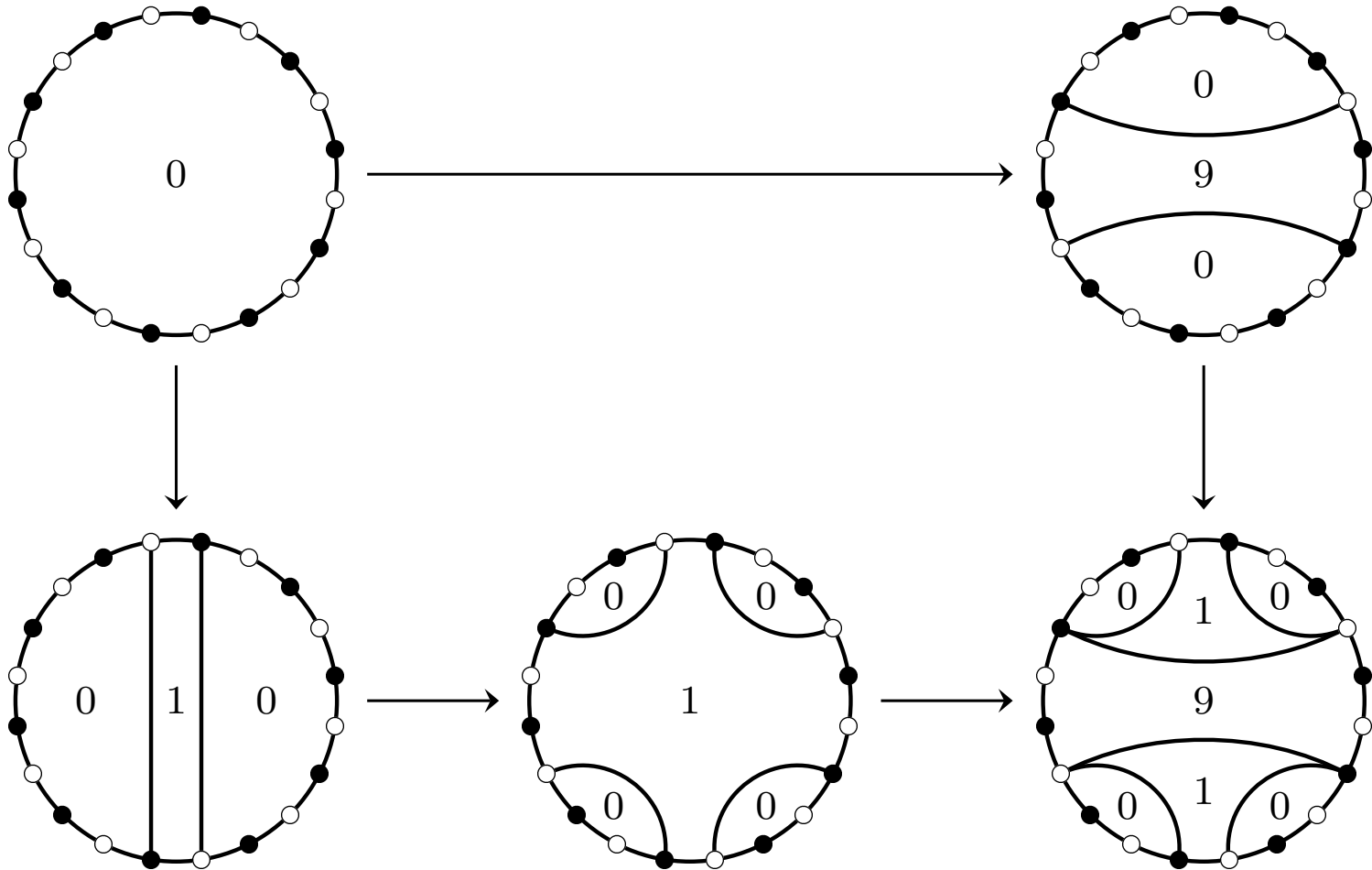
A simple example

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A simple example

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Warning. It is *not* true that if $x \leq y \leq z$ and (x, z) is semi-simple, then (x, y) or (y, z) is semi-simple.

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Explanation. Between any two distinct real numbers there is another real number. The semi-simple pairs behave like a preferred small set of positive normal forms.

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Definition. Let $x \in K$ and $a \in \mathbb{R}$. Then $(x, x\Delta^a)$ is the semi-simple pair where all critical values move to the right by a .

A simple example

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3. Let $x, y \in K$, $x < y$, $d(x, y) = a$. Consider the path from x to y

$$\begin{cases} [0, a] \longrightarrow K \\ t \longmapsto x\Delta^t \wedge y. \end{cases}$$

There is a finite sequence $x = x_0, \dots, x_k = y$ such that (x_i, x_{i+1}) is semi-simple and the path passes through all x_i .

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4. These paths satisfy a grid property. □

A simple example

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Suggestion. Study *Garside spaces* instead of say CAT(0) spaces.

Definition. The *mapping class group* $\text{MCG}(S)$ of a surface S is H/H_0 where H is the topological group of (orientation preserving) self-homeomorphisms of S , preserving the boundary pointwise, and $H_0 \subset H$ is the connected component of 1 in H .

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Fact 3. *If S is compact then $\text{MCG}(S)$ is finitely presented.* \square

Fact 4. *Let $g \geq 0$. The set of isomorphism classes of genus g Riemann surfaces can be given the structure of an algebraic orbifold (moduli space); its fundamental group is $\text{MCG}(S)$.* \square

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The quotient $\text{MCG}(S)/I(S)$ is an arithmetic group and infinite in general.

The general set-up

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Example 1. $\ker \phi = (\pi_1 S)'$.

Example 2. $S = \mathbb{C} \setminus \{\alpha_1, \dots, \alpha_n\}$,

$$\phi(\gamma) = \frac{1}{2\pi i} \int_{\gamma} \sum_i \operatorname{dlog}(x - \alpha_i).$$

So $\operatorname{im} \phi = \mathbb{Z}!$.

Definition. An element of K is a cell decomposition of S with $Q_0 \cup Q_1$ for vertices (as before); the height function is now a function

$$h: \left\{ \text{regions of } \tilde{S} \right\} \longrightarrow \mathbb{R}$$

such that, for all regions R and all $\gamma \in \pi_1 S$:

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Also, a universal cover \tilde{S} of S and a function $f: \tilde{S} \rightarrow \mathbb{C}$ such that $df = \omega$ and $f(p) \in \mathbb{R}$ for one (hence all) critical points p . □

The general set-up

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Definition/Theorem. The remaining definitions and theorems are as in the special case of $S = \mathbb{C}$. □

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- We can replace \mathbb{R} by any totally ordered abelian group. In our case, $2\pi\mathbb{Z}$ suffices.