

A Garside like structure on the framed mapping class group

Daan Krammer

June 22, 2007

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- We want to do something similar for the mapping class group. The role of S_n should be played by something like $\mathrm{Sp}(2g, \mathbb{Z})$.
- The ordering on S_n defined by

$$x \leq xy \iff \ell(xy) = \ell(x) + \ell(y)$$

(ℓ =length with respect to $\{(1, 2), (2, 3), \dots, (n - 1, n)\}$) is the well-known *weak Bruhat* ordering.

- (1) Garside graphs

Plan

3

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- (2) Laminations

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- (5) How good is this Garside structure?

Garside graphs

4

Definition (Garside graphs). Let V be a set, $E \subset V \times V$ a binary relation (or directed graph). We write $x \longrightarrow y$ or $y \longleftarrow x$ for $(x, y) \in E$. We call (V, E) a *Garside graph* if the following hold.

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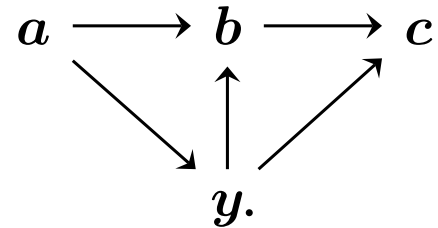
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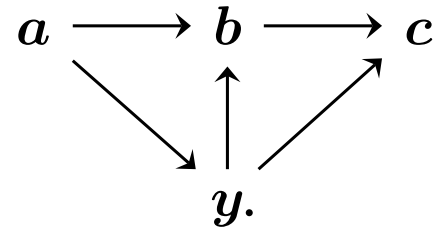
- (5) Let $a \longrightarrow x \longrightarrow c$. Then there exists $b \in V$ such that (universal property) for all $y \in V$, if $a \longrightarrow y \longrightarrow c$ then



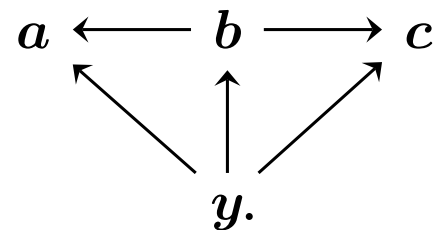
Garside graphs

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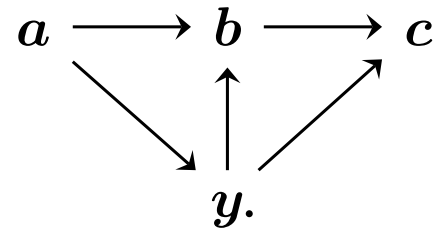


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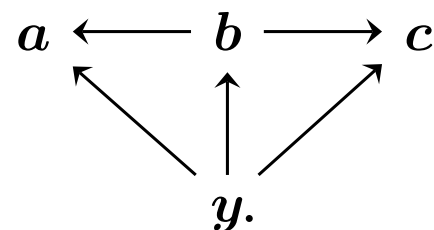


Garside graphs

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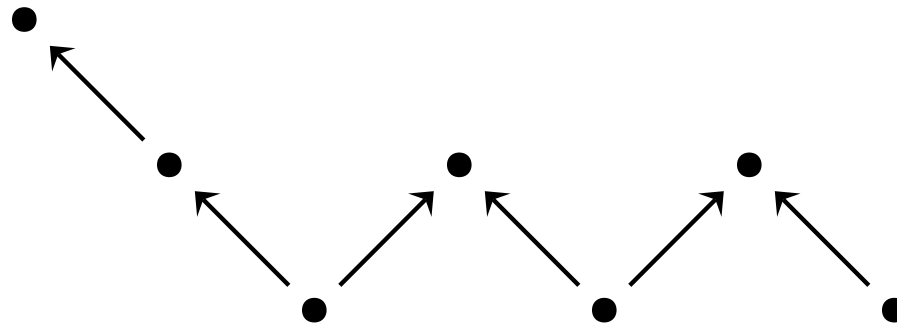
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Proposition. *Any two vertices are connected by a distinguished path, which is unique up to repeating the endpoints.*

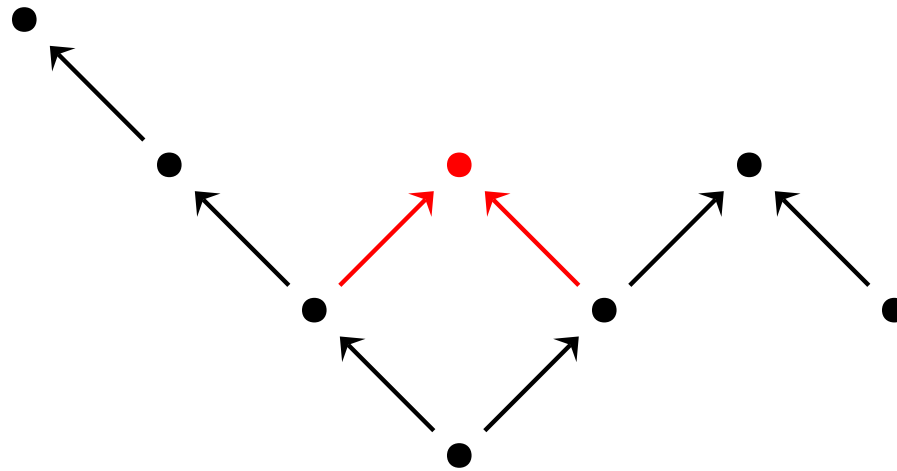
Garside graphs

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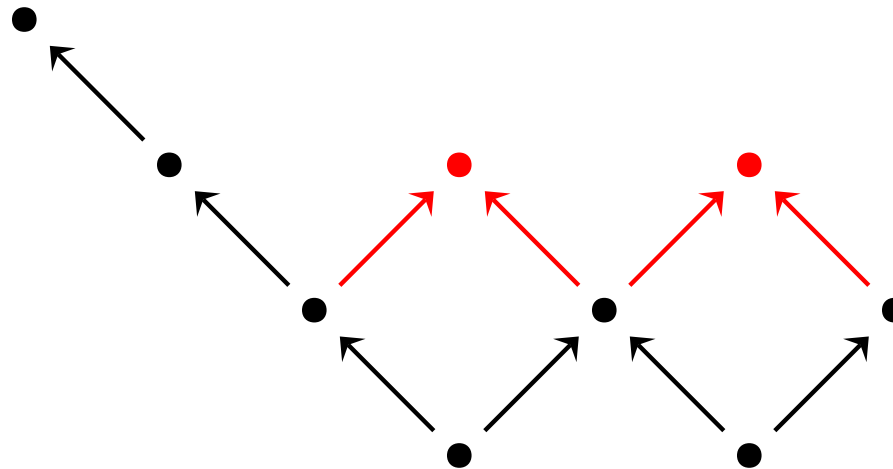
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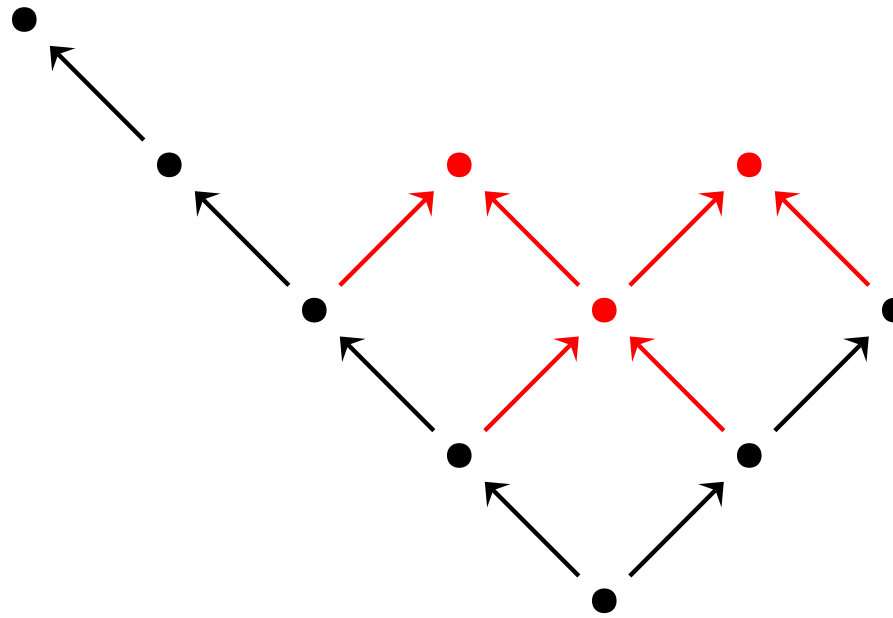
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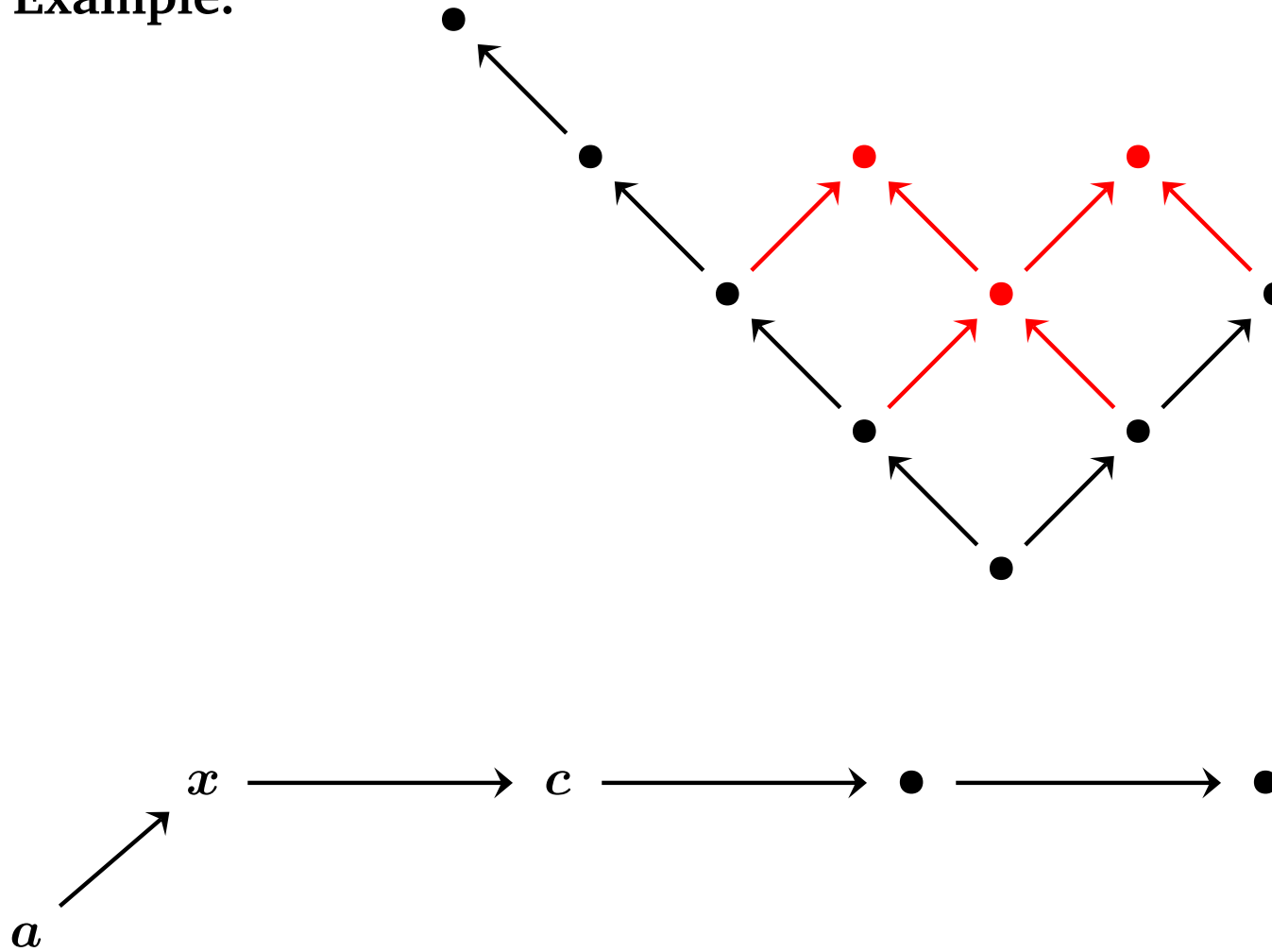
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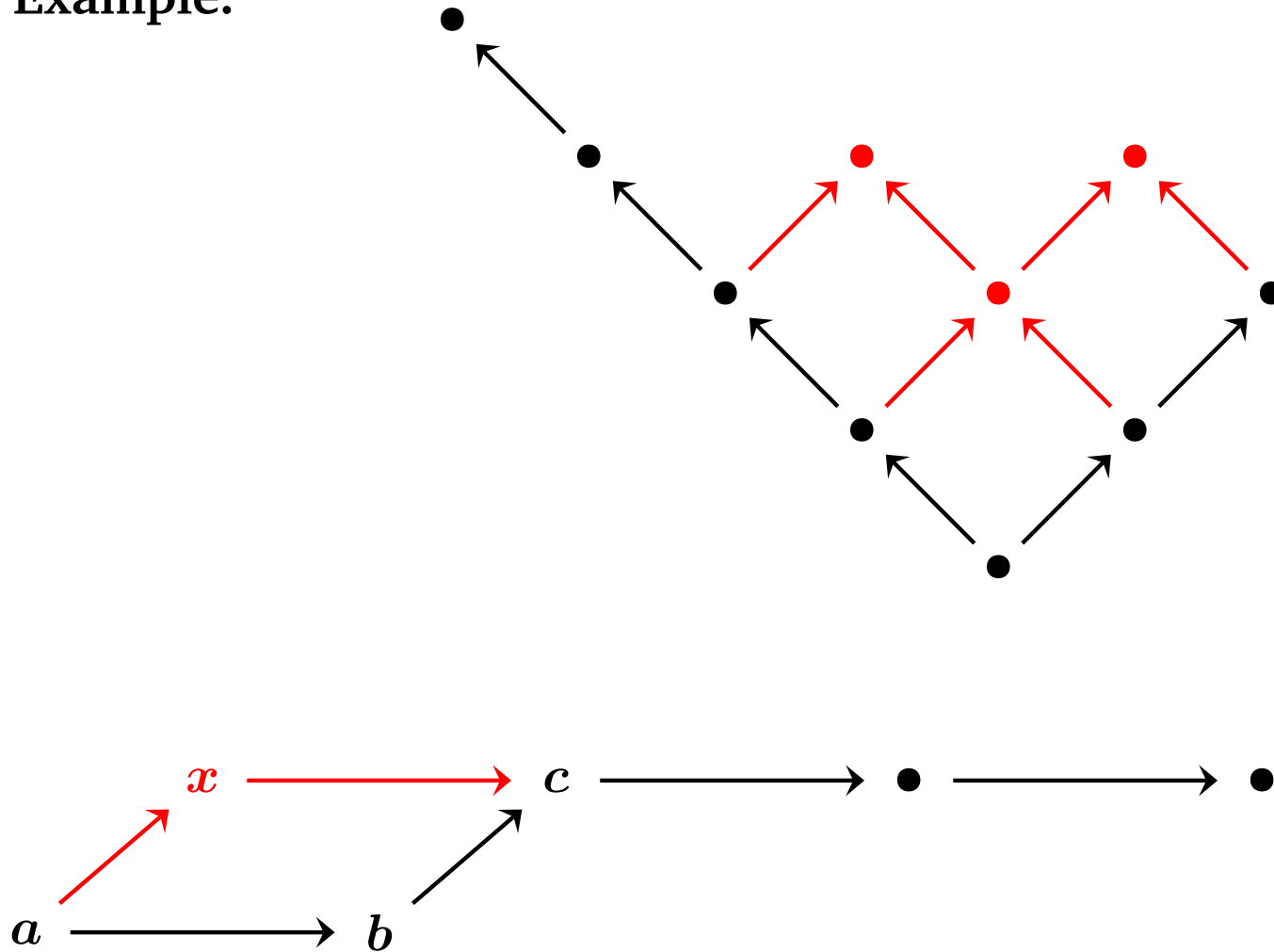
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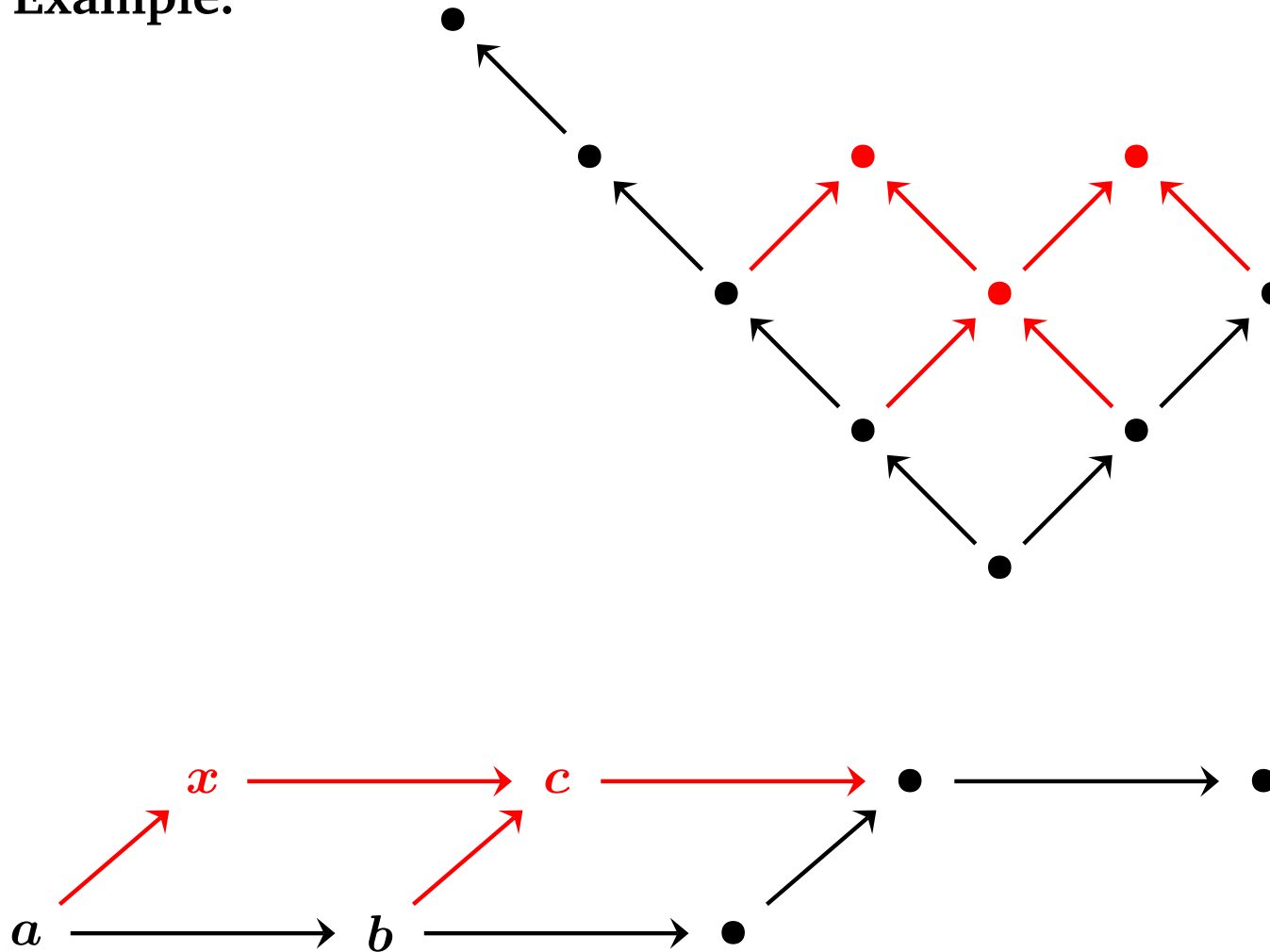
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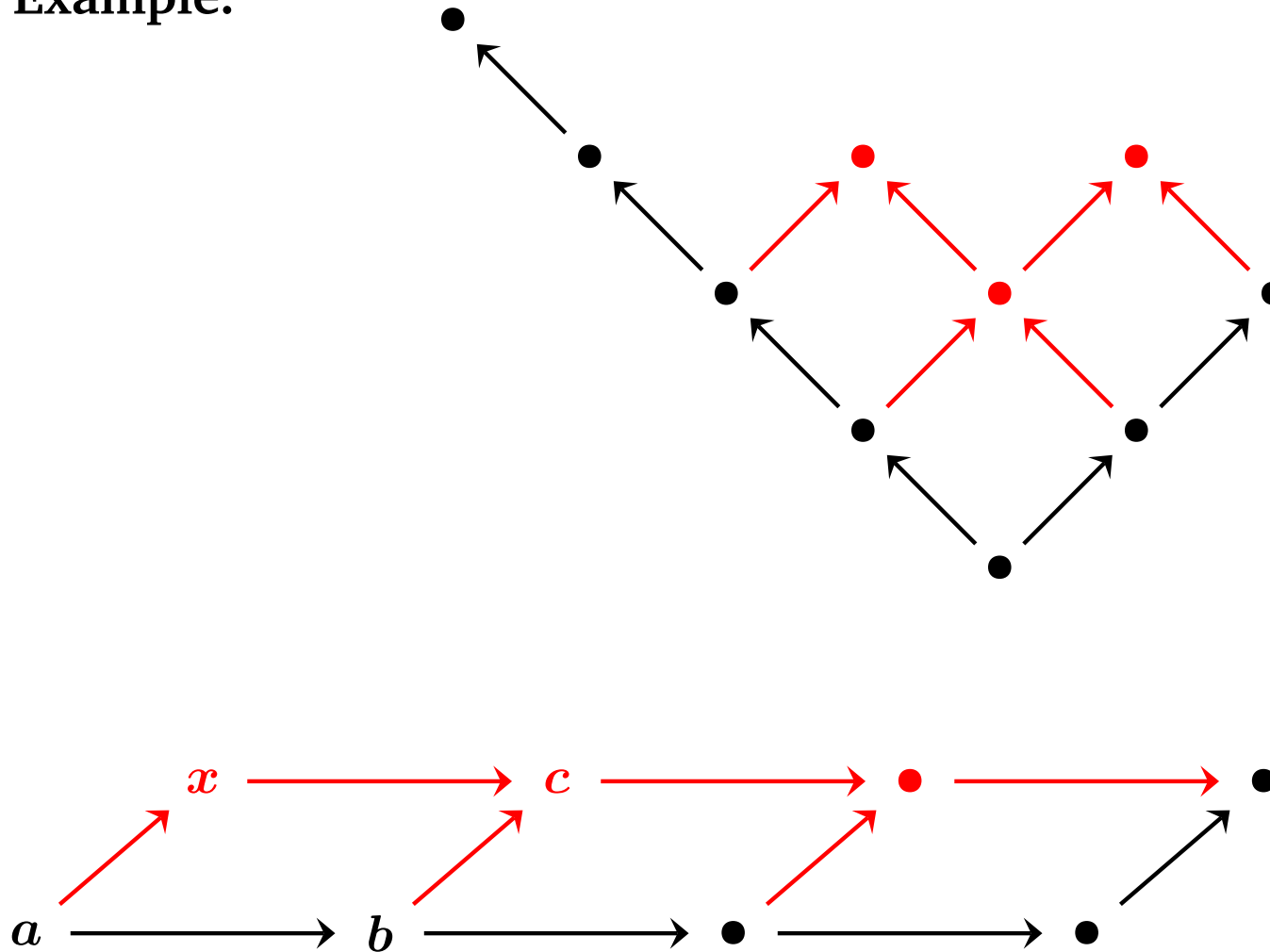
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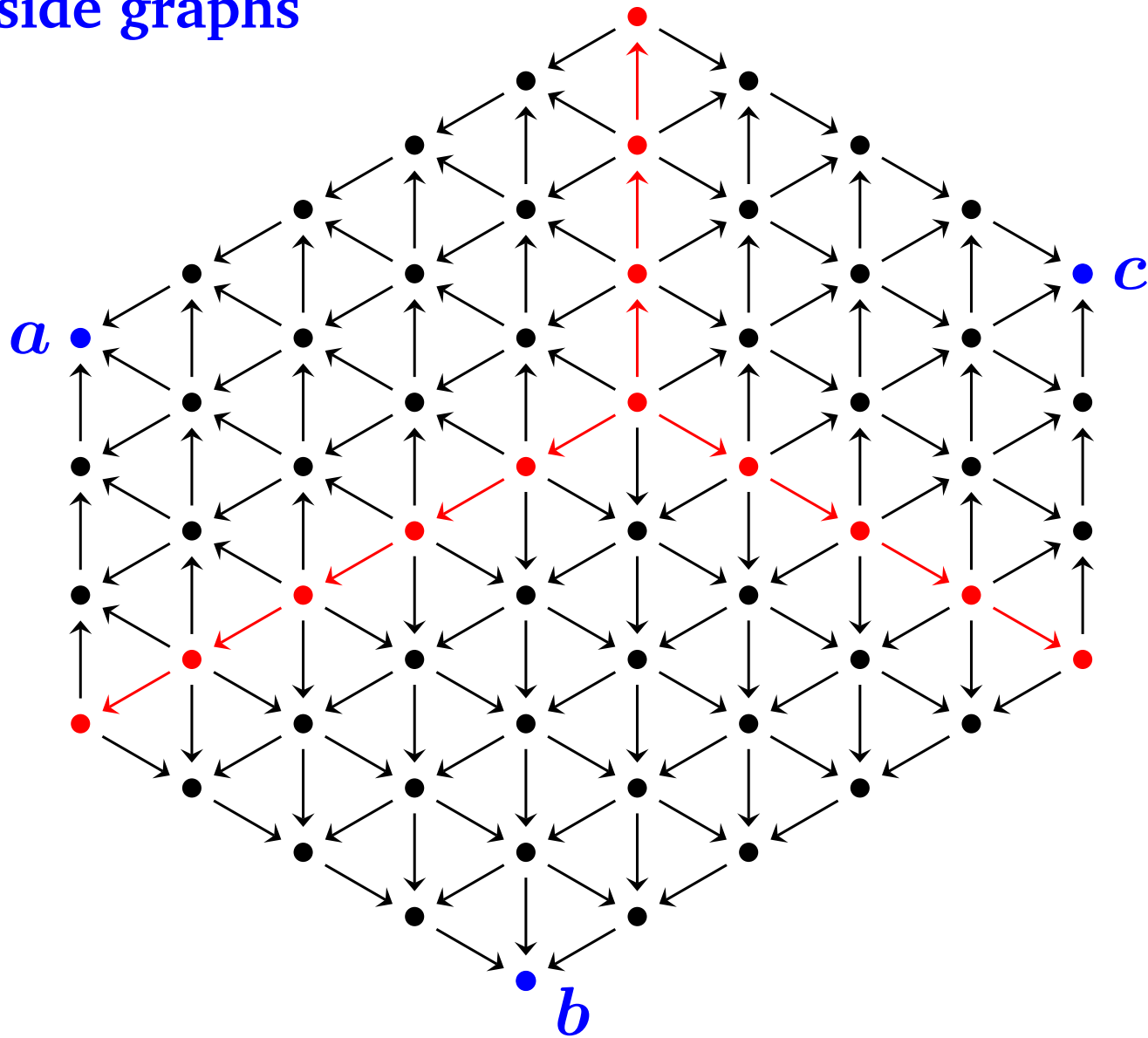
Definition. A set of vertices is *convex* if it contains a distinguished path as soon as it contains its endpoints.

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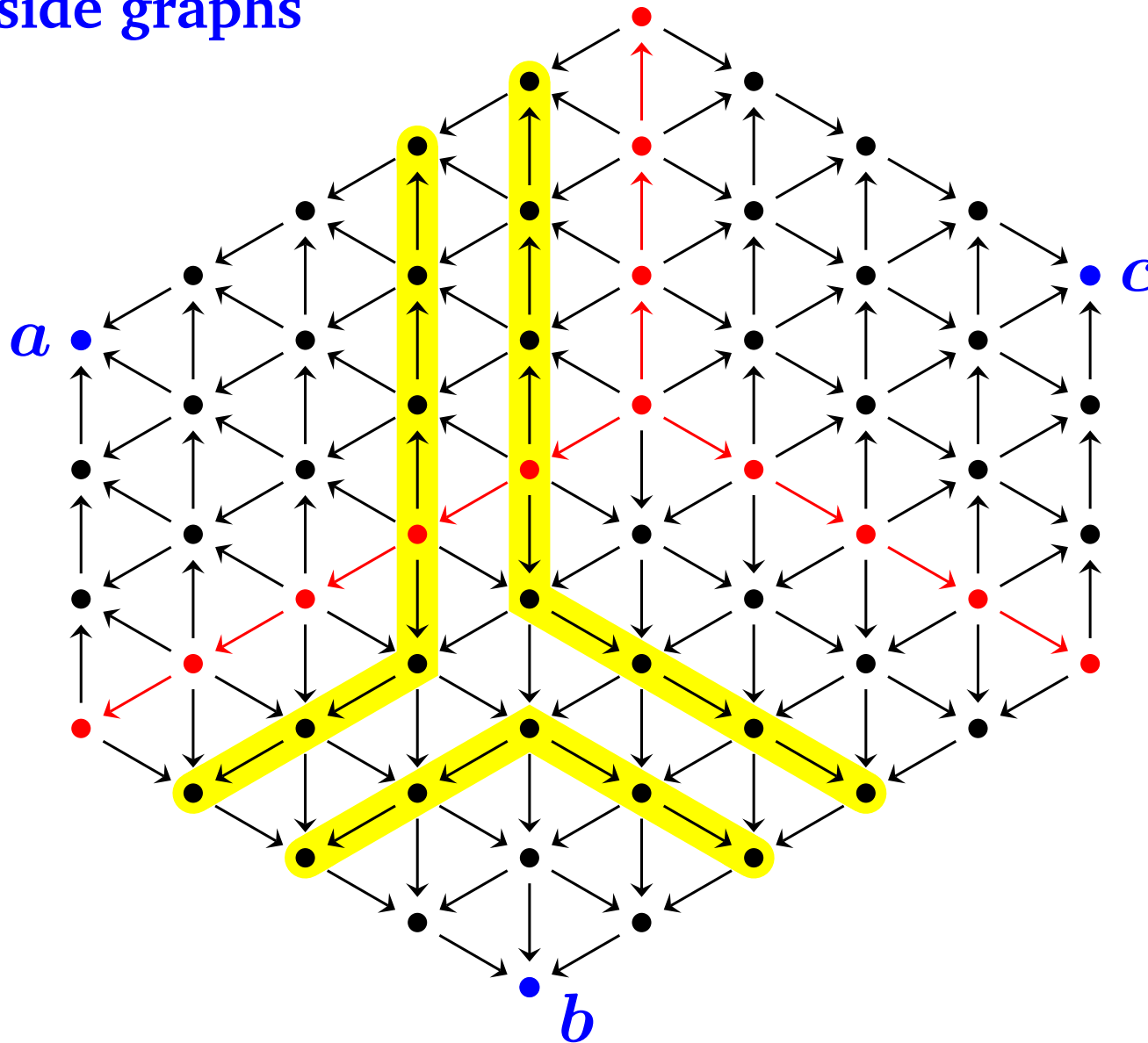
Proposition (Grid property). *The convex hull of any three $a, b, c \in V$ looks as follows (up to vertex repetition).*

Garside graphs

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Garside graphs



Yellow stripes are typical distinguished paths. They go straight ahead except for possibly once making one of the three turns shown.

Definition. Let K be an ordered field (for example, $K = \mathbb{R}$) and $n \geq 0$. A *realisation* of a Garside graph (V, E) is, for each vertex $v \in V$, a non-empty convex subset $C(v) \subset K^n$ with the following properties.

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- A realisation for B_n exists (covariant under a linear representation).
- A Garside graph is a combinatorial analog to (a convex subset of) a real vector space.

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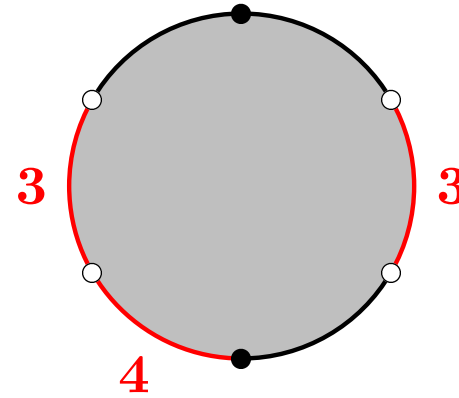
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- (3) The set of laminations on a surface S is essentially independent of the hyperbolic metric on S . □

Main construction

12

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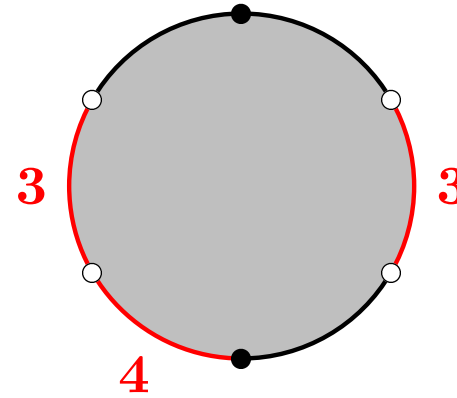


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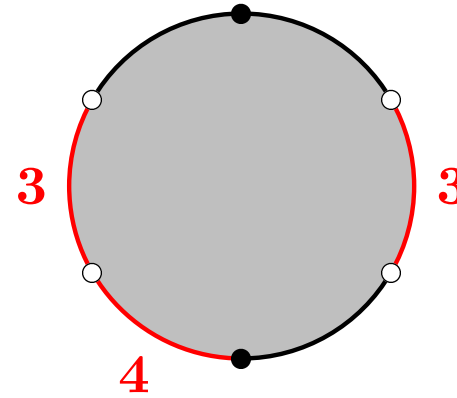


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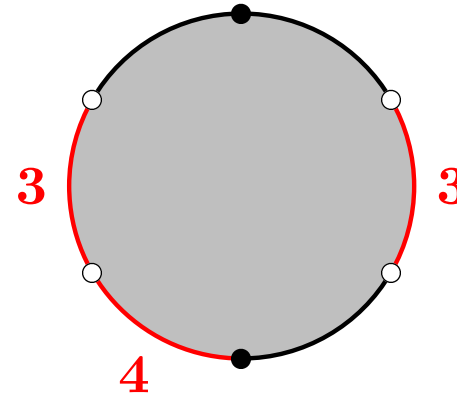


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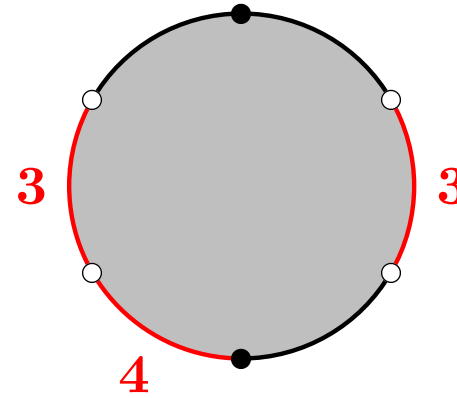


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- (5) G is the mapping class group of all of the above. □

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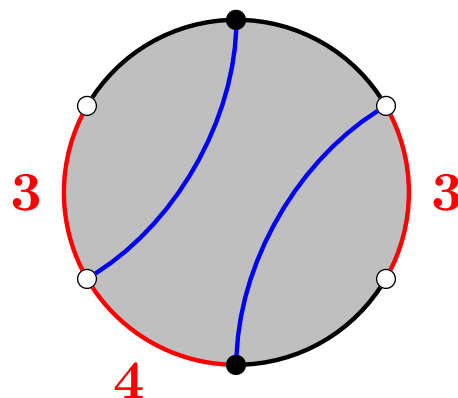
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Example.



Black and red are fixed. Blue is a choice of a lamination in W . □

There are three levels of goodness for laminations.

- Best: let $W_0 \subset W$ denote the laminations whose leaves (together with cusps) are compact intervals. These are cell decompositions of the surface with cusps for vertices.

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Note that if S is simply connected then $W = W_0$ and W is finite.

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- For all $e \in \pi_0(\partial S \setminus Q)$ one has that ∂e is single-coloured if and only if $e \in R$ and $w(e)$ is odd; and

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- some numerical condition coming from the Euler characteristic.

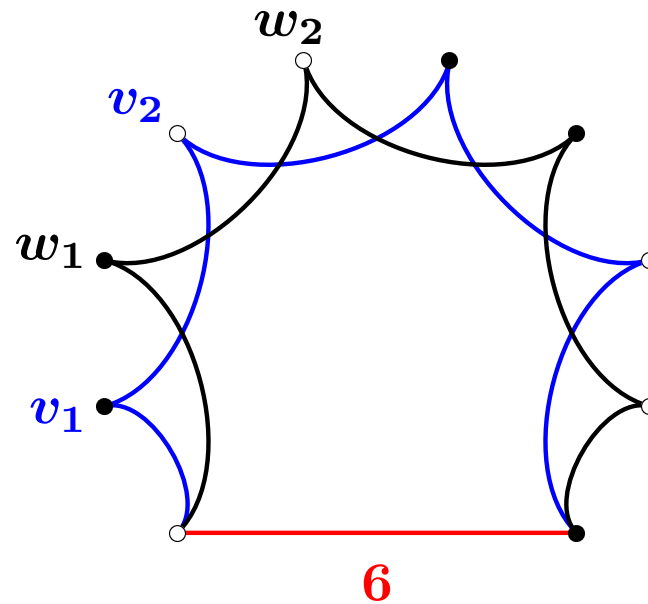
From now we assume these conditions to hold.

Definition. Let $E' \subset W \times W$ be the binary relation consisting of those (v, w) such that for all special edges (here red) one has:

Main construction

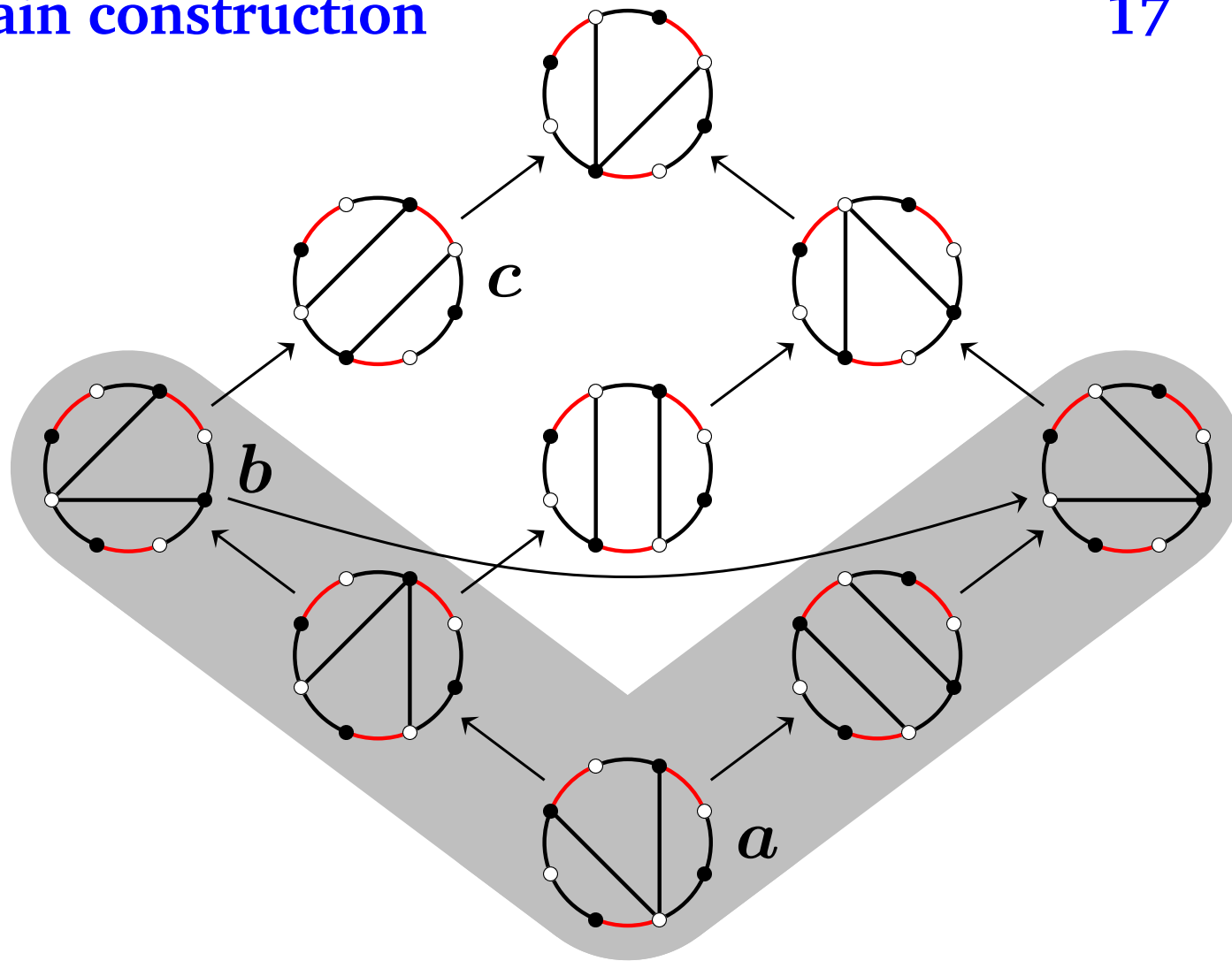
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$$v_1 \leq w_1 < v_2 \leq w_2 < \dots .$$

□



Example. A full (W, E') which is Garside. We only show the *indecomposable* arrows. The gray region is the set of vertices x such that $a \longrightarrow x$. An example of a distinguished path is abc .

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Flips are precisely the indecomposable arrows (see previous page).

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In general however, W is locally infinite and the ordering on W is not generated by the flips.

Main theorem. *In general, (W, E') is a locally Garside graph, that is, a disjoint union of Garside graphs.*

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The main point is to prove that $A(x)$ is a *semi-lattice*, that is, any two $y, z \in A(x)$ have a least common multiple or *join* $y \vee z$.

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Suppose for simplicity $x \in W_0$. Then x defines a cell decomposition x' of the universal cover of S . For every finite union x_0 of regions of x' one has a map $A(x) \rightarrow A(x_0)$.

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Proof. Let $x \in W$. Let $A(x)$ denote the set of ends of arrows beginning at x .

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Suppose for simplicity $x \in W_0$. Then x defines a cell decomposition x' of the universal cover of S . For every finite union x_0 of regions of x' one has a map $A(x) \rightarrow A(x_0)$.

Also $A(x)$ is the set of $\pi_1(S)$ -invariant elements of the lattice $\varinjlim_{x_0} A(x_0)$ hence $A(x)$ is itself a lattice. □

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What is H ?

Definition. Let S be a surface with non-empty boundary.

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- (4) The *framed mapping class group* is defined to be $F = \text{Stab}_G(\text{one framing})$. □

Theorem/Conjecture. *We have $H = F$, that is, $\text{Stab}_G(V) = \text{Stab}_G(\text{one framing})$.*

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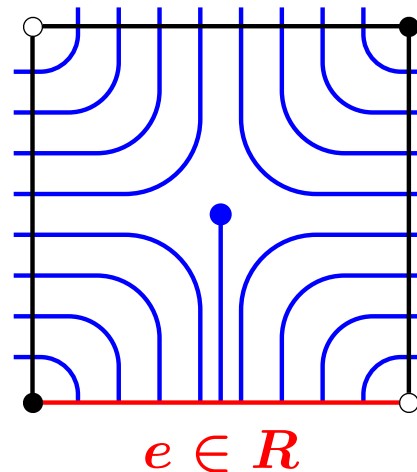
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Framed mapping class group

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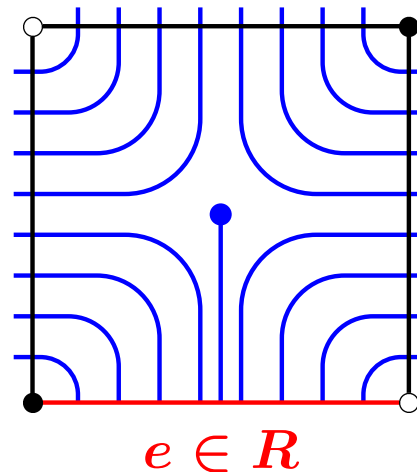
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Proof of \supset . ? □

Framed mapping class group

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Suggestion. Find a presentation for the framed mapping class group.

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Conjecture. We have $\text{diam } V/H < \infty$. □

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Proposition. Let $(x, y), (x, z) \in E \cap (V_1 \times V_1)$. If $y \neq z$ then there is no g in the Torelli group such that $gy = z$ (as promised). □