Garside theory

Daan Krammer

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Abstract

These are very preliminary notes on Garside theory. One of their aims is possible use in a common project with David Bessis, Patrick Dehornoy, François Digne and Jean Michel.

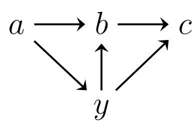
Contents

1 Axioms

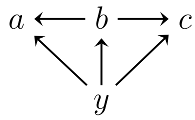
Definition 1. Let (V, E) be graph, that is, V is a set (of vertices) and $E \subset V \times V$ is a set of edges (or a binary relation). We write $x \to y$ for $(x, y) \in E$.

In (2)–(10) we list some properties that may or may not be satisfied.

- We have $(x, x) \in E$ for all $x \in V$. We call (x, x) a *loop*. (2)
- The graph is connected (as unoriented graph). (3)
- The fundamental group of the graph is generated by those conjugacy classes associated with closed (unoriented) paths of at most 3 edges. (4)
- There exists a (necessarily unique) ordering $\leq$ on V generated by E. (5)
- Let $a \to x \to c$. Then there exists $b \in V$ such that $a \to b \to c$ and if $y \in V$ is such that $a \to y \to c$ then $y \to b$. (6)



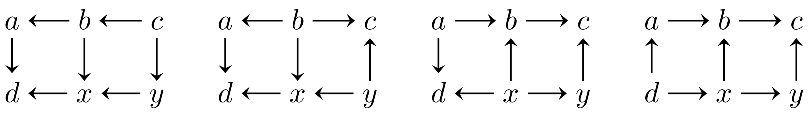
- Let $a \leftarrow x \rightarrow c$. Then there exists $b \in V$ such that $a \leftarrow b \rightarrow c$ and if $y \in V$ is such that $a \leftarrow y \rightarrow c$ then $y \rightarrow b$. (7)



We call (a, b, c) a *greedy path of type 1* (of length 2) if the conclusion of 6 holds. We also call (c, b, a) a *greedy path of type 2.* We call (a, b, c) (hence (c, b, a)) a *greedy path of type 3* if the conclusion of 7 holds.

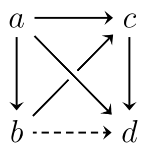
It may happen that something is a greedy path of one type, not another, even though the arrows (are allowed to) point in the required direction. Here is the simplest example. Let $a \rightarrow b$ but $a \neq b$. Then $a \leftarrow a \rightarrow b$ and the triple (a, a, b) is greedy of type 3. At the same time we have $a \rightarrow a \rightarrow b$ and (a, a, b) is not greedy of type 1 (because that would be (a, b, b)). The conclusion is that a statement of the form 'this path is greedy' is only well-defined if the directions of the arrows have been specified.

- Let $a \rightarrow x \leftarrow c$. Then there exists y with $a \leftarrow y \rightarrow c$. (8)
- In each of the 4 diagrams (9)



(9a) (9b) (9c) (9d)

  the following holds. If abc, dxb, xyc are greedy (relative the arrows of the diagram) then so is dxy.
- If the solid arrows in the diagram (10)



  exist, then so does the dashed arrow.

Definition 11. For $k, \ell \geq 0$, we write $X(k, \ell)$ for the set of those $(x_{-k}, x_{-k+1}, \ldots, x_{\ell-1}, x_\ell) \in V^{k+\ell+1}$ such that $x_{i-1} \rightarrow x_i$ for all $i \in \{1, \ldots, \ell\}$ and $x_{-i+1} \rightarrow x_{-i}$ for all $i \in \{1, \ldots, k\}$.

We define relations $\circ\!\!\longrightarrow_i$ $(-k < i < \ell)$ on $X(k, \ell)$ by:

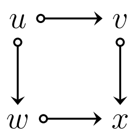
$$(a_{-k}, \ldots, a_\ell) \circ\!\!\longrightarrow_i (b_{-k}, \ldots, b_\ell) \iff a_i \rightarrow b_i \text{ and } a_j = b_j \text{ for all } j \neq i.$$

Note that these imply $a_{-k} = b_{-k}$ and $a_\ell = b_\ell$. We write $a \circ\!\!\longrightarrow b$ if and only if $a \circ\!\!\longrightarrow_i b$ for some i. An element $a \in X(k, \ell)$ is called *greedy* if for all b we have $a \circ\!\!\longrightarrow b \Rightarrow a = b$. As noted above, an element of $X(k, \ell) \cap X(k', \ell')$ can be greedy in one not the other.

From now on we assume that (V, E) satisfies (2)–(10).

2 Uniqueness of greedy paths

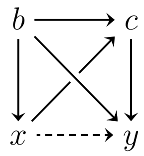
Lemma 12. *Let $u, v, w \in X(k,\ell)$ be such that $u \circ\!\!\longrightarrow v$ and $u \circ\!\!\longrightarrow w$. Then there exists $x \in X(k,\ell)$ such that $v \circ\!\!\longrightarrow x$ and $w \circ\!\!\longrightarrow x$.*



Proof. Suppose $u \circ\!\!\longrightarrow_i v$ and $u \circ\!\!\longrightarrow_j w$. If $|i-j| > 1$ then it is clear.

Suppose next $i = j$. We may suppose $(k, i, \ell) \in \{(-1, 0, 1), (0, 1, 2)\}$. First suppose it is $(0, 1, 2)$. Write $u = (a, u_0, c)$. Define b as in (6). Then (6) proves that $x := (a, b, c)$ has the required properties. If $(k, i, \ell) = (-1, 0, 1)$ then the argument is similar and uses (7) instead of (6).

Suppose now $|i - j| = 1$, say, $j = i + 1$. Then we may suppose $(k, i, j, \ell) \in \{(-1, 0, 1, 2), (0, 1, 2, 3)\}$. First suppose it is $(0, 1, 2, 3)$. Write $u = (a, b, c, d)$, $v = (a, x, c, d)$, $w = (a, b, y, d)$. Then the solid arrows of



exist. By (10), the dashed arrow exists. So $x := (a, x, y, d)$ has the required properties. If $(k, i, j, \ell) = (-1, 0, 1, 2)$ then the argument is similar. This finishes the proof. □

Definition 13. Let $\overset{*}{\circ\!\!\longrightarrow}$ be the transitive closure of $\circ\!\!\longrightarrow$. Let $\approx$ be the equivalence relation generated by $\circ\!\!\longrightarrow$.

It is clear that $\overset{*}{\circ\!\!\longrightarrow}$ is an ordering. An element $u \in X(k,\ell)$ is greedy if and only if it is maximal in this ordering.

Lemma 14. *(a). Every equivalence class $C \subset X(k,\ell)$ has finite upper bounds, that is, for all $u, v \in C$ there exists $w \in C$ with $u \overset{*}{\circ\!\!\longrightarrow} w$ and $v \overset{*}{\circ\!\!\longrightarrow} w$.*

(b). Every greedy path is an upper bound (with respect to $\overset{*}{\circ\!\!\longrightarrow}$) of all equivalent elements.

Proof. Part (b) follows immediately from (a). Proof of (a). Let $u, v \in X(k,\ell)$ be equivalent, that is, there exist

$$u = u_0,\ u_1,\ \ldots,\ u_n = v, \qquad u_i \in X(k,\ell)$$

such that for i one has $u_i \circ\!\!\longrightarrow u_{i+1}$ or $u_{i+1} \circ\!\!\longrightarrow u_i$. By induction on n, we shall prove that $\{u, v\}$ has an upper bound.

For $n = 0$ there is nothing to prove. Assume that it is tue for $n - 1$. Then $\{u_0, u_1\}$ has an upper bound w. If $u_n \circ\!\!\longrightarrow u_{n+1}$ then w is an upper bound of u and v, so suppose $u_{n+1} \circ\!\!\longrightarrow u_n$.

Since $u_{n-1} \overset{*}{\circ\!\!\longrightarrow} w$ there exists a diagram as follows.

$$\begin{array}{ccccccc} u_{n-1} =: x_0 & \circ\!\!\longrightarrow & x_1 & \circ\!\!\longrightarrow \cdots \circ\!\!\longrightarrow & x_m := w \\ \circ\!\!\downarrow & & & & \\ v = u_n =: y_0 & & & & \end{array} \tag{15}$$

Using lemma 12 recursively we can extend (15) to a diagram as follows.

$$\begin{array}{ccccc} u_{n-1} =: x_0 & \circ\!\!\longrightarrow & x_1 & \circ\!\!\longrightarrow \cdots \circ\!\!\longrightarrow & x_m := w \\ \circ\!\!\downarrow & & \circ\!\!\downarrow & & \circ\!\!\downarrow \\ v = u_n =: y_0 & \circ\!\!\longrightarrow & y_1 & \circ\!\!\longrightarrow \cdots \circ\!\!\longrightarrow & y_m =: z \end{array}$$

So $v \overset{*}{\circ\!\!\longrightarrow} z$ and also $u = u_0 \overset{*}{\circ\!\!\longrightarrow} w \overset{*}{\circ\!\!\longrightarrow} z$. □

3 Existence of greedy paths

Lemma 16. *Every two vertices can be connected by a greedy path.*

Proof. Let $x, y \in V$. By (8) there exist $k, \ell \in \mathbb{Z}_{\geq 0}$ and $(a_{-k}, \ldots, a_\ell) \in X(k, \ell)$ such that $(x, y) = (a_{-k}, a_\ell)$. By induction on $m := k + \ell$ we prove that it is equivalent to a greedy path. For $m = 0$ this is trivial. Suppose $m > 0$ and it is true for $m - 1$. After interchanging k with ℓ if necessary we may suppose $k > 0$. By the induction hypothesis, there exists a greedy path $(b_{1-k}, \ldots, b_\ell)$ from $a_{1-k} = b_{1-k}$ to $a_\ell = b_\ell$.

$$\begin{array}{l} b_{1-k} \longleftarrow b_{2-k} \longleftarrow \cdots \longleftarrow b_0 \longrightarrow \cdots \longrightarrow b_{\ell-1} \longrightarrow b_\ell = y \\ \downarrow \\ x \end{array}$$

Using (6) and (7) we can uniquely extend this to a diagram

$$\begin{array}{ccccccccccccc} & b_{1-k} & \longleftarrow & b_{2-k} & \longleftarrow \cdots \longleftarrow & b_0 & \longrightarrow & b_1 & \longrightarrow \cdots \longrightarrow & b_{\ell-1} & \longrightarrow & b_\ell = y \\ & \downarrow & & \downarrow & & \downarrow & & \uparrow & & \uparrow & & \uparrow \\ x = & c_{-k} & \longleftarrow & c_{1-k} & \longleftarrow \cdots \longleftarrow & c_{-1} & \longleftarrow & c_0 & \longrightarrow \cdots \longrightarrow & c_{\ell-2} & \longrightarrow & c_{\ell-1} \end{array}$$

such that (b_t, c_{t-1}, c_{t-2}) is greedy (relative the above orientations of the arrows) whenever $2 - k \leq t \leq \ell$. Then (c_t, c_{t+1}, c_{t+2}) is greedy (relative the above orientations of the arrows) whenever $-k \leq t \leq \ell - 2$: if $-k \leq t \leq -3$ this follows from (9a); if $t = -2$ this follows from (9b); if $t = -1$ this follows from (9c); if $0 \leq t \leq \ell - 3$ this follows from (9d); and if $t = \ell - 2$ this follows from the construction.

Then $(c_{-k}, \ldots, c_\ell)$ is a greedy path from x to y. □

Question 17. Can you reduce the set of axioms such that the above results are still true?