A Garside like structure on the framed mapping class group

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Introduction

- **Theorem (Garside, 1969).** *The braid group B_n can be presented by*

  Generators: $\{rx \mid x \in S_n\}$ *(a copy of the set S_n).*

  Relations: $r(xy) = (rx)(ry)$ *whenever* $\ell(xy) = \ell(x) + \ell(y)$.

Moreover, the Cayley graph of (B_n, rS_n) is a ***Garside graph*** *(soon to be defined).* □

- We want to do something similar for the mapping class group. The role of S_n should be played by something like $\mathrm{Sp}(2g, \mathbb{Z})$.

- The ordering on S_n defined by

$$x \le xy \quad \Longleftrightarrow \quad \ell(xy) = \ell(x) + \ell(y)$$

(ℓ=length with respect to $\{(1,2),(2,3),\dots,(n-1,n)\}$) is the well-known ***weak Bruhat*** ordering.

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(1) Garside graphs

(2) Laminations

(3) Main construction

(4) Framed mapping class group

(5) How good is this Garside structure?

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Definition (Garside graphs). Let V be a set, $E \subset V \times V$ a binary relation (or directed graph). We write $x \longrightarrow y$ or $y \longleftarrow x$ for $(x, y) \in E$. We call (V, E) a ***Garside graph*** if the following hold.

(1) It is connected.

(2) $\pi_1(V, E)$ is generated by closed paths of at most 3 edges.

(3) $x \longrightarrow x$ for all $x \in V$.

(4) There exists an ordering $\leq$ on V generated by E.

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(5) Let $a \longrightarrow x \longrightarrow c$. Then there exists $b \in V$ such that (universal property) for all $y \in V$, if $a \longrightarrow y \longrightarrow c$ then

$$\begin{tikzcd} a \arrow[r] \arrow[dr] & b \arrow[r] & c \\ & y. \arrow[u] \arrow[ur] & \end{tikzcd}$$

(6) Let $a \longleftarrow x \longrightarrow c$. Then there exists $b \in V$ such that (universal property) for all $y \in V$, if $a \longleftarrow y \longrightarrow c$ then

$$\begin{tikzcd} a & b \arrow[l] \arrow[r] & c \\ & y. \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

(7) Let $a \longrightarrow x \longleftarrow c$. Then there exists $b \in V$ such that $a \longleftarrow b \longrightarrow c$. □

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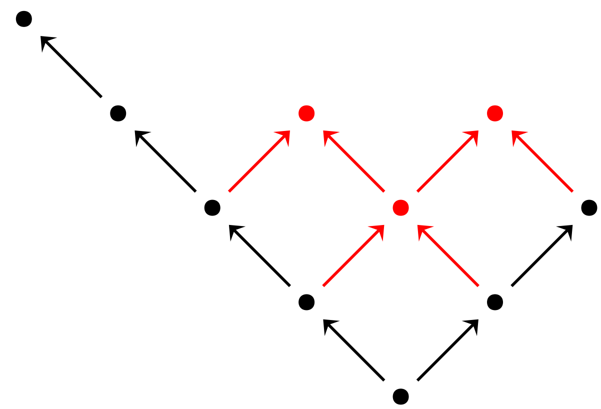
Question. Are there global consequences?

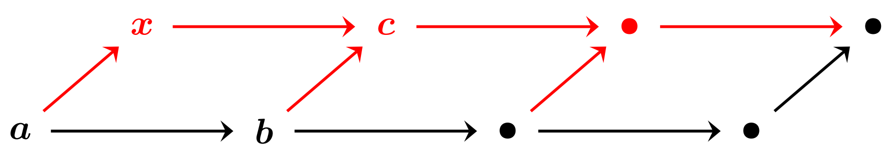
Yes!

Definition. In (5) or (6) we call (a, b, c) and (c, b, a) ***distinguished paths*** (of length 2). We call $(a_0, \ldots, a_n)$ a distinguished path if (a_i, a_{i+1}, a_{i+2}) is for all i.

Proposition. *Any two vertices are connected by a distinguished path, which is unique up to repeating the endpoints.*

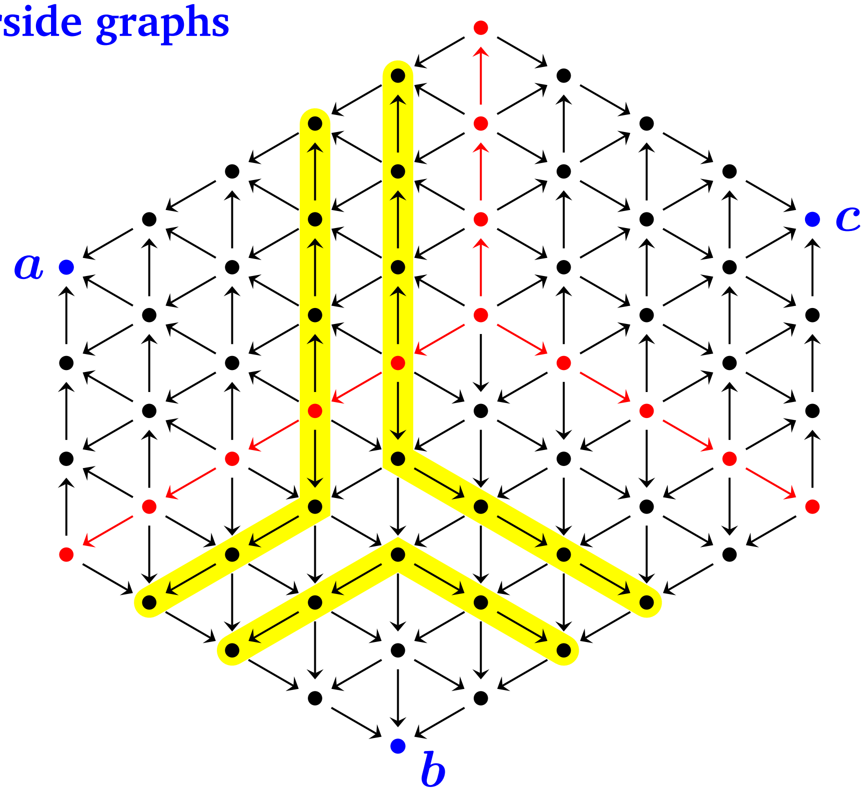
Example.





Definition. A set of vertices is ***convex*** if it contains a distinguished path as soon as it contains its endpoints.

Proposition (Grid property). *The convex hull of any three* $a, b, c \in V$ *looks as follows (up to vertex repetition).*



Yellow stripes are typical distinguished paths. They go straight ahead except for possibly once making one of the three turns shown. □

Definition. Let K be an ordered field (for example, $K = \mathbb{R}$) and $n \geq 0$. A ***realisation*** of a Garside graph (V, E) is, for each vertex $v \in V$, a non-empty convex subset $C(v) \subset K^n$ with the following properties.

(1) $C(v) \cap C(w) = \varnothing$ if $v \neq w$.

(2) Let $I \subset V$. Then I is convex $\iff \bigsqcup_{v \in I} C(v)$ is convex. □

- A realisation for B_n exists (covariant under a linear representation).

- A Garside graph is a combinatorial analog to (a convex subset of) a real vector space.

Laminations

Definition. A ***Lamination*** on a complete hyperbolic surface S is a closed subset L which is a disjoint union of full geodesics (called ***leaves***). □

Basic properties of laminations.

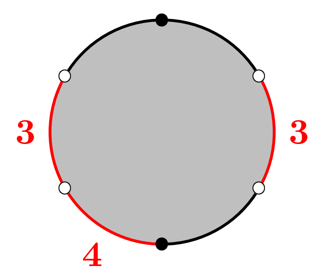
(1) A lamination is a disjoint union of full geodesics in only one way.

(2) A component of $S \setminus L$ is called a ***region***; its boundary is a disjoint union of ***boundary leaves***.

(3) The set of laminations on a surface S is essentially independent of the hyperbolic metric on S. □

Main construction

From now on we fix the following.



(1) S is a compact oriented connected surface with non-empty boundary ∂S.

(2) $Q \subset \partial S$ is a finite set (of cusps) meeting each boundary component, and $c\colon Q \to \{\text{white,black}\}$ is a colouring.

(3) $R \subset \pi_0(\partial S \setminus Q)$ is a specified set of boundary edges called ***special edges*** (in red).

(4) $w\colon R \to \mathbb{Z}_{>2}$ is a fixed map of ***weights***.

(5) G is the mapping class group of all of the above. □

Definition. Let W be the set of oriented laminations of S with the following properties.

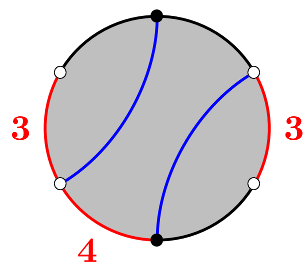
(1) The lamination is with respect to one (hence any) complete hyperbolic metric on $S \setminus Q$ with geodesic boundary. The elements of Q are cusps.

(2) Every boundary edge (component of $\partial S \setminus Q$) is a leaf.

(3) Let T be a region. Then $\bar{T}$ contains a unique special edge e. Moreover, T is simply connected and has precisely $w(e)$ boundary leaves.

(4) The orientation is from white cusps to black cusps (special edges are not oriented). □

Example.



Black and red are fixed. Blue is a choice of a lamination in W. □

There are three levels of goodness for laminations.

- Best: let $W_0 \subset W$ denote the laminations whose leaves (together with cusps) are compact intervals. These are cell decompositions of the surface with cusps for vertices.
- Worse: let $W_1 \subset W$ denote the laminations whose leaves are compact (intervals or circles). So $W_0 \subset W_1 \subset W$.
- Worst: W.

Note that if S is simply connected then $W = W_0$ and W is finite.

Main construction

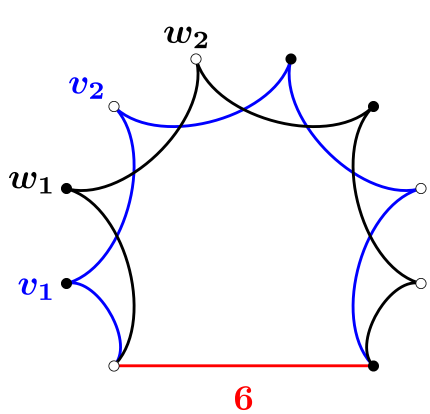
Note: $W \neq \varnothing \iff$

- For all $e \in \pi_0(\partial S \setminus Q)$ one has that ∂e is single-coloured if and only if $e \in R$ and $w(e)$ is odd; and
- some numerical condition coming from the Euler characteristic.

From now we assume these conditions to hold.

Main construction

Definition. Let $E' \subset W \times W$ be the binary relation consisting of those (v, w) such that for all special edges (here red) one has:



$$v_1 \leq w_1 < v_2 \leq w_2 < \cdots . \qquad \square$$

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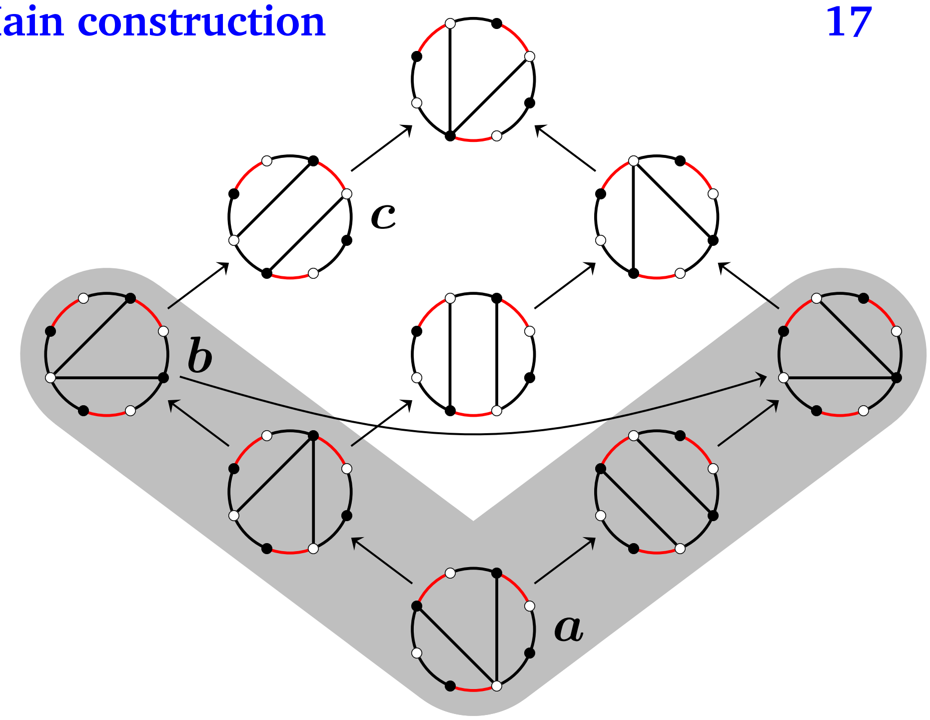

Example. A full (W, E') which is Garside. We only show the ***indecomposable*** arrows. The gray region is the set of vertices x such that $a \longrightarrow x$. An example of a distinguished path is abc.

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Definition. Let T_1, T_2 be regions of $v \in W$ with a common boundary leaf ℓ.

Then one can obtain another $w \in W$ by rotating ℓ inside $\overline{T_1 \cup T_2}$ counterclockwise by one "click", provided it doesn't slide along special edges.

Then $(v, w) \in W$. We call (v, w) a ***flip***. □

Flips are precisely the indecomposable arrows (see previous page).

Theorem. *If S is simply connected then (W, E') is a Garside graph.*

Proof. What makes this doable is that W is finite. So $\leq$ on W is generated by the flips. The desired result is known to be equivalent to a condition on triples of flips (Dehornoy) which one checks by brute force. $\square$

In general however, W is locally infinite and the ordering on W is not generated by the flips.

Main theorem. *In general, (W, E') is a* ***locally Garside graph****, that is, a disjoint union of Garside graphs.*

Proof. Let $x \in W$. Let $A(x)$ denote the set of ends of arrows beginning at x.

The main point is to prove that $A(x)$ is a ***semi-lattice***, that is, any two $y, z \in A(x)$ have a least common multiple or ***join*** $y \vee z$.

Suppose for simplicity $x \in W_0$. Then x defines a cell decomposition x' of the universal cover of S. For every finite union x_0 of regions of x' one has a map $A(x) \to A(x_0)$.

Also $A(x)$ is the set of $\pi_1(S)$-invariant elements of the lattice $\lim\limits_{x_0} A(x_0)$ hence $A(x)$ is itself a lattice. $\square$

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We shall shortly see that (W, E') is disconnected. Only connected components of a locally Garside graph are useful, so in any case we need to choose a component of (W, E').

Choose $(V, E) \subset (W, E')$ to be a component meeting W_0 and let $H \subset G$ be its stabiliser.

What is H?

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Definition. Let S be a surface with non-empty boundary.

(1) Let US be the space of nonzero tangent vectors in TS.

(2) We have a special element $x \in \pi_1(US)$ going round the origin in positive direction in one (hence any) fiber.

(3) A ***framing*** of S is a group homomorphism

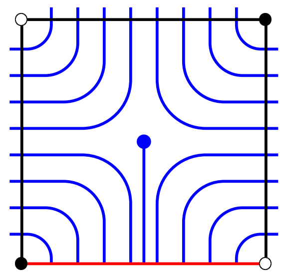
$$\begin{aligned} \pi_1(US) &\longrightarrow \mathbb{Z} \\ x &\longmapsto 1. \end{aligned}$$

Equivalently, a nowhere vanishing vector field up to homotopy.

(4) The ***framed mapping class group*** is defined to be $F = \mathrm{Stab}_G(\text{one framing})$. $\square$

Theorem/Conjecture. *We have $H = F$, that is,* $\mathrm{Stab}_G(V) = \mathrm{Stab}_G(\text{one framing})$.

Proof of $\subset$. We need to attach a unique framing to a component V of W. The blue lines in



(whose orientations are determined by the colours of the cusps) define a framing which is indeed independent under passing through edges. $\square$

Proof of $\supset$. ? $\square$

Suggestion. Find a presentation for the framed mapping class group.

How good is this Garside structure? 25

How good is this Garside structure?

Note: V is uncountable.

By a generality, there is a ***countable*** convex H-invariant $V_2 \subset V$, for example, the convex hull of one H-orbit in V. Then V_2 is a Garside graph.

Can you explicitly describe one such V_2?

Main conjecture. *$V_1 := V \cap W_1$ is convex (hence a Garside graph).* □

Conjecture. *We have* $\operatorname{diam} V/H < \infty$. □

How good is this Garside structure? 26

How many edges are there? What about $\mathrm{Sp}(2g, \mathbb{Z})$?

Definition. The ***Torelli group*** is the group of elements $g \in G$ whose action on $H_1(S)$ is trivial. □

Proposition. *Let $(x, y), (x, z) \in E \cap (V_1 \times V_1)$. If $y \neq z$ then there is no g in the Torelli group such that $gy = z$ (as promised).* □