

A Garside like structure on the framed mapping class group

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Introduction

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- **Theorem (Garside, 1969).** *The braid group B_n can be presented by*

Generators: $\{rx \mid x \in S_n\}$ (a copy of the set S_n).

Relations: $r(xy) = (rx)(ry)$ whenever
 $\ell(xy) = \ell(x) + \ell(y)$.

Moreover, the Cayley graph of (B_n, rS_n) is a Garside graph (soon to be defined). □

- We want to do something similar for the mapping class group. The role of S_n should be played by something like $\mathrm{Sp}(2g, \mathbb{Z})$.
- The ordering on S_n defined by

$$x \leq xy \iff \ell(xy) = \ell(x) + \ell(y)$$

(ℓ =length with respect to $\{(1, 2), (2, 3), \dots, (n-1, n)\}$) is the well-known *weak Bruhat* ordering.

Plan

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- (1) Garside graphs
- (2) Laminations
- (3) Main construction
- (4) Framed mapping class group
- (5) How good is this Garside structure?

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Garside graphs

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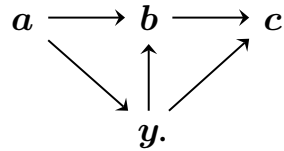
Definition (Garside graphs). Let V be a set, $E \subset V \times V$ a binary relation (or directed graph). We write $x \longrightarrow y$ or $y \longleftarrow x$ for $(x, y) \in E$. We call (V, E) a *Garside graph* if the following hold.

- (1) It is connected.
- (2) $\pi_1(V, E)$ is generated by closed paths of at most 3 edges.
- (3) $x \longrightarrow x$ for all $x \in V$.
- (4) There exists an ordering \leq on V generated by E .

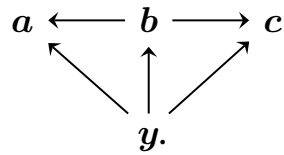
Garside graphs

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- (5) Let $a \longrightarrow x \longrightarrow c$. Then there exists $b \in V$ such that (universal property) for all $y \in V$, if $a \longrightarrow y \longrightarrow c$ then



- (6) Let $a \longleftarrow x \longrightarrow c$. Then there exists $b \in V$ such that (universal property) for all $y \in V$, if $a \longleftarrow y \longrightarrow c$ then



- (7) Let $a \longrightarrow x \longleftarrow c$. Then there exists $b \in V$ such that $a \longleftarrow b \longrightarrow c$. □

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Garside graphs

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Question. Are there global consequences?

Yes!

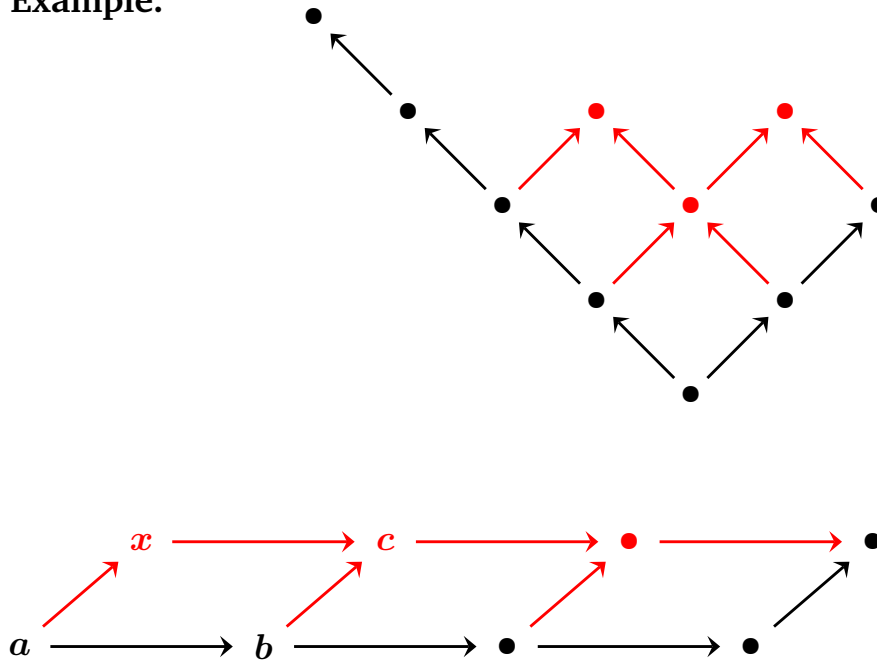
Definition. In (5) or (6) we call (a, b, c) and (c, b, a) *distinguished paths* (of length 2). We call (a_0, \dots, a_n) a distinguished path if (a_i, a_{i+1}, a_{i+2}) is for all i .

Proposition. Any two vertices are connected by a distinguished path, which is unique up to repeating the endpoints.

Garside graphs

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Example.



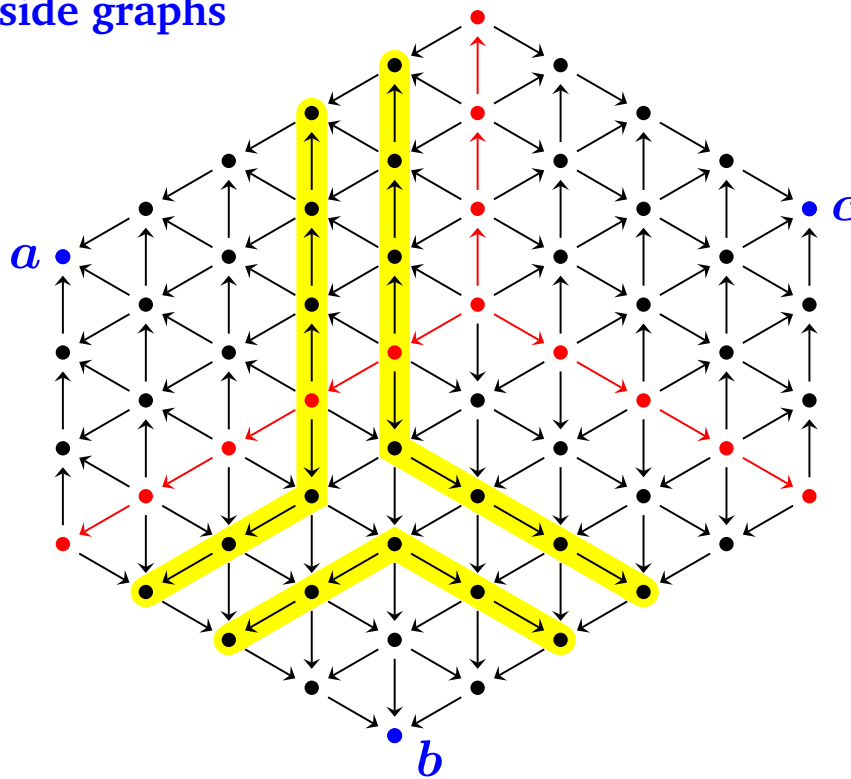
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Garside graphs

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Definition. A set of vertices is **convex** if it contains a distinguished path as soon as it contains its endpoints.

Proposition (Grid property). *The convex hull of any three $a, b, c \in V$ looks as follows (up to vertex repetition).*



Yellow stripes are typical distinguished paths. They go straight ahead except for possibly once making one of the three turns shown.

A Garside like structure on the framed mapping class group – p.9/26 □

Definition. Let K be an ordered field (for example, $K = \mathbb{R}$) and $n \geq 0$. A *realisation* of a Garside graph (V, E) is, for each vertex $v \in V$, a non-empty convex subset $C(v) \subset K^n$ with the following properties.

- (1) $C(v) \cap C(w) = \emptyset$ if $v \neq w$.
- (2) Let $I \subset V$. Then I is convex $\iff \bigsqcup_{v \in I} C(v)$ is convex. □

- A realisation for B_n exists (covariant under a linear representation).
- A Garside graph is a combinatorial analog to (a convex subset of) a real vector space.

Definition. A *Lamination* on a complete hyperbolic surface S is a closed subset L which is a disjoint union of full geodesics (called *leaves*). □

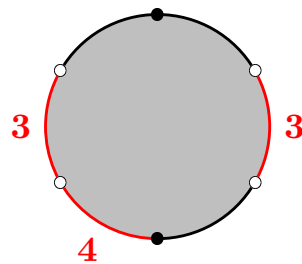
Basic properties of laminations.

- (1) A lamination is a disjoint union of full geodesics in only one way.
- (2) A component of $S \setminus L$ is called a *region*; its boundary is a disjoint union of *boundary leaves*.
- (3) The set of laminations on a surface S is essentially independent of the hyperbolic metric on S . □

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Main construction

From now on we fix the following.



- (1) S is a compact oriented connected surface with non-empty boundary ∂S .
- (2) $Q \subset \partial S$ is a finite set (of cusps) meeting each boundary component, and $c: Q \rightarrow \{\text{white,black}\}$ is a colouring.
- (3) $R \subset \pi_0(\partial S \setminus Q)$ is a specified set of boundary edges called *special edges* (in red).
- (4) $w: R \rightarrow \mathbb{Z}_{>2}$ is a fixed map of *weights*.
- (5) G is the mapping class group of all of the above. □

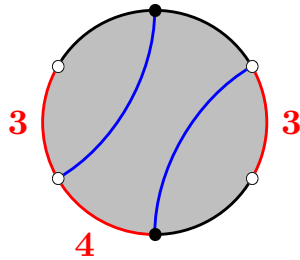
Main construction

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Definition. Let \mathcal{W} be the set of oriented laminations of S with the following properties.

- (1) The lamination is with respect to one (hence any) complete hyperbolic metric on $S \setminus Q$ with geodesic boundary. The elements of Q are cusps.
- (2) Every boundary edge (component of $\partial S \setminus Q$) is a leaf.
- (3) Let T be a region. Then \bar{T} contains a unique special edge e . Moreover, T is simply connected and has precisely $w(e)$ boundary leaves.
- (4) The orientation is from white cusps to black cusps (special edges are not oriented). □

Example.



Black and red are fixed. Blue is a choice of a lamination in \mathcal{W} . □

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Main construction

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There are three levels of goodness for laminations.

- Best: let $\mathcal{W}_0 \subset \mathcal{W}$ denote the laminations whose leaves (together with cusps) are compact intervals. These are cell decompositions of the surface with cusps for vertices.
- Worse: let $\mathcal{W}_1 \subset \mathcal{W}$ denote the laminations whose leaves are compact (intervals or circles). So $\mathcal{W}_0 \subset \mathcal{W}_1 \subset \mathcal{W}$.
- Worst: \mathcal{W} .

Note that if S is simply connected then $\mathcal{W} = \mathcal{W}_0$ and \mathcal{W} is finite.

Main construction

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Note: $W \neq \emptyset \iff$

- For all $e \in \pi_0(\partial S \setminus Q)$ one has that ∂e is single-coloured if and only if $e \in R$ and $w(e)$ is odd; and
- some numerical condition coming from the Euler characteristic.

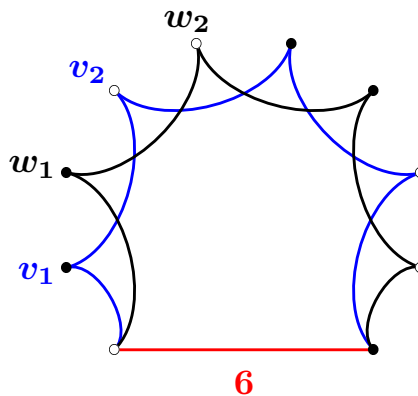
From now we assume these conditions to hold.

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Main construction

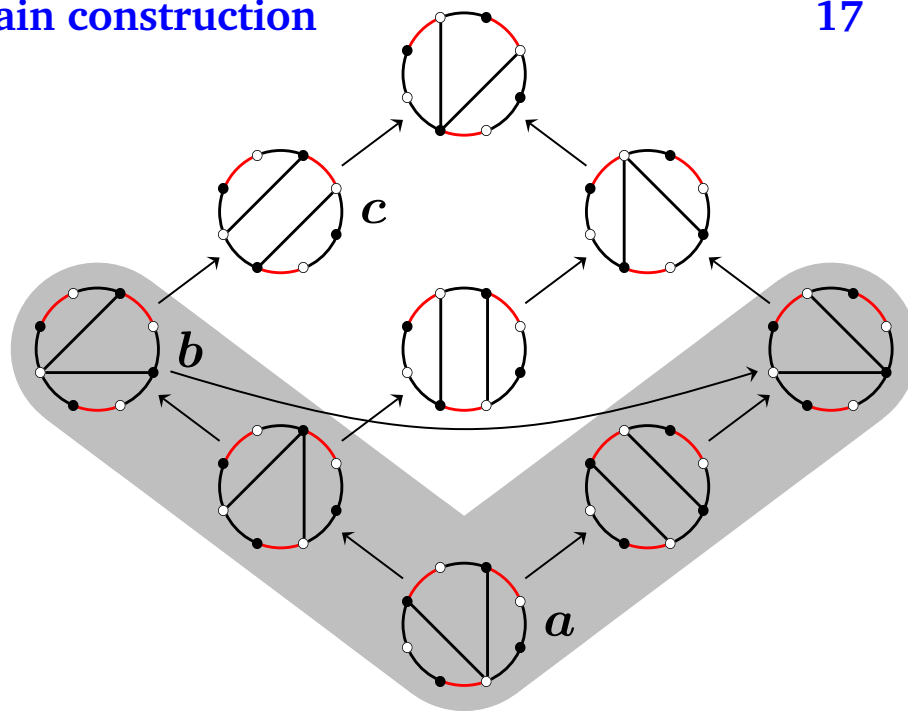
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Definition. Let $E' \subset W \times W$ be the binary relation consisting of those (v, w) such that for all special edges (here red) one has:



$$v_1 \leq w_1 < v_2 \leq w_2 < \dots$$

□



Example. A full (W, E') which is Garside. We only show the *indecomposable* arrows. The gray region is the set of vertices x such that $a \rightarrow x$. An example of a distinguished path is abc .

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Definition. Let T_1, T_2 be regions of $v \in W$ with a common boundary leaf ℓ .

Then one can obtain another $w \in W$ by rotating ℓ inside $\overline{T_1 \cup T_2}$ counterclockwise by one “click”, provided it doesn’t slide along special edges.

Then $(v, w) \in W$. We call (v, w) a *flip*. □

Flips are precisely the indecomposable arrows (see previous page).

Main construction

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Theorem. *If S is simply connected then (W, E') is a Garside graph.*

Proof. What makes this doable is that W is finite. So \leq on W is generated by the flips. The desired result is known to be equivalent to a condition on triples of flips (Dehornoy) which one checks by brute force. \square

In general however, W is locally infinite and the ordering on W is not generated by the flips.

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Main construction

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Main theorem. *In general, (W, E') is a locally Garside graph, that is, a disjoint union of Garside graphs.*

Proof. Let $x \in W$. Let $A(x)$ denote the set of ends of arrows beginning at x .

The main point is to prove that $A(x)$ is a **semi-lattice**, that is, any two $y, z \in A(x)$ have a least common multiple or **join** $y \vee z$.

Suppose for simplicity $x \in W_0$. Then x defines a cell decomposition x' of the universal cover of S . For every finite union x_0 of regions of x' one has a map $A(x) \rightarrow A(x_0)$.

Also $A(x)$ is the set of $\pi_1(S)$ -invariant elements of the lattice $\varinjlim_{x_0} A(x_0)$ hence $A(x)$ is itself a lattice. \square

Main construction

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We shall shortly see that (W, E') is disconnected. Only connected components of a locally Garside graph are useful, so in any case we need to choose a component of (W, E') .

Choose $(V, E) \subset (W, E')$ to be a component meeting W_0 and let $H \subset G$ be its stabiliser.

What is H ?

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Framed mapping class group

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Definition. Let S be a surface with non-empty boundary.

- (1) Let US be the space of nonzero tangent vectors in TS .
- (2) We have a special element $x \in \pi_1(US)$ going round the origin in positive direction in one (hence any) fiber.
- (3) A *framing* of S is a group homomorphism

$$\begin{aligned}\pi_1(US) &\longrightarrow \mathbb{Z} \\ x &\longmapsto 1.\end{aligned}$$

Equivalently, a nowhere vanishing vector field up to homotopy.

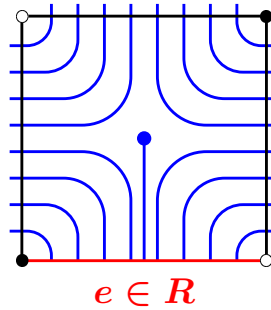
- (4) The *framed mapping class group* is defined to be $F = \text{Stab}_G(\text{one framing})$. □

Framed mapping class group

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Theorem/Conjecture. We have $H = F$, that is,
 $\text{Stab}_G(V) = \text{Stab}_G(\text{one framing})$.

Proof of \subset . We need to attach a unique framing to a component V of W . The blue lines in



(whose orientations are determined by the colours of the cusps) define a framing which is indeed independent under passing through edges. \square

Proof of \supset . ? \square

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Framed mapping class group

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Suggestion. Find a presentation for the framed mapping class group.

How good is this Garside structure?

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How good is this Garside structure?

Note: V is uncountable.

By a generality, there is a *countable* convex H -invariant $V_2 \subset V$, for example, the convex hull of one H -orbit in V . Then V_2 is a Garside graph.

Can you explicitly describe one such V_2 ?

Main conjecture. $V_1 := V \cap W_1$ is convex (hence a Garside graph). □

Conjecture. We have $\text{diam } V/H < \infty$. □

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How good is this Garside structure?

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How many edges are there? What about $\text{Sp}(2g, \mathbb{Z})$?

Definition. The *Torelli group* is the group of elements $g \in G$ whose action on $H_1(S)$ is trivial. □

Proposition. Let $(x, y), (x, z) \in E \cap (V_1 \times V_1)$. If $y \neq z$ then there is no g in the Torelli group such that $gy = z$ (as promised). □