A Garside type structure on the Torelli group

Daan Krammer
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Garside structures on B_n 2

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- 1998, Birman, Ko, Lee (BKL). New Garside structure on B_n (to be generalised today).
- Recently, weak versions of "Garside groups" were considered. We shall weaken much further.

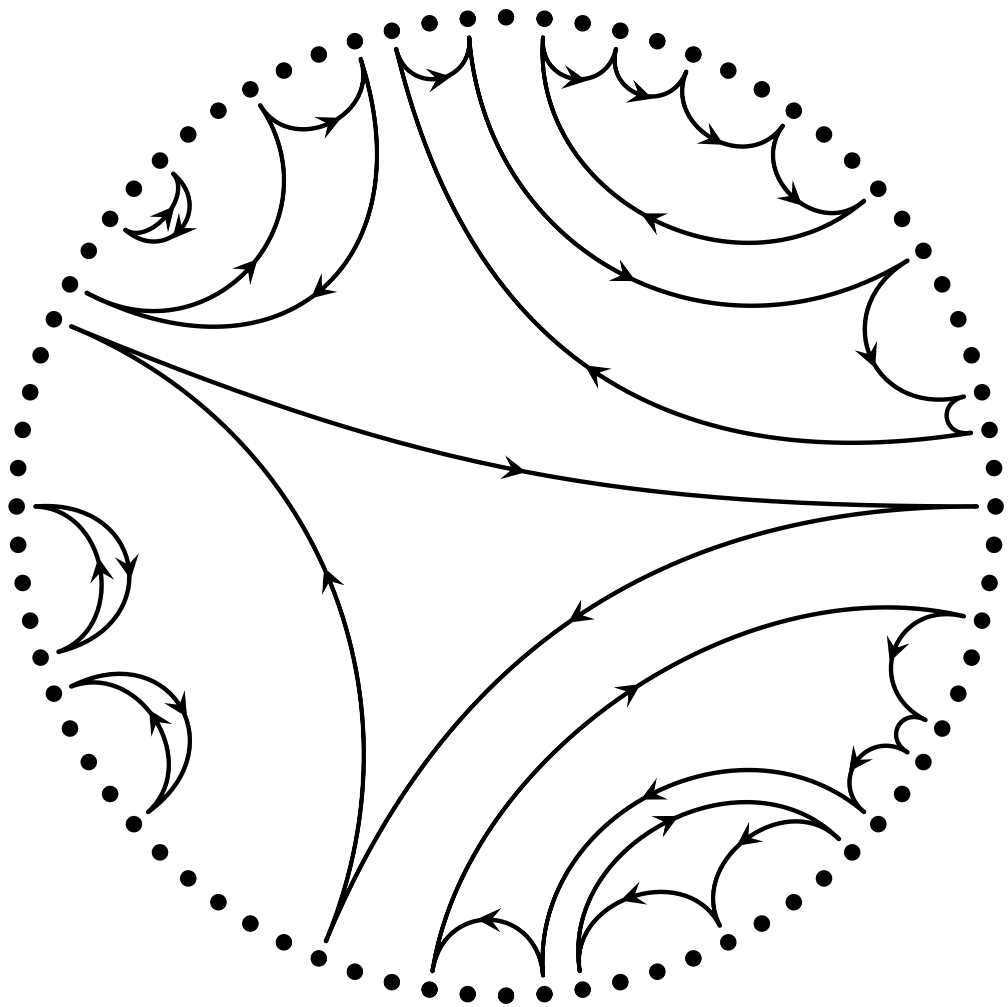
Garside structures on B_n 3

Plan of the talk:

- BKL Garside structure.
- Simple example.
- Crash course on Torelli groups.
- General set-up.
- Why does it generalise BKL?

Birman, Ko, Lee's structure on B_n

Definition. A ***(BKL) simple braid*** is a braid like the following example:



Birman, Ko, Lee's structure on B_n

Definition.

1. $\Omega :=$ {simple braids}.
2. $B_n^+ :=$ submonoid $\langle \Omega \rangle$ of B_n.
3. $\leq$ ordering on B_n: $x \leq xy \Leftrightarrow y \in B_n^+$.

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Definition. A ***lattice*** is an ordered set such that for all $x, y \in L$ there is a least common upper bound or ***join*** $x \vee y$ and a greatest common lower bound or ***meet*** $x \wedge y$.

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Theorem. $(B_n, \leq)$ *is a lattice.*

Theorem. *Every $x \in B_n$ can uniquely be written in the* ***symmetric normal form*** $x_k^{-1} \cdots x_1^{-1} y_1 \cdots y_\ell$ *defined by the following properties:*

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Theorem. *Every* $x \in B_n$ *can uniquely be written in the* ***symmetric normal form*** $x_k^{-1} \cdots x_1^{-1}\, y_1 \cdots y_\ell$ *defined by the following properties:*

1. x_i *and* y_j *are nontrivial simple braids.*
2. *The greatest simple braid* $\leq x_i\, x_{i+1}$ *(which always exists) equals* x_i*. Same for* y_i*,* y_{i+1}*.*
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- Better than automaticity: Grid property to be explained next.

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Definition. Let $a, b \in B_n$. The ***distinguished path*** from a to b is $\{a_0, \ldots, a_r\} \subset B_n$ defined by:

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Definition. A subset of B_n is ***convex*** if it contains an entire distinguished path as soon as it contains its endpoints.

Definition. The ***Cayley graph*** is the graph with vertex set B_n and edges $\{a, ax\}$ whenever $a \in B_n$ and x is a simple braid.

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Theorem (Grid property, K–Dehornoy). *The convex hull of three points in B_n is a planar graph. (This can and should be made more precise).*

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We say that BKL discovered a ***Garside structure*** on B_n. All of the above properties are true for all Garside groups.

A simple example

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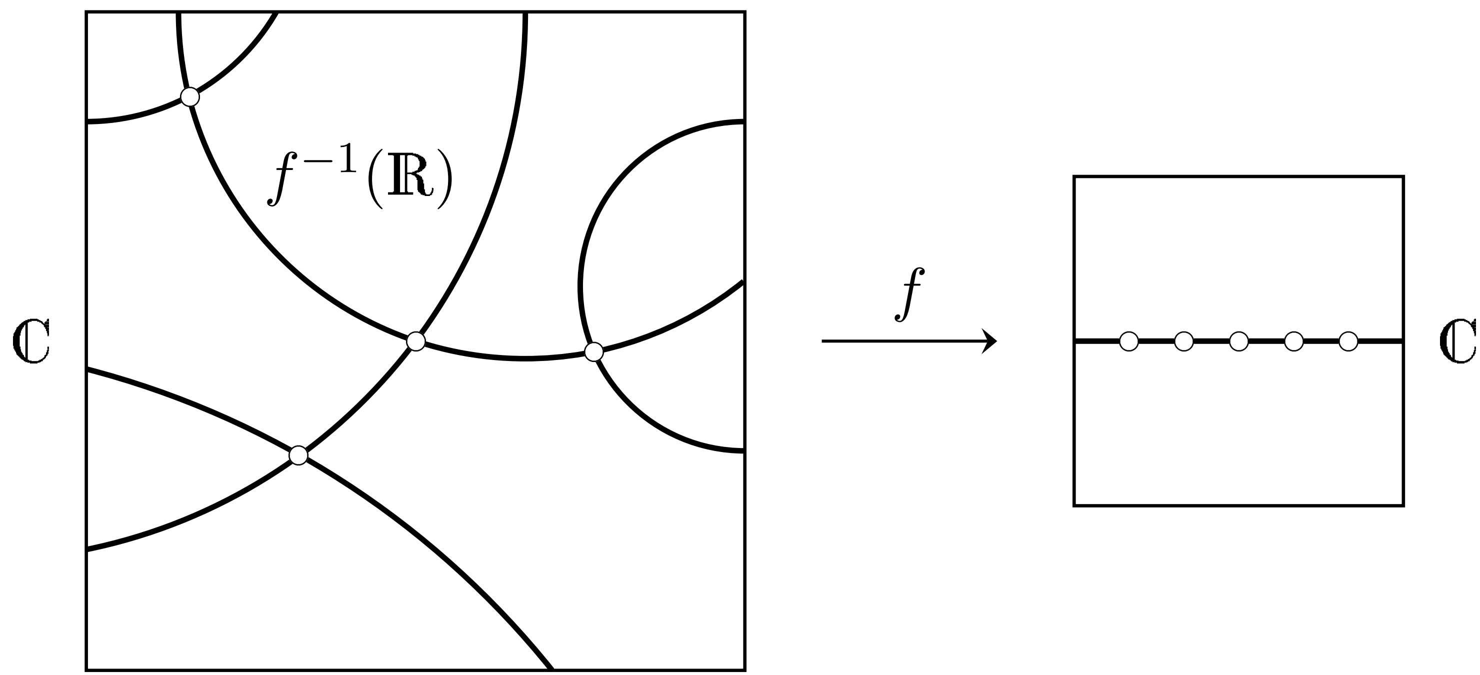

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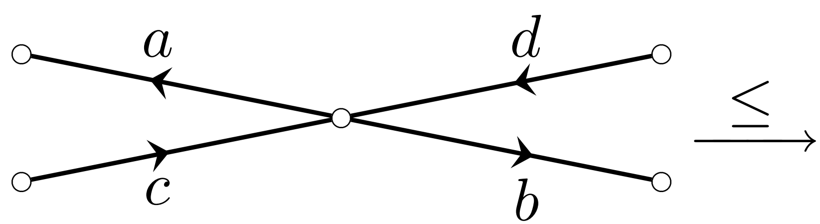

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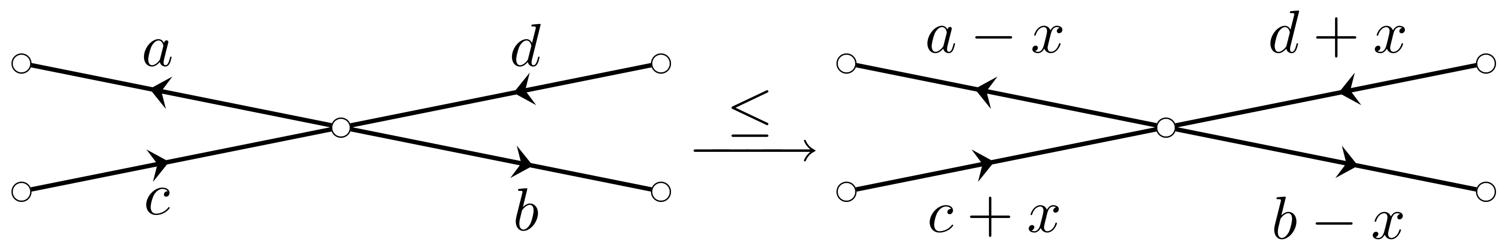

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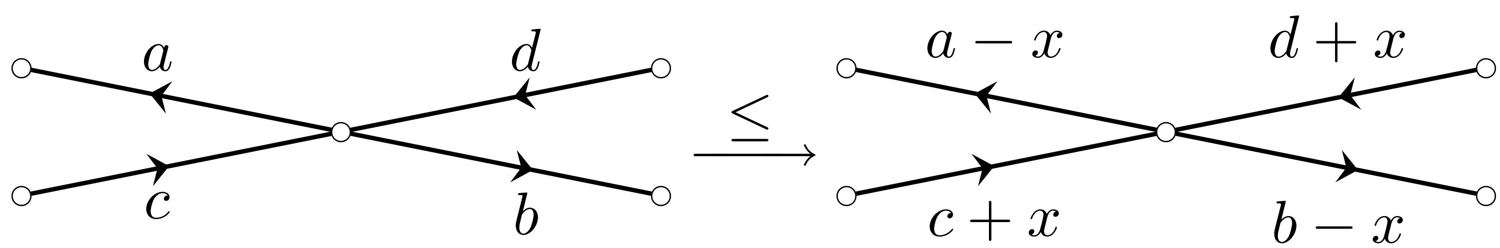

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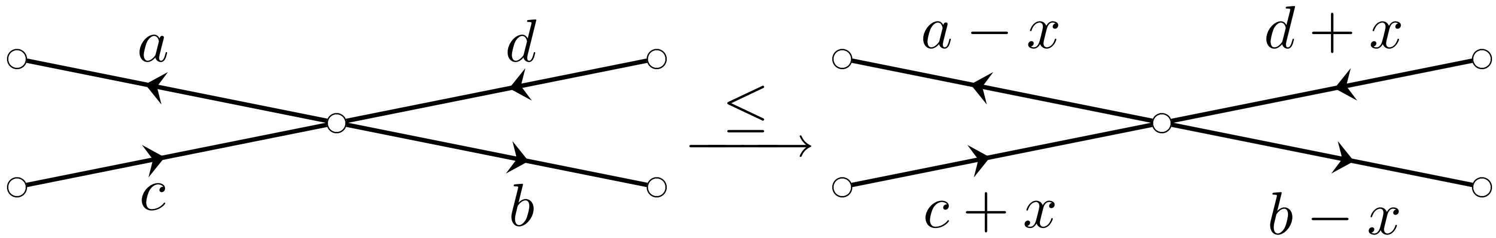

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We will soon see examples of critical points bumping into each other or splitting.

From now on, we mostly ***work dually***. Critical points become 2-cells (called ***regions***), and so on.

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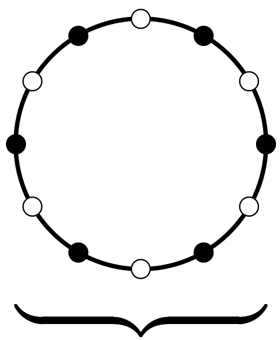
Example of an element of K:



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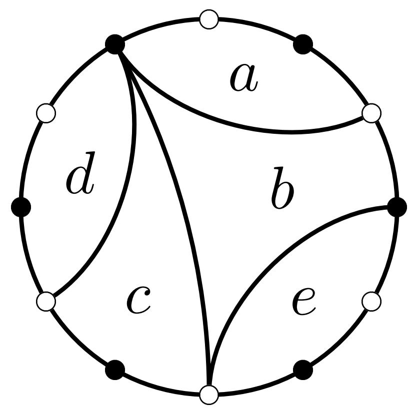

$$\begin{cases} a, \ldots, e \in \mathbb{R} \\ a \le b \le c \le d \\ e \le b \end{cases}$$

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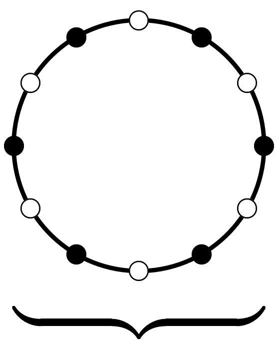

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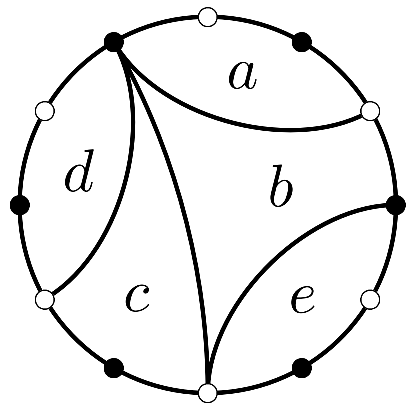

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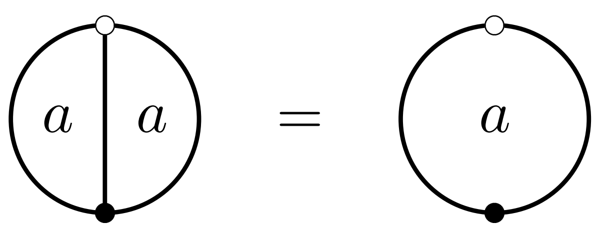
Always modulo removing arcs between regions of equal heights:



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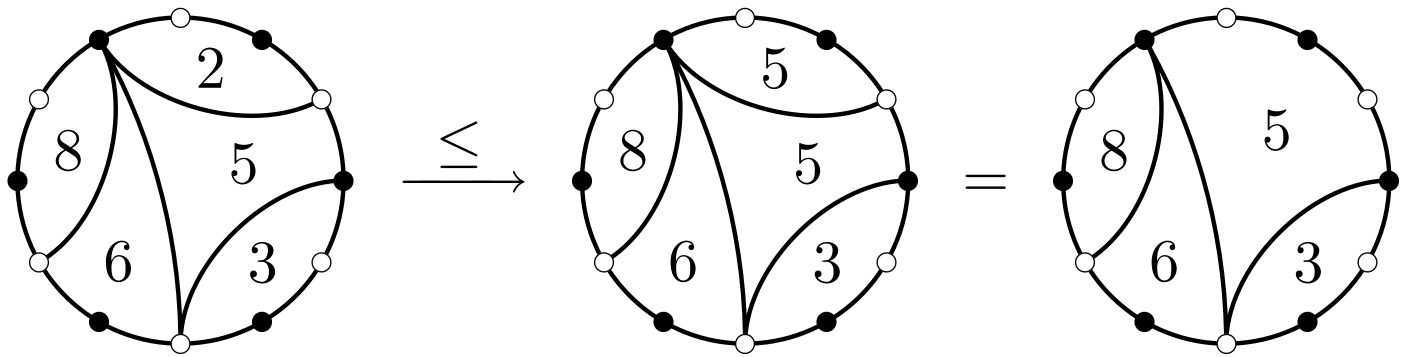
Example of a generator of the ordering:



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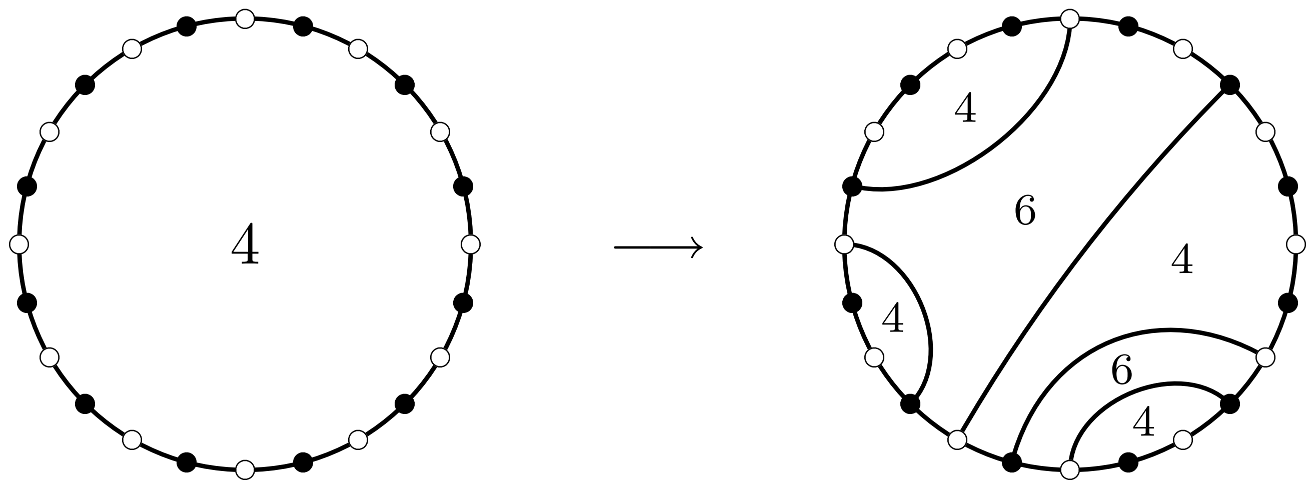

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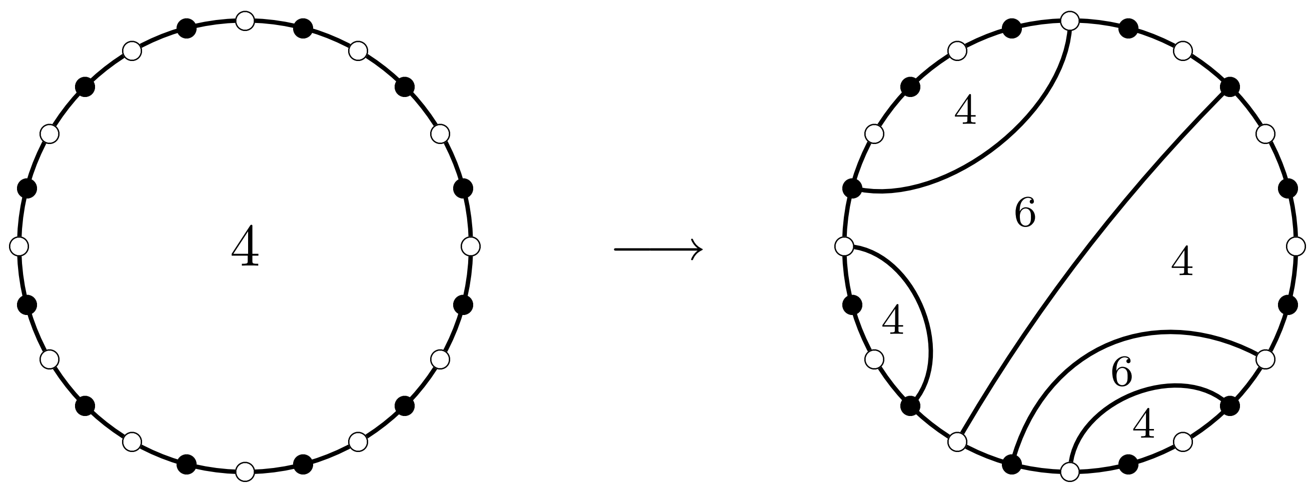
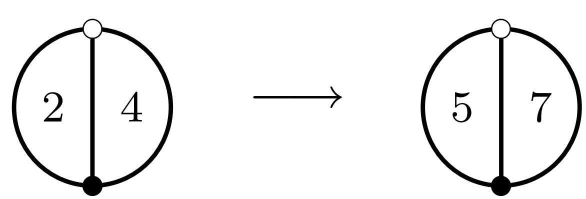


Example. Another example of a semi-simple pair $(x, y) = (x \to y)$, which is allowed even though $5 > 4$:



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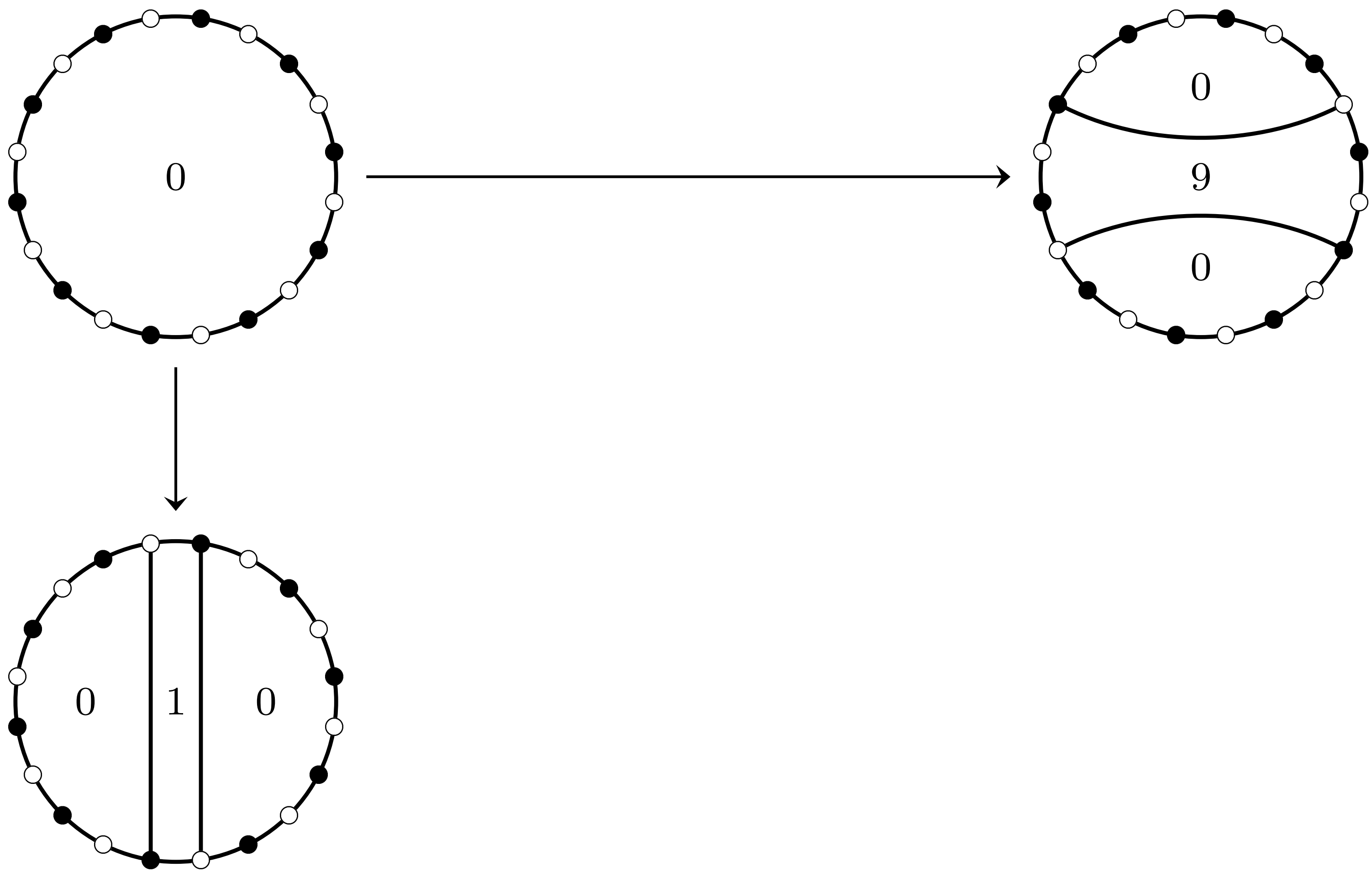

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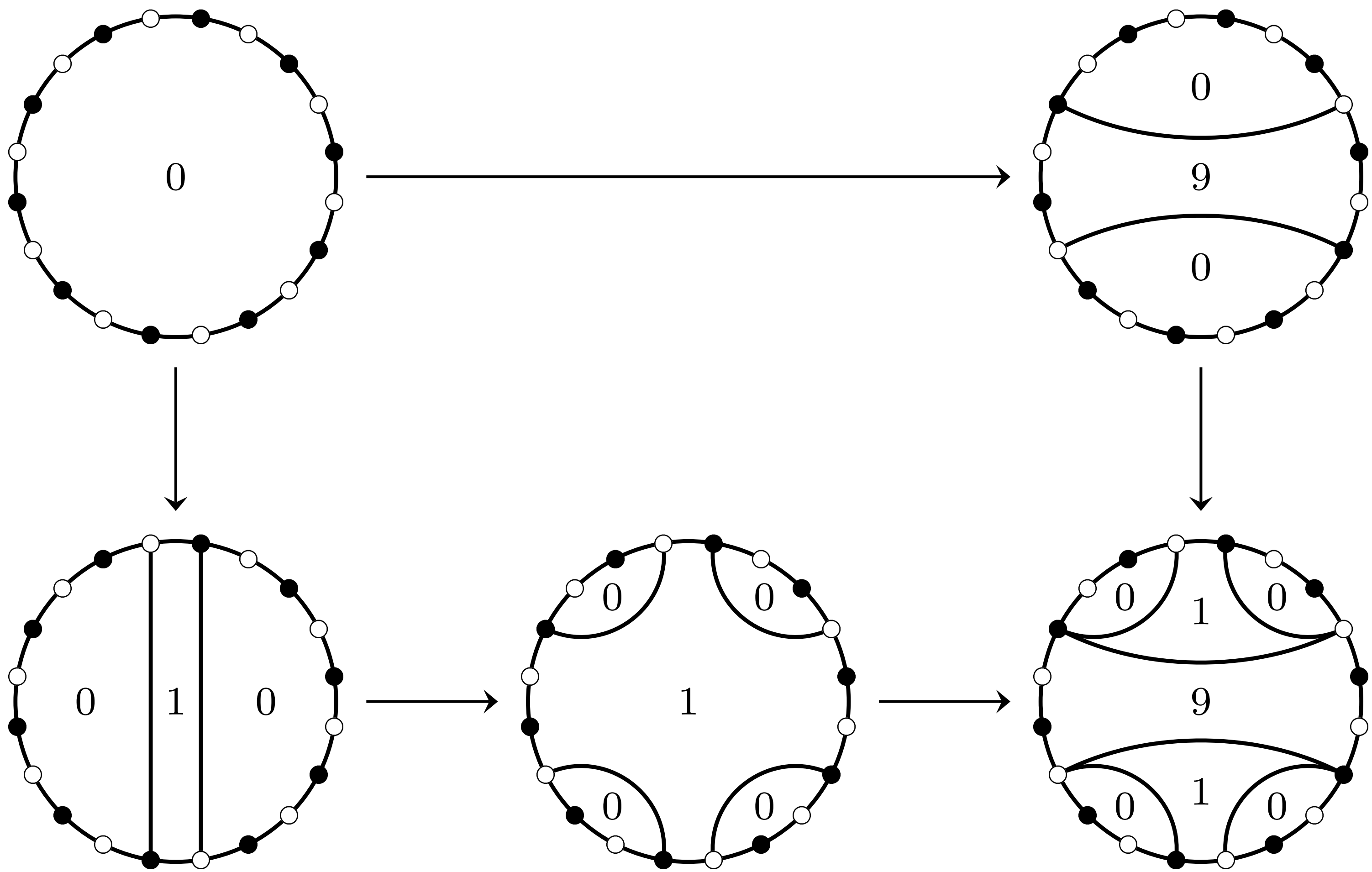

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Warning. It is ***not*** true that if $x \leq y \leq z$ and (x, z) is semi-simple, then (x, y) or (y, z) is semi-simple.

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Explanation. Between any two distinct real numbers there is another real number. The semi-simple pairs behave like a preferred small set of positive normal forms.

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Definition. Let $x \in K$ and $a \in \mathbb{R}$. Then $(x, x\Delta^a)$ is the semi-simple pair where all critical values move to the right by a.

A simple example

Theorem. *1.* $(K, \leq)$ *is a lattice.*

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3. *Let $x, y \in K$, $x < y$, $d(x, y) = a$. Consider the path from x to y*

$$\begin{cases} [0, a] \longrightarrow K \\ \qquad t \longmapsto x\Delta^t \wedge y. \end{cases}$$

There is a finite sequence $x = x_0, \ldots, x_k = y$ such that (x_i, x_{i+1}) is semi-simple and the path passes through all x_i.

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4. *These paths satisfy a grid property.* $\square$

A simple example

Suggestion. Study ***Garside spaces*** instead of say CAT(0) spaces.

Definition. The ***mapping class group*** $\mathrm{MCG}(S)$ of a surface S is H/H_0 where H is the topological group of (orientation preserving) self-homeomorphisms of S, preserving the boundary pointwise, and $H_0 \subset H$ is the connected component of 1 in H.

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Crash course on Torelli group

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Fact 4. *Let $g \geq 0$. The set of isomorphism classes of genus g Riemann surfaces can be given the structure of an algebraic orbifold (moduli space); its fundamental group is $\mathrm{MCG}(S)$.* $\square$

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The quotient $\mathrm{MCG}(S)/I(S)$ is an arithmetic group and infinite in general.

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Example 1. $\ker\phi = (\pi_1 S)'$.

Example 2. $S = \mathbb{C} \setminus \{\alpha_1, \ldots, \alpha_n\}$,

$$\phi(\gamma) = \frac{1}{2\pi i} \oint_\gamma \sum_i \mathrm{dlog}(x - \alpha_i).$$

So $\operatorname{im}\phi = \mathbb{Z}$!.

The general set-up

Definition. An element of K is a cell decomposition of S with $Q_0 \cup Q_1$ for vertices (as before); the height function is now a function

$$h\colon \left\{\text{regions of } \widetilde{S}\right\} \longrightarrow \mathbb{R}$$

such that, for all regions R and all $\gamma \in \pi_1 S$:

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Also, a universal cover $\widetilde{S}$ of S and a function $f\colon \widetilde{S} \to \mathbb{C}$ such that $df = \omega$ and $f(p) \in \mathbb{R}$ for one (hence all) critical points p. $\square$

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Observation. The Torelli group $I(S)$ acts on K.

K looks a lot like Teichmüller space T (:=orbifold universal cover of moduli space=space of homotopy classes of complex structures on S); K and T are both simply connected spaces on which $I(S)$ act.

Definition/Theorem. The remaining definitions and theorems are as in the special case of $S = \mathbb{C}$. $\square$

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- The ordering on K is generated by moving critical values around the unit circle in positive direction.
- We can replace $\mathbb{R}$ by any totally ordered abelian group. In our case, $2\pi\mathbb{Z}$ suffices.