Annals of Mathematics, **155** (2002), 131–156

Braid groups are linear

By Daan Krammer*

Abstract

In a previous work [11], the author considered a representation of the braid group ρ: $B_n \to \mathrm{GL}_m(\mathbb{Z}[q^{\pm 1}, t^{\pm 1}])$ $(m = n(n-1)/2)$, and proved it to be faithful for $n = 4$. Bigelow [3] then proved the same representation to be faithful for all n by a beautiful topological argument. The present paper gives a different proof of the faithfulness for all n. We establish a relation between the Charney length in the braid group and exponents of t. A certain B_n-invariant subset of the module is constructed whose properties resemble those of convex cones. We relate line segments in this set with the Thurston normal form of a braid.

Contents

1. Introduction

Statement and history of the problem. A group is said to be linear if it is isomorphic to a subgroup of $\mathrm{GL}(n, K)$ for some natural number n and some field K. An interesting question asks whether the braid group is linear.

One of the most famous representations of the braid group is the Burau representation $B_n \to \mathrm{GL}_{n-1}(\mathbb{Z}[q^{\pm 1}])$. It is easily shown to be faithful for $n \leq 3$. Moody [15] proved the Burau representation to be unfaithful for $n \geq 9$. This bound was improved to $n \geq 6$ by Long and Paton [13] and to $n \geq 5$ by Bigelow [2]. It is still unknown whether the Burau representation of B_4 is faithful.

*Financially supported by the Swiss National Science Foundation.

One of the braid group representations, previously studied by Lawrence [12], was proved to be faithful by the author [11] in the case of B_4. Shortly thereafter, Bigelow [3] found a proof that the same representation is faithful for all n by a beautiful topological argument. The present article deals with again the same representation.

More on the history of the linearity problem for braid groups can be found in Birman's review [4].

The representation. The representation of our interest will be denoted $\rho\colon B_n \to \mathrm{GL}(V)$, where V is an m-dimensional free module over some ring R, with $m = n(n-1)/2$. It depends on two invertible elements $q, t \in R$. There are many definitions of this representation. This paper follows an elementary route by exhibiting the entries of the involved matrices, and completely avoids topological arguments. Other definitions include a second homology group [11], [12] and a pictorial approach [3], [11]. Zinno [19] recently showed the representation to be a summand of the Birman-Wenzl algebra [6].

Combinatorial preliminaries. Our linearity proof for the braid group involves a solution to the word problem in the braid group. Among the solutions to the word problem we mention Artin's one [1] (B_n is isomorphic to a subgroup of $\mathrm{Aut}(F_n)$) and a solution based on Thurston's boundary of Teichmüller space [17]. Neither solution is relevant to this paper. Important for us is a third, again totally different solution due to Garside ([10], see also [9], [7]).

For $1 \le i < j \le n$, let $s(i,j) = s_{ij}$ denote the permutation (called a reflection) in the symmetric group S_n interchanging i with j and preserving the rest. The set of reflections in S_n will be denoted by Ref. Let $\ell\colon S_n \to \mathbb{Z}_{\ge 0}$ denote the length function with respect to $\{s_{12}, s_{23}, \ldots, s_{n-1,n}\} \subset S_n$.

The braid group B_n admits a presentation by generators $\{rx \mid x \in S_n\}$ and relations $r(xy) = (rx)(ry)$ whenever $\ell(xy) = \ell(x) + \ell(y)$. The positive braid monoid B_n^+ is by definition the submonoid of B_n generated by $\Omega := r(S_n)$. For $x \in B_n^+$ there exists a unique longest $y \in S_n$ with $x \in (ry)B_n^+$, notation: $ry = \mathrm{LF}(x)$. We will make use of the following proposition, which is implied by Garside's results.

Proposition A (See 2.1). *Let B_n act on a set U. Suppose we are given nonempty disjoint subsets $C_x \subset U$ $(x \in \Omega)$ with $xC_y \subset C_{\mathrm{LF}(xy)}$ for all $x, y \in \Omega$. Then the B_n-action on U is faithful.*

Later on, we will apply Proposition A by putting $U = V$. The central question is to find C_x satisfying the assumption of Proposition A. The C_x we use will be convex in some sense.

For any $x \in B_n^+$ there is a unique $(x_1, \ldots, x_k) \in \Omega^k$ such that $x_1 \cdots x_k = x$ and $\mathrm{LF}(x_i x_{i+1}) = x_i$ for all i, and $x_k = 1$. It is called the greedy form of x and is due to Garside [10]. Thurston [7] showed that any braid $x \in B_n$ can uniquely be written $x = y^{-1}z$ with $y, z \in B_n^+$ such that there is no $w \in B_n^+ - \{1\}$ with $\{y, z\} \subset wB_n^+$. Writing $(y_1, \ldots, y_k)$ for the greedy form for y and $(z_1 \ldots, z_\ell)$ for the greedy form for z then gives $x = y_k^{-1} \cdots y_1^{-1} z_1 \cdots z_\ell$; this is called the Thurston normal form. Closely related is the length function $\ell_\Omega \colon B_n \to \mathbb{Z}_{\geq 0}$ with respect to Ω. Charney [8] showed that the growth function $\sum_{x \in B_n} z^{\ell_\Omega(x)} \in \mathbb{Z}[[z]]$ is rational. We call ℓ_Ω the Charney length function.

Faithfulness. We will throughout make use of a certain basis $\{x_s \mid s \in \mathrm{Ref}\}$ of V, and will identify an element of $\mathrm{End}(V)$ with its matrix with respect to this basis. Thus, $\mathrm{End}(V)$ is identified with $M_m(R)$, the size m matrix algebra over R.

We will observe that $\rho B_n^+ \subset M_m(\mathbb{Z}[q, q^{-1}, t])$; i.e., for positive braids x, the entries of ρx do not involve negative powers of t.

Henceforth, we assume $R = \mathbb{R}[t^{\pm 1}]$, $q \in \mathbb{R}$ and $0 < q < 1$. Then for all positive braids $x \in B_n^+$, the entries of ρx are in $\mathbb{R}_{\geq 0} + t\,\mathbb{R}[t]$. This observation is the most important step of the faithfulness proof of ρ. A faithfulness proof of the braid group seems to be impossible without some kind of inequalities involved (think of convex cones), and the foregoing observation fulfills this need.

Let $M_m(\{0,1\})$ denote the set of size m square matrices with entries in $\{0,1\} \subset \mathbb{Z}$. Multiplication in $M_m(\{0,1\})$ is defined as follows. Given two elements, one first multiplies them in $M_m(\mathbb{Z})$, then replaces all positive entries by one, leaving zero entries untouched. This multiplication makes $M_m(\{0,1\})$ into a monoid. We have a monoid homomorphism $B_n^+ \to M_m(\{0,1\})$, the image of $x \in B_n^+$ being obtained from ρx by setting $t = 0$ and then replacing positive entries by one. Now $M_m(\{0,1\})$ is finite; the combinatorics of the homomorphism $B_n^+ \to M_m(\{0,1\})$ are crucial in the correct definition of C_x, which is briefly as follows.

Define

$$\begin{aligned}
\mathrm{HP} &= \left\{ A \subset \mathrm{Ref} \;\middle|\; s_{ij}, s_{jk} \in A \Rightarrow s_{ik} \in A \text{ whenever } 1 \leq i < j < k \leq n \right\}, \\
L(x) &= \left\{ s_{ij} \;\middle|\; 1 \leq i < j \leq n,\ x^{-1}i > x^{-1}j \right\}, \quad (x \in S_n).
\end{aligned}$$

We will see that for any $A \in \mathrm{HP}$ there is a (unique) greatest $B \in L(S_n)$ with $B \subset A$. Notation: $B = \mathrm{Pro}(A)$. For $x \in \Omega$, one defines $C_x \subset V$ to be the set of those vectors $\sum_{s \in \mathrm{Ref}} a_s x_s$ with $a_s \in \mathbb{R}_{\geq 0} + t\,\mathbb{R}[t]$ and such that on putting $A := \{s \in \mathrm{Ref} \mid a_s \in t\,\mathbb{R}[t]\}$ one has $A \in \mathrm{HP}$ and $x = r\,L^{-1}\,\mathrm{Pro}(A)$.

Clearly, it is a purely combinatorial issue whether $xC_y \subset C_{\mathrm{LF}(xy)}$ for all $x, y \in \Omega$ (the condition of Proposition A). It turns out to be correct, whence by Proposition A:

THEOREM B (See 4.6). *The representation* ρ: $B_n \to \mathrm{GL}(V)$ *is faithful, even if* q *is a real number with* $0 < q < 1$.

Theorems C and D below state two closely related properties of the representation. They are new and will be proved in Section 6.

THEOREM C (See 6.1). *Let* $x \in B_n$, *and consider the Laurent expansion of* ρx *with respect to* t:

$$\rho x = \sum_{i=k}^{\ell} A_i(q)\, t^i, \quad A_i \in M_m(\mathbb{Z}[q^{\pm 1}]), \quad A_k \neq 0, \quad A_\ell \neq 0.$$

(a) *Then* $\ell_\Omega(x) = \max(\ell - k, \ell, -k)$.

(b) *If, in addition,* $x \in B_n^+ - \Delta B_n^+$, *then* $k = 0$ *and* $\ell = \ell_\Omega(x)$.

We define an ordering on $R = \mathbb{R}[t^{\pm 1}]$ by giving a nonzero element of it the same sign as its trailing coefficient (the coefficient for the least occurring exponent of t). We write $C_1 = C$ and $U = \cup_{x \in B_n} xC$. The following result shows that U has properties resembling those of convex cones in real vector spaces, and moreover connects the Thurston normal form with line segments in U.

THEOREM D (See 6.3).

(a) *The* xC *(with* $x \in B_n$*) are disjoint.*

(b) *Let* $(\tilde{y}_1, \ldots, \tilde{y}_k)$ *be a Thurston normal form; i.e., there are greedy* $(u_1, \ldots, u_s)$, $(v_1, \ldots, v_t)$ *with* $(u_s^{-1}, \ldots, u_1^{-1}, v_1, \ldots, v_t) = (\tilde{y}_1, \ldots, \tilde{y}_k)$, *and* $u_s, v_t \neq 1$, *and there is no* $w \in B_n^+ - \{1\}$ *such that* $\{u_1, v_1\} \subset wB_n^+$. *Let* $\tilde{x}_0, \ldots, \tilde{x}_k \in B_n$ *be such that* $\tilde{x}_i = \tilde{x}_{i-1}\tilde{y}_i$ $(1 \leq i \leq k)$. *Then*

$$\frac{t^i \tilde{x}_0 C + \tilde{x}_k C}{t^i + 1} \subset \begin{cases} \tilde{x}_0 C, & i \leq -s; \\ \tilde{x}_{i+s} C, & -s \leq i \leq t; \\ \tilde{x}_k C, & t \leq i. \end{cases}$$

(c) *The set* U *is closed under addition and scalar multiplication by positive elements of* R.

Comparison of three methods. In a previous paper [11], the representation ρ is proved to be faithful for $n = 4$ by a somewhat different method. I do not know whether this method works for $n > 4$. The differences and similarities between this method and the method of the present paper are as follows. Briefly, the roles (not the meanings) of q and t are interchanged.

One of our results, Theorem C, relates the exponents of t with the Charney length function. In [11] one finds a (for $n > 4$ conjectural) relation between the exponents of q and the length function with respect to some other generating subset $Q \subset B_n$ with cardinality

$$|Q| = \frac{1}{n+1}\binom{2n}{n}.$$

A basic reference to Q, which is also known as the set of band generators, is [5]. The present paper assumes q to be a real number with $0 < q < 1$; in [11], t is a real number with $0 < t < 1$.

The present paper studies the set

$$\bigoplus_{s\in\mathrm{Ref}} \Big(\mathbb{R}_{\geq 0} + t\,\mathbb{R}[t]\Big)\, x_s \subset V,$$

which is essentially a simplicial cone. If $t = 1$, then the B_n-module V can be shown to be the symmetric square of the Burau module, so that 'the cone of positive semi-definite elements' makes sense. In [11], a generalization of the cone of positive semi-definite elements is studied. This convex cone is not simplicial at all; rather, it is given by finitely many nonlinear algebraic inequalities.

A third method of proof was found by Bigelow [3]. His beautiful and strikingly short proof involves neither a solution to the word problem, nor a basis of the module. In Bigelow's proof, both q and t are variables. The total ordering on $\langle q, t\rangle$ he uses makes q "more important" than t, so that his method is closer to having t constant than to having q constant.

It seems to be interesting to combine the three approaches into one theory, which presumably involves both generating sets Q and Ω.

Overview. The paper is built as follows. There are two sets of combinatorial results. The first set is mainly due to Garside, Thurston and Charney and is collected in Section 2. The second set might be new and is treated in Section 5. In Section 3, we define the representation and establish a few identities. An overview of the faithfulness proof (but more detailed than in the introduction) can be found in Section 4. Section 6 is devoted to proving Theorems C and D.

Acknowledgements. The author gratefully acknowledges the support by the Swiss National Science Foundation. Many thanks to Hanspeter Kraft for his warm support.

2. Combinatorial preliminaries

This section collects some combinatorial properties of braid groups mainly due to Garside, Thurston and Charney. For proofs, we refer to [10], [9], [7], [8], [14]; remaining statements are left to the reader to prove.

The braid group B_n is defined to be the fundamental group of $\{X \subset \mathbb{C}\colon |X| = n\}$, the set of n-element subsets of $\mathbb{C}$, with its obvious topology. Artin proved that the braid group B_n admits a finite presentation (called the *Artin presentation*) with generators $\sigma_1,\dots,\sigma_{n-1}$ and relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad (1 \le i \le n-1), \tag{1}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad (|i-j| > 1). \tag{2}$$

(We will view σ_i as an element of the braid group.)

Let S_n denote the symmetric group on $I_n = \{1, 2, \dots, n\}$ (action from the left). For $1 \le i < j \le n$, let $s_{ij} = s(i, j) \in S_n$ denote the permutation (called a *reflection*) interchanging i with j and fixing the other elements of I_n. Put $s_i = s_{i,i+1}$ and $S = \{s_1, \dots, s_{n-1}\}$. (The pair (S_n, S) is known as a Coxeter system of type A_{n-1}.) By Ref we will denote the set of reflections in S_n.

Let $\ell\colon S_n \to \mathbb{Z}_{\ge 0}$ denote the length function with respect to S; i.e., $\ell(x)$ is the smallest natural number k such that there exist $x_1, \dots, x_k \in S$ with $x = x_1 \cdots x_k$. The symmetric group S_n contains a unique longest element w_0, given by $w_0(i) = n + 1 - i$.

The braid group B_n admits a presentation with generators $\{rx \mid x \in S_n\}$ and relations $r(xy) = (rx)(ry)$ whenever $\ell(xy) = \ell(x) + \ell(y)$. We will view rx as an element of B_n, and we denote the image of $r\colon S_n \to B_n$ by Ω. There exists a well-known homomorphism $B_n \to S_n$ defined by $rx \mapsto x$ ($x \in S_n$). One can identify $r(s_i)$ with σ_i in the Artin presentation of the braid group. The element $\Delta := r(w_0)$ is known as the *half-twist*.

The submonoid of B_n generated by Ω will be denoted B_n^+ (this includes 1). Elements of the braid group B_n are called *braids* and elements of B_n^+ are called *positive braids*. Recall the length function $\ell\colon S_n \to \mathbb{Z}_{\ge 0}$. By the same symbol, we will denote the *length function* $\ell\colon B_n^+ \to \mathbb{Z}_{\ge 0}$, which is the (unique) monoid homomorphism with $\ell(rx) = \ell(x)$ for all $x \in S_n$. Let Ω_k denote the set of elements of Ω of length k.

A *smallest* (respectively, *greatest*) element of a (partially) ordered set is an element which is smaller (respectively, greater) than any other element. A smallest or greatest element does not necessarily exist, but if it exists, then it is unique.

Define an ordering on B_n^+ by $x \le y \Leftrightarrow y \in xB_n^+$. Restriction of this ordering yields an ordering on Ω, and thereby on S_n. The ordering on S_n can equivalently be given by $x \le xy$ if and only if $\ell(xy) = \ell(x) + \ell(y)$; it is known as the weak Bruhat ordering. The ordered set S_n has a smallest element 1 and a greatest element w_0. The smallest element of Ω is also denoted 1, and its greatest element is Δ.

It can be shown that for any $x \in B_n^+$, the set $\{y \in \Omega \mid y \le x\}$ has a greatest element. It will be denoted by $\mathrm{LF}(x)$ (Left most Factor). A sequence $(x_1, \ldots, x_k) \in \Omega^k$ is said to be (*left*) *greedy* if $\mathrm{LF}(x_i x_{i+1}) = x_i$ for all $i = 1, \ldots, k-1$. For any $x \in B_n^+$, there is a unique greedy sequence $(x_1, \ldots, x_k)$ with $x_1 \cdots x_k = x$ and $x_k \neq 1$. It is called the (*left*) *greedy form* for x.

An important identity reads

$$\mathrm{LF}(xy) = \mathrm{LF}(x\,\mathrm{LF}(y)) \tag{3}$$

for all $x, y \in B_n^+$. It implies that the map $B_n^+ \times \Omega \to \Omega$ defined by $(x, y) \mapsto \mathrm{LF}(xy)$ is an action of the monoid B_n^+ on Ω.

The following proposition singles out an aspect of the word problem which will be used in the present paper. A similar result can be found in [11]. Its proof is a simple application of Garside's results described above. The result gives a sufficient condition on a B_n-action on any set to be faithful. Later on, the set will be chosen to be a module.

PROPOSITION 2.1. *Let B_n act on a set U. Suppose we are given subsets $C_x \subset U$ $(x \in \Omega)$.*

(a) *If the inclusion $xC_y \subset C_{\mathrm{LF}(xy)}$ holds for all pairs $(x, y) \in \Omega_1 \times \Omega$, then it holds for all pairs in $B_n^+ \times \Omega$.*

(b) *Assume the following*:

  (1) *The C_x are nonempty and* (*pairwise*) *disjoint.*

  (2) *The properties of* (a) *hold.*

Then the B_n-action on U is faithful.

Proof. (a) We will show the desired result by induction on $\ell(x)$. If $\ell(x) \le 1$, there is nothing to prove. Now let $\ell(x) > 1$, say $x = uv$, $u, v \in B_n^+ - \{1\}$. Then $xC_y = u(vC_y) \subset u(C_{\mathrm{LF}(vy)}) \subset C_{\mathrm{LF}(u\,\mathrm{LF}(vy))} = C_{\mathrm{LF}(uvy)} = C_{\mathrm{LF}(xy)}$. (The two inclusions follow from the induction hypothesis. The middle equality follows from (3).) This proves the induction step and thereby part (a).

(b) Let $\mathrm{Sym}(U)$ denote the group of permutations of U, and let π: $B_n \to \mathrm{Sym}(U)$ denote the action. Write xu instead of $(\pi x)u$ $(x \in B_n,\ u \in U)$. It is known that for any $z \in B_n$ there are $x, y \in B_n^+$ with $z = xy^{-1}$. Our proposition will therefore be proved if we show that for any $x, y \in B_n^+$, if $\pi(x) = \pi(y)$ then $x = y$. We will show this by induction on $\ell(x) + \ell(y)$.

Suppose $x, y \in B_n^+$ with $\pi(x) = \pi(y)$. If $\ell(x) + \ell(y) = 0$ then $x = 1$ and $y = 1$, so certainly $x = y$. Consider now the case $\ell(x) + \ell(y) > 0$. It is given that C_1 is nonempty; choose any $u \in C_1$. By (a), we have $xu \in xC_1 \subset C_{\mathrm{LF}(x)}$ and similarly $yu \in C_{\mathrm{LF}(y)}$. We have $\pi x = \pi y$, whence $xu = yu$. It follows that $xu \in C_{\mathrm{LF}(x)} \cap C_{\mathrm{LF}(y)}$. By assumption (1), all C_z are disjoint however. It follows that $\mathrm{LF}(x) = \mathrm{LF}(y)$. Write $z = \mathrm{LF}(x)$, and define $x', y' \in B_n^+$ by $x = zx'$, $y = zy'$. Note $z \neq 1$, because otherwise $x = y = 1$, contradicting the fact that $\ell(x) + \ell(y) > 0$. It follows that $\ell(x') + \ell(y') < \ell(x) + \ell(y)$. The induction assumption thus yields $x' = y'$ and hence $x = y$. This proves the induction step and thereby part (b) of the proposition. □

The results in this section so far suffice to understand the faithfulness proof in Sections 4, 5. We now turn to some more combinatorial results which will be used in the proof of 6.1.

The *Charney length function* is the length function $\ell_\Omega\colon B_n \to \mathbb{Z}_{\geq 0}$ with respect to Ω; i.e., $\ell_\Omega(x)$ is the smallest natural number k such that there exist $x_1, \ldots, x_k \in \Omega \cup \Omega^{-1}$ with $x = x_1 \cdots x_k$.

The center of B_n is isomorphic to $\mathbb{Z}$ and, if $n \geq 3$, generated by Δ^2. We have a bijection $\mathbb{Z} \times (B_n^+ - \Delta B_n^+) \to B_n$ defined by $(k, x) \mapsto \Delta^k x$.

From the Artin presentation of the braid group, it follows that there exists an automorphism of B_n which takes any σ_i to its inverse. We will denote this automorphism by $x \mapsto \overline{x}$.

The following theorem collects some combinatorial results.

Theorem 2.2 (Garside, Thurston, Charney).

(a) *Let* $(x_1, \ldots, x_k)$ *denote the greedy form of some positive braid* $x \in B_n^+$. *Then* $\ell_\Omega(x) = k$.

(b) *Let* $x \in B_n$. *Then there are unique* $y = y_x$ *and* $z = z_x$ *both in* B_n^+ *with* $x = y^{-1}z$ *such that there is no* $w \in \Omega_1$ *with* $\{y, z\} \subset wB_n^+$. *They satisfy* $\ell_\Omega(x) = \ell_\Omega(y) + \ell_\Omega(z)$.

(c) *Let* $x \in B_n^+ - \Delta B_n^+$ *with* $\ell_\Omega(x) = k$. *Then* $\ell_\Omega(\Delta^\ell x) = \max(k + \ell, k, -\ell)$ *for all* $\ell \in \mathbb{Z}$.

(d) *Let* $x \in B_n^+ - \Delta B_n^+$ *with* $\ell_\Omega(x) = k$. *Then* $\Delta^k \overline{x} \in B_n^+ - \Delta B_n^+$ *and* $\ell_\Omega(\Delta^k \overline{x}) = k$.

(e) *The growth function*

$$\sum_{x \in B_n} z^{\ell_\Omega(x)} \in \mathbb{Z}[[z]]$$

is rational.

(f) *There exists an algorithm that on input* $n \in \mathbb{Z}_{\geq 0}$ *and* $x \in B_n$ *computes* (*the greedy forms of*) y_x, z_x (*as defined in* (b)) *and* $\ell_\Omega(x)$. *The time the algorithm takes is bounded by a polynomial in* $n + \ell_\Omega(x)$.

Charney's result Theorem 2.2(e) becomes even more remarkable if one knows that for most other finite generating subsets of B_n (including the Artin generating set $\{\sigma_1, \ldots, \sigma_{n-1}\} = \Omega_1$) it is still unknown whether the growth function with respect to it is rational. This should not be confused with Deligne's result [9] that the growth function of positive braids

$$\sum_{x \in B_n^+} z^{\ell(x)}$$

is rational (see also [18]).

In contrast to Theorem 2.2(f) Paterson and Razborov [16] proved that computing the length of a braid in B_n with respect to Ω_1 (with n variable) is an NP-complete problem.

3. The representation

Let R denote a commutative ring and $q, t \in R$ two invertible elements. Let V denote the free R-module with basis $\{x_s \mid s \in \mathrm{Ref}\}$. Thus, the dimension of V is $m := |\mathrm{Ref}| = n(n-1)/2$. We will also write x_{ij} instead of $x_{s(i,j)}$ where $1 \leq i < j \leq n$. We define a representation ρ: $B_n \to \mathrm{GL}(V)$ as follows (action of $\mathrm{GL}(V)$ on V from the left; instead of $(\rho x)v$, we use the simpler notation xv, $x \in B_n$, $v \in V$):

$$
\begin{aligned}
&\sigma_k\, x_{k,k+1} = tq^2\, x_{k,k+1}; && \\
&\sigma_k\, x_{ik} = (1-q)\, x_{ik} + q\, x_{i,k+1}, && i<k; \\
&\sigma_k\, x_{i,k+1} = x_{ik} + tq^{k-i+1}(q-1)\, x_{k,k+1}, && i<k; \\
&\sigma_k\, x_{kj} = tq(q-1)\, x_{k,k+1} + q\, x_{k+1,j}, && k+1<j; \\
&\sigma_k\, x_{k+1,j} = x_{kj} + (1-q)\, x_{k+1,j}, && k+1<j; \\
&\sigma_k\, x_{ij} = x_{ij}, && i<j<k \ \text{ or } \ k+1<i<j; \\
&\sigma_k\, x_{ij} = x_{ij} + tq^{k-i}(q-1)^2\, x_{k,k+1}, && i<k<k+1<j.
\end{aligned}
$$

It should be proved here that these formulas do indeed define a representation, i.e., that they respect relations (1) and (2) in the Artin presentation of the braid group, and that $\rho\sigma_k$ is invertible. This is a straightforward though tedious task which we leave to the reader.

Remark. In [11] the author uses another basis $\{v_{ij} \mid 1 \le i < j \le n\}$ of the same module V. Its relation with $\{x_{ij}\}_{ij}$ is given by

$$(4) \qquad v_{ij} = x_{ij} + (1-q) \sum_{i<k<j} x_{kj}, \qquad x_{ij} = v_{ij} + (q-1) \sum_{i<k<j} q^{k-1-i}\, v_{kj}.$$

In [11] one can also find a topological interpretation of v_{ij}. Combination with (4) then results in a topological interpretation of x_{ij}. In the present paper, we will not consider any other bases than $\{x_{ij}\}_{ij}$ and its dual. A quicker proof of our formulas defining a representation is obtained if one is willing to accept the formulas with respect to $\{v_{ij}\}$ in [11], by combining with (4).

Let V^* denote the dual of V and let $\langle \cdot, \cdot \rangle \colon V^* \times V \to R$ denote the pairing. Let $\{x_s^* \mid s \in \mathrm{Ref}\}$ denote the basis dual to $\{x_s\}_s$, and write $x_{ij}^* = x_{s(i,j)}^*$. In formula:

$$V^* = \bigoplus_{s \in \mathrm{Ref}} R x_s^*, \qquad \langle x_r^*, x_s \rangle = \delta_{rs}.$$

Let $\mathrm{End}(V)$ act on V^* on the right by $\langle uA, v \rangle = \langle u, Av \rangle$ for all $A \in \mathrm{End}(V)$, $(u,v) \in V^* \times V$. Again, we will write vx instead of $v(\rho x)$ ($x \in B_n$, $v \in V^*$).

The braid group action on V^* is given by

$$(5) \quad x_{k,k+1}^* \sigma_k = t\Bigg[q^2\, x_{k,k+1}^* + q(q-1) \sum_{k+1<b} x_{kb}^* + (q-1) \sum_{a<k} q^{k-a+1}\, x_{a,k+1}^* + (q-1)^2 \sum_{a<k<k+1<b} q^{k-a} x_{ab}^* \Bigg]$$

and

$$(6) \qquad x_{ij}^* \sigma_k = \begin{cases} (1-q)\, x_{ik}^* + x_{i,k+1}^*, & i<k,\ j=k; \\ q\, x_{ik}^*, & i<k,\ j=k+1; \\ x_{k+1,j}^*, & i=k,\ j>k+1; \\ (1-q)\, x_{k+1,j}^* + q\, x_{kj}^*, & i=k+1,\ j>k+1; \\ x_{ij}^*, & \{i,j\} \cap \{k,k+1\} = \emptyset. \end{cases}$$

The results in this section so far are sufficient background for reading the faithfulness proof in the next two sections. The remainder of this section deals with a few identities which will be used in Section 6 to prove some more properties of the representation.

We define a linear map $T(u) \colon V \to V$ depending on a parameter u by

$$T(u) \colon x_{ij} \mapsto \sum_{i<k<\ell<j} (1-u)^2 u^{i+\ell}\, x_{k\ell} + \sum_{i=k<\ell<j} (1-u) u^{i+\ell}\, x_{k\ell} + \sum_{i<k<\ell=j} (1-u) u^{i+\ell}\, x_{k\ell} + \sum_{i=k<\ell=j} u^{i+\ell}\, x_{k\ell},$$

each sum ranging over those (k,ℓ) with $1 \le k < \ell \le n$ satisfying the indicated inequalities and identities.

An obvious total ordering on the basis $\{x_{ij}\}$ makes $T(u)$ into a triangle matrix with powers of u on the diagonal. In particular, $T(u)$ is invertible if u is. (A fact which we will not need is that $T(u)T(u^{-1}) = 1$.)

In order to indicate the dependence of ρ on q,t, we write $\rho(x,q,t)$ ($x \in B_n$). Thus, for any two invertible elements q',t' in some ring, $\rho(x,q',t')$ is obtained from the matrix of ρx with respect to $\{x_{ij}\}_{ij}$ by entry-wise replacing q by q' and t by t'.

Recall the automorphism of B_n denoted $x \mapsto \overline{x}$ and mapping each σ_i to its inverse.

Lemma 3.1. *For all $x \in B_n$, one has* $T(q)\,\rho(x,q^{-1},t^{-1})\,T(q)^{-1} = \rho(\overline{x},q,t)$.

Proof. The proof is straightforward and left to the reader. We only remark that it suffices to give a proof for $x \in \Omega_1$, because both sides of the desired identity are group homomorphism images of x. □

If the braid group is viewed as the mapping class group of the punctured disk, all punctures being real, then the matrix T corresponds to complex conjugation. For the application of 3.1 we have in mind (6.1), it is important that $T(q)$ does not involve t.

Lemma 3.2. *We have* $\Delta\, x_{n+1-j,n+1-i} = tq^{i+j-1}\, x_{ij}$ *whenever* $1 \le i < j \le n$.

Proof. Perhaps the best way to prove it is by having a topological interpretation of the representation. (See for example [12] or [11].) We will follow a more elementary path instead, which completely avoids topological arguments. Basically the idea is to multiply the matrices $\rho\sigma_i$ according to a factorization of Δ like $\Delta = (\sigma_1 \cdots \sigma_n)(\sigma_1 \cdots \sigma_{n-1}) \cdots (\sigma_1\sigma_2)\sigma_1$. We reduce the amount of calculations as follows.

Define $A \in \mathrm{GL}(V)$ by the expected formula for Δ: $A\, x_{n+1-j,n+1-i} = tq^{i+j-1}\, x_{ij}$. Our goal is then to prove $A = \rho\Delta$.

Claim 1. $A(\rho\sigma_k)A^{-1} = \rho\sigma_{n-k}$ whenever $1 \le k \le n-1$. This is a straightforward computation, which will be left to the reader.

Claim 2. The centralizer in $\mathrm{GL}(V)$ of ρB_n consists of the scalar matrices only. (This is closely related to the irreducibility of V, which was established by Zinno [19].) In order to prove Claim 2, let $B \in \mathrm{GL}(V)$ commute with each element of ρB_n. We must then show B to be a scalar matrix. Note that $x_{k,k+1}$ is an eigenvector of σ_k with eigenvalue tq^2. From our formula for the

σ_k-action on V^*, it readily follows that the eigenvalue tq^2 is simple; indeed, all remaining eigenvalues depend on q only. Since v_{12} is an eigenvector of σ_1 with simple eigenvalue, and B commutes with σ_1, one has $Bx_{12} = \lambda x_{12}$ for some invertible $\lambda \in R$. After multiplying B with a scalar matrix, we may assume $Bx_{12} = x_{12}$; our task is then to show $B = 1$. Using the identities $\sigma_k\, x_{1k} = (1-q)\, x_{1k} + q\, x_{1,k+1}$ $(1 < k)$ one inductively finds $Bx_{1j} = x_{1j}$. Using the identities $\sigma_k\, x_{k+1,j} = x_{kj} + (1-q)\, x_{k+1,j}$ $(k+1 < j)$, one inductively finds $Bx_{ij} = x_{ij}$. This shows that $B = 1$, thus proving Claim 2.

Note $\Delta\sigma_k\Delta^{-1} = \sigma_{n-k}$ for all k. Thus, the property of A formulated in Claim 1 is also satisfied by $\rho\Delta$. In other words, $A^{-1}(\rho\Delta)$ is in the centralizer of ρB_n. By Claim 2, $A^{-1}(\rho\Delta)$ is scalar, so it suffices to show that $Ax_{n-1,n} = \Delta x_{n-1,n}$. We use the following factorization of Δ:

$$\Delta = \sigma_1(\sigma_2\sigma_1)(\sigma_3\sigma_2)\cdots(\sigma_{n-1}\sigma_{n-2})\Delta_{n-2}$$

where

$$\Delta_{n-2} = (\sigma_1\cdots\sigma_{n-2})(\sigma_1\cdots\sigma_{n-3})\cdots(\sigma_1\sigma_2)\sigma_1.$$

First of all, $\Delta_{n-2}x_{n-1,n} = x_{n-1,n}$. Moreover, for $3 \le k \le n$, one has

$$\begin{aligned}\sigma_{k-1}\sigma_{k-2}x_{k-1,k} &= \sigma_{k-1}(x_{k-2,k} + (1-q)\, x_{k-1,k}) \\ &= (x_{k-2,k-1} + tq^2(q-1)\, x_{k-1,k}) \\ &\quad + (1-q)tq^2x_{k-1,k} = x_{k-2,k-1}.\end{aligned}$$

It follows that $\Delta x_{n-1,n} = \sigma_1(\sigma_2\sigma_1)(\sigma_3\sigma_2)\cdots(\sigma_{n-1}\sigma_{n-2})\Delta_{n-2}x_{n-1,n} = \sigma_1 x_{12} = tq^2x_{12} = Ax_{n-1,n}$. This finishes the proof. □

4. Faithfulness

Recall our representation ρ: $B_n \to \mathrm{GL}(V)$. We often tacitly identify ρx with its matrix with respect to the basis $\{x_{ij}\}_{ij}$.

Observe that for any positive braid x, the entries of the matrix of ρx are in $\mathbb{Z}[q, q^{-1}, t]$. This follows from the matrices given in Section 3 and the fact that B_n^+ is generated by $\Omega_1 = \{\sigma_1, \ldots, \sigma_{n-1}\}$.

From now on, we take the one-variable Laurent polynomial ring $R = \mathbb{R}[t^{\pm 1}]$ for base ring, and $q \in \mathbb{R} \subset R$ with $0 < q < 1$. Put

$$V_1 := \bigoplus_{s\in\mathrm{Ref}} \mathbb{R}[t]\, x_s \subset V = \bigoplus_{s\in\mathrm{Ref}} \mathbb{R}[t^{\pm 1}]\, x_s.$$

We thus have $B_n^+V_1 \subset V_1$.

Note that all entries of $\rho\sigma_k$ are in $\{0, 1, q, 1-q\} + t\,\mathbb{Z}[q, q^{-1}, t]$. By our assumption that q is a real number with $0 < q < 1$, they are in $\mathbb{R}_{\ge 0} + t\,\mathbb{R}[t]$.

On putting

$$\begin{aligned} V_2 &= \bigoplus_{s\in\mathrm{Ref}} \Big(\mathbb{R}_{\geq 0} + t\,\mathbb{R}[t]\Big)\, x_s = \Big(\bigoplus_{s\in\mathrm{Ref}} \mathbb{R}_{\geq 0}\, x_s\Big) \oplus tV_1 \\ &= \Big\{ v \in V \;\Big|\; \text{for all } s \in \mathrm{Ref}\colon \langle x_s^*, v\rangle \in \mathbb{R}_{\geq 0} + t\,\mathbb{R}[t] \Big\}, \end{aligned}$$

we have $B_n^+ V_2 \subset V_2$.

Definition. For $A \subset \mathrm{Ref}$, define

$$D_A = \Big\{ v \in V_2 \;\Big|\; \text{for all } s \in \mathrm{Ref}\colon \langle x_s^*, v\rangle \in t\,\mathbb{R}[t] \Leftrightarrow s \in A \Big\}.$$

Thus, V_2 is the disjoint union of the D_A ($A \subset \mathrm{Ref}$).

Let $x \in B_n^+$, $A \subset \mathrm{Ref}$. Then there is a unique $B \subset \mathrm{Ref}$ with $xD_A \subset D_B$. (A formula for B is given in 4.1.) Notation: $B = xA$. Let 2^{Ref} denote the power set of Ref. The map $B_n^+ \times 2^{\mathrm{Ref}} \to 2^{\mathrm{Ref}}$, $(x, A) \mapsto xA$ defines an action of B_n^+ on 2^{Ref}. (This follows from the facts that ρ is a representation and that ρB_n^+ preserves V_2.) An explicit formula for the B_n^+-action on 2^{Ref} is as follows.

Lemma 4.1. *Let $A \subset \mathrm{Ref}$ and $1 \leq k \leq n-1$. Then $\sigma_k A$ is the set of those $s(i,j)$ with $1 \leq i < j \leq n$ and*

$$\begin{cases} \text{true statement}, & i = k,\ j = k+1; \\ \{s(i,k), s(i,k+1)\} \subset A, & i < k,\ j = k; \\ s(i,k) \in A, & i < k,\ j = k+1; \\ s(k+1,j) \in A, & i = k,\ j > k+1; \\ \{s(k+1,j), s(k,j)\} \subset A, & i = k+1,\ j > k+1; \\ s(i,j) \in A, & \{i,j\} \cap \{k,k+1\} = \emptyset. \end{cases}$$

Proof. This is readily obtained from the formulas for the σ_k-action on V^*, (5) and (6). $\square$

If one were given the formula of Lemma 4.1 only, it would not be obvious that it defines a B_n^+-action on 2^{Ref}; but we get the proof of this fact for free as a consequence of our representation.

The notation xA ($A \subset \mathrm{Ref}$) can have rather different meanings according to whether $x \in S_n$ or $x \in B_n^+$. If $x \in S_n$ then $xA = \{xa \mid a \in A\}$, involving multiplication in the symmetric group. For $x \in B_n^+$, the notation refers to the B_n^+-action on 2^{Ref}.

Note that the B_n^+-action on 2^{Ref} preserves inclusions, i.e., for $A \subset B \subset \mathrm{Ref}$ and $x \in B_n^+$ one has $xA \subset xB$.

Definition. We define a map $L\colon S_n \to 2^{\mathrm{Ref}}$ by

$$L(x) = \left\{ s(i,j) \;\middle|\; 1 \le i < j \le n,\ x^{-1}i > x^{-1}j \right\}.$$

Note: $\ell(x) = |L(x)|$. Moreover, for all $x, y \in S_n$, we have

$$\begin{aligned} x \le xy &\iff \ell(xy) = \ell(x) + \ell(y) \\ &\iff L(xy) = L(x) \cup x\, L(y)\, x^{-1} \iff L(x) \subset L(xy). \end{aligned} \tag{7}$$

The image of L will be denoted by $L(S_n)$. As L is injective, one may identify S_n with $L(S_n)$. We will however distinguish them in our notation, because otherwise it would cause confusion.

It can be shown that the B_n^+-action on 2^{Ref} does not preserve $L(S_n)$. (See 4.7 for more details.) In particular, the obvious definition $C_x = D_{Lr^{-1}x}(?)$ does not satisfy the conditions of 2.1(b). Therefore we need a new idea, which is as follows.

Definition. A set $A \subset \mathrm{Ref}$ is said to be a *half-permutation*[1] if, whenever $1 \le i < j < k \le n$, one has

$$s(i,j),\ s(j,k) \in A \Longrightarrow s(i,k) \in A.$$

We will denote the set of half-permutations by HP.

Every element of $L(S_n)$ is a half-permutation. A fact which we shall not need is that a subset $A \subset \mathrm{Ref}$ is in $L(S_n)$ if and only if both A and $(\mathrm{Ref} - A)$ are half-permutations. This explains the terminology of half-permutations. There is another interpretation of half-permutations as follows. There is a bijection from the set of (partial) orderings $<_0$ on $I_n = \{1, \dots, n\}$ with $(i <_0 j) \Rightarrow (i < j)$, to HP, which takes $<_0$ to $\{s(i,j) \mid i <_0 j\}$.

We next record a few combinatorial results whose proofs are deferred to the next section for the sake of readability.

Lemma 4.2. *Let $x \in B_n^+$, $A \in \mathrm{HP}$. Then $xA \in \mathrm{HP}$.*

Proof. For a proof, see 5.2. □

Recall that a *greatest* element in a (partially) ordered set is an element greater than all other elements.

Lemma/Definition 4.3. *For every half-permutation A there is a greatest (with respect to inclusion) $B \in L(S_n)$ with $B \subset A$. Notation: $B = \mathrm{Pro}(A)$.*

Proof. See 5.4. □

[1] Half-permutations are more commonly called *closed sets*.

Definition. Let GB (Greatest Braid) denote the map

$$\mathrm{GB} = r\,L^{-1}\,\mathrm{Pro}\colon \mathrm{HP} \to \Omega.$$

Moreover, for $x \in \Omega$, define

$$C_x = \cup \left\{ D_A \;\middle|\; A \in \mathrm{HP},\ \mathrm{GB}(A) = x \right\}.$$

Recall the B_n^+-action on Ω defined by $B_n^+ \times \Omega \to \Omega$, $(x, y) \mapsto \mathrm{LF}(xy)$.

LEMMA 4.4. *The map* GB: HP $\to \Omega$ *is* B_n^+*-equivariant. In formula, if* $x \in B_n^+$, $A \in \mathrm{HP}$, $y = \mathrm{GB}(A)$, *then* $\mathrm{GB}(xA) = \mathrm{LF}(xy)$.

Proof. See 5.6. □

Since GB: HP $\to \Omega$ is surjective (indeed, $\mathrm{GB}(L(r^{-1}x)) = x$ for all $x \in \Omega$), one may call HP a refinement of Ω.

LEMMA 4.5.

(a) *The* C_x *are disjoint and nonempty.*

(b) *We have* $xC_y \subset C_{\mathrm{LF}(xy)}$ *for all* $(x, y) \in B_n^+ \times \Omega$.

Proof. (a) The set C_x is nonempty because $\emptyset \neq D_{L(r^{-1}x)} \subset C_x$. That the C_x are disjoint is trivial.

(b) The definition of C_y reads $C_y = \cup\{D_A \mid A \in \mathrm{HP},\ \mathrm{GB}(A) = y\}$. We must therefore show $xD_A \subset C_{\mathrm{LF}(xy)}$ whenever $A \in \mathrm{HP}$, $\mathrm{GB}(A) = y$. Our proof is summarized by the following chain:

$$xD_A \overset{(1)}{\subset} D_{xA} \overset{(2)}{\subset} C_{\mathrm{GB}(xA)} \overset{(3)}{=} C_{\mathrm{LF}(xy)}.$$

Here, (1) follows from the definition of the B_n^+-action on 2^{Ref}. In order to justify (2), note that $xA \in \mathrm{HP}$ by Lemma 4.2. By 4.3 then, $\mathrm{Pro}(xA)$ is defined and hence so are $\mathrm{GB}(xA)$ and the right-hand side of (2). Inclusion (2) follows by definition of C_z. Identity (3) is Lemma 4.4. This finishes the proof of (b).□

THEOREM 4.6. *The representation* ρ: $B_n \to \mathrm{GL}(V)$ *is faithful.*

Proof. This is an immediate consequence of 4.5 and 2.1(b) (with $U = V$). □

The considerations of this section can be illustrated by the commutative diagram of B_n^+-equivariant maps in Figure 1 below. The arrows pointing to the left are inclusions.

$$\begin{array}{ccccccccc}
(v \in D_A) & & V_2 & \longleftarrow & \coprod_{A \in \mathrm{HP}} D_A & \xrightarrow{\mathrm{id}} & \coprod_{x \in \Omega} C_x & & (v \in C_x) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
A & & 2^{\mathrm{Ref}} & \longleftarrow & \mathrm{HP} & \xrightarrow{\mathrm{GB}} & \Omega & & x
\end{array}$$

Figure 1. A B_n^+-equivariant commutative diagram

Remark 4.7. The (easy) proof of 4.5 shows a more general statement as follows. Let $\mathrm{HP}' \subset 2^{\mathrm{Ref}}$ denote a B_n^+-invariant subset, and $\mathrm{GB}'\colon \mathrm{HP}' \to \Omega$ a surjective B_n^+-equivariant map. Then the sets $C'_x := \cup\{D_A \mid A \in \mathrm{HP}',\ \mathrm{GB}'(A) = x\}$ satisfy the same conclusion of 4.5, thus proving once more that ρ is faithful. In this section, we have constructed one such a pair $(\mathrm{HP}', \mathrm{GB}')$, namely, $(\mathrm{HP}, \mathrm{GB})$. The following questions arise. Are there more such pairs $(\mathrm{HP}', \mathrm{GB}')$? Is there a best pair, whatever that means? Any pair with the desired properties cannot involve $\mathrm{HP}' = L(S_n)$, because $L(S_n)$ is not B_n^+-invariant, as is proved by the following counterexample: If $n = 4$, then $A := \{s_{13}, s_{14}, s_{23}, s_{24}\} \in L(S_n)$, but $\sigma_2 A = \{s_{23}, s_{14}\} \notin L(S_n)$. Another solution is given by $\mathrm{HP}' = \mathrm{HP}_0$, the smallest B_n^+-invariant subset of HP containing $L(S_n)$, and $\mathrm{GB}' = \mathrm{GB}_0$, the appropriate restriction of GB. For example, if $n = 4$, then $|\mathrm{HP}_0| = 25$ and $\mathrm{HP}_0 = L(S_n) \cup \{\{s_{23}, s_{14}\}\}$. The sets HP_0 seem to be rather messy, and $(\mathrm{HP}, \mathrm{GB})$ is a comfortable solution after all.

5. Half-permutations

The aim of this section is to prove some combinatorial results among which are the promised lemmas of the previous section.

Remark 5.1. There exists an involutory automorphism of the system $(S_n,\ S,\ B_n^+,\ r\colon S_n \to B_n^+,\ \mathrm{Ref},\ B_n^+ \times 2^{\mathrm{Ref}} \to 2^{\mathrm{Ref}},\ \mathrm{HP},\ L,\ \mathrm{Pro},\ \mathrm{GB})$, defined by conjugation by w_0 in S_n, or by Δ in B_n. (Note: Δ^2 is central in B_n.) The easy proof is left to the reader. For example, this involution maps s_k to s_{n-k}. Especially the fact that the involution preserves the B_n^+-action on 2^{Ref} is remarkable. This symmetry will prove useful as it can be used to reduce the number of cases in a few case-by-case proofs.

Lemma 5.2. *Let $x \in B_n^+$, $A \in \mathrm{HP}$. Then $xA \in \mathrm{HP}$.*

Proof. One may suppose $x \in \Omega_1$, say $x = \sigma_k$. Let $1 \le p < q < r \le n$. We must prove (H): $s(p,q),\ s(q,r) \in xA \Rightarrow s(p,r) \in xA$. Modulo the symmetry of 5.1, there are five cases to consider, as shown in the first two columns of

Figure 2. For each of these cases, the table in Figure 2 gives a statement in terms of A which is equivalent to a given one among $s_{pq} \in xA$, $s_{qr} \in xA$, $s_{pr} \in xA$. This table is a consequence of 4.1. Using the table, one readily verifies (H). As an example, we do Case 4:

$$\begin{aligned} s(p,q),\ s(q,r) \in xA \ &\Rightarrow\ s(p,q),\ s(p,k+1),\ s(k+1,r) \in A \\ &\Rightarrow\ s(p,k+1),\ s(k+1,r) \in A \\ &\Rightarrow\ s(p,r) \in A \Rightarrow s(p,r) \in xA, \end{aligned}$$

which proves case 4. □

		$s(p,q) \in xA$ $\iff \ldots$	$s(q,r) \in xA$ $\iff \ldots$	$s(p,r) \in xA$ $\iff \ldots$
1	$\{p,q,r\} \cap \{k,k+1\} = \emptyset$	$s(p,q) \in A$	$s(q,r) \in A$	$s(p,r) \in A$
2	$r = k$	$s(p,q) \in A$	$s(q,r) \in A$ and $s(q,k+1) \in A$	$s(p,r) \in A$ and $s(p,k+1) \in A$
3	$q < k,\ r = k+1$	$s(p,q) \in A$	$s(q,k) \in A$	$s(p,k) \in A$
4	$q = k,\ r > k+1$	$s(p,q) \in A$ and $s(p,k+1) \in A$	$s(k+1,r) \in A$	$s(p,r) \in A$
5	$q = k,\ r = k+1$	$s(p,q) \in A$ and $s(p,r) \in A$	true	$s(p,q) \in A$

Figure 2. To the proof of 5.2

Lemma 5.3. *Let* $A \in \mathrm{HP}$, $x \in S_n$, $L(x) \subset A$, $B = x^{-1}(A - L(x))x$. *Then* $B \in \mathrm{HP}$.

Proof. First we prove the lemma for the case $\ell(x) = 1$, say $x = s_k$. Notice that $x^2 = 1$. Let $1 \leq p < q < r \leq n$. We must show (H): $s_{pq},\ s_{qr} \in B \Rightarrow s_{pr} \in B$. First consider the case where $\{p,q,r\} \cap \{k, k+1\}$ consists of at most one element. Write $(p', q', r') = (xp, xq, xr)$. Then $p' < q' < r'$. From $s(p,q) \in B$ we find $s(p',q') = x\,s(p,q)\,x \in A$; similarly $s(q,r) \in B$ implies $s(q',r') = x\,s(q,r)\,x \in A$. As A is a half-permutation and $p' < q' < r'$, it follows that $s(p',r') \in A$. Hence $s(p,r) = x\,s(p',r')\,x \in B$, thus proving (H) if $|\{p,q,r\} \cap \{k,k+1\}| \leq 1$. Because of the symmetry of 5.1, it remains only to consider the case $q = k$, $r = k+1$. Then the left-hand side of (H) implies $s_{qr} \in B$, whence $x \in B$, whence $x \in xBx = A - L(x) = A - \{x\}$, a contradiction. This proves (H) in the case $q = k$, $r = k+1$, thus establishing the lemma for $\ell(x) = 1$.

We finish the proof of the lemma by induction on $\ell(x)$. For $\ell(x) \leq 1$ there is nothing left to prove. Suppose $u \leq uv = x$ with $u, v \in S_n - \{1\}$. Recall (7) that $L(x)$ is the disjoint union of $L(u)$ with $u\,L(v)\,u^{-1}$. Since $L(x) \subset A$, we have $L(u) \subset A$. Applying the induction hypothesis to (A, u) shows that $C := u^{-1}(A - L(u))u$ is a half-permutation. From $L(x) \subset A$ we find $L(v) \subset C$. Applying the induction hypothesis to (C, v) then yields

$$\begin{aligned} \mathrm{HP} \ni v^{-1}(C - L(v))v &= v^{-1}(u^{-1}(A - L(u))u - L(v))v \\ &= v^{-1}u^{-1}(A - L(u) - u\,L(v)\,u^{-1})uv \\ &= x^{-1}(A - L(x))x = B. \end{aligned}$$

This proves the induction step and hence the lemma. □

LEMMA/DEFINITION 5.4. *For every half-permutation A there is a greatest (with respect to inclusion) $B \in L(S_n)$ with $B \subset A$. Notation: $B = \mathrm{Pro}(A)$.*

Proof. Recall (7) that for $x, y \in S_n$ we have $x \leq y \Leftrightarrow L(x) \subset L(y)$. So an equivalent formulation of the lemma is that $P := \{y \in S_n \mid L(y) \subset A\}$ contains a greatest element. This is the formulation which we will prove.

Note that the ordering on P is generated by $x \leq xs$ whenever true, with $x, xs \in P$, $s \in S$. Let $x, xs, xt \in P$ with $x \leq xs$, $x \leq xt$, $s, t \in S$ $(s \neq t)$. Since P is finite and has a smallest element, it suffices (by a well-known elementary result on partial ordered sets) to show that there exists then $y \in P$ with $xs, xt \leq y$. Let $m_{st} \in \{2, 3\}$ denote the order of st, and put $y = xst$ if $m_{st} = 2$, and $y = xsts$ if $m_{st} = 3$. It is well-known that $xs, xt \leq y$. We claim that $y \in P$. The lemma would clearly follow from this claim. We consider two cases according to the value of m_{st}.

Case 1. $m_{st} = 2$. Then $L(y) = L(xs) \cup L(xt) \subset A$ whence $y \in P$.

Case 2. $m_{st} = 3$. Write $s = s_k$, $t = s_{k+1}$. Define $C = x^{-1}(A - L(x))x$. Let $\amalg$ denote disjoint union. For any $u \in S_n$ with $x \leq xu$ we have $xu \in P \Leftrightarrow L(xu) \subset A \Leftrightarrow L(x) \amalg x\,L(u)\,x^{-1} \subset A \Leftrightarrow x\,L(u)\,x^{-1} \subset A - L(x) \Leftrightarrow L(u) \subset x^{-1}(A - L(x))x = C$, thus showing (for any $u \in S_n$ with $x \leq xu$):

$$xu \in P \Longleftrightarrow L(u) \subset C. \tag{8}$$

Applying (8) to $u = s, t$ gives $s, t \in C$; i.e., $s(k, k+1)$, $s(k+1, k+2) \in C$. By 5.3, we have $C \in \mathrm{HP}$, which means that we may conclude $s(k, k+2) \in C$. Hence $L(sts) = \{s(k, k+1), s(k+1, k+2), s(k, k+2)\} \subset C$. Applying (8) in the reverse direction to $u = sts$ we find $y = xsts \in P$. This finishes case 2 and thereby the proof of the lemma. □

Notice that Pro is a projection; i.e., $\mathrm{Pro}^2 = \mathrm{Pro}$. Moreover, $\mathrm{Pro}\,L = L$.

As to the following lemma, we will only make use of the special case of (a) where $\ell(x) = 1$. We prove the entire lemma because it appears to have interest of its own. Recall the B_n^+-action on 2^{Ref} defined in the previous section (or by 4.1), and which preserves HP by 5.2.

LEMMA 5.5.

(a) *Let* $A \in \mathrm{HP}$, $x \in S_n$. *Then* $(rx)A$ *equals the greatest* (*with respect to inclusion*) *half-permutation* B *with*

$$L(x) \subset B \subset L(x) \cup xAx^{-1}. \tag{9}$$

(*In particular, a greatest such half-permutation exists.*)

(b) *For* $x, y \in S_n$ *with* $x \le xy$ *we have* $(rx)L(y) = L(xy)$. *In particular* (*for* $y = 1$), $(rx)\emptyset = L(x)$.

Proof. We start by proving (a) if $\ell(x) = 1$. Write $x = s_k$, and note $L(x) = \{x\}$. By 5.2, we have $(rx)A \in \mathrm{HP}$. From the definition of $(rx)A$ one readily finds $\{x\} \subset (rx)A \subset \{x\} \cup xAx$.

It remains to show, for any half-permutation B, that (9) implies $B \subset (rx)A$. Suppose $s_{ij} \in B$, $1 \le i < j \le n$. We must prove $s_{ij} \in (rx)A$. We consider four cases.

Case 1. $i = k$, $j = k+1$. Then $s_{ij} \in (rx)A$ by 4.1.

Case 2. $i < k$, $j = k+1$. We have $x \ne s_{ij} \in B \subset \{x\} \cup xAx$, whence $s_{ij} \in xAx$, whence $s_{ik} = xs_{ij}x \in A$, whence $s_{ij} = s_{i,k+1} \in (rx)A$ by 4.1.

Case 3. $i < k$, $j = k$. Then similarly to Case 2, we have $x \ne s_{ij} \in B \subset \{x\} \cup xAx$, whence $s_{ij} \in xAx$ and $s_{i,k+1} = xs_{ij}x \in A$. Moreover, as $s_{ik},\ s_{k,k+1} \in B$ and $B \in \mathrm{HP}$, we also have $s_{i,k+1} \in B$. In Case 2 we already saw that $s_{i,k+1} \in B$ implies $s_{ik} \in A$. Summarizing, we have $s_{ik}, s_{i,k+1} \in A$ whence $s_{ij} = s_{ik} \in (rx)A$ by 4.1. This finishes Case 3.

Case 4. $\{i,j\} \cap \{k,k+1\} = \emptyset$. Then $s_{ij} \in B$ readily implies $s_{ij} \in A$ and hence $s_{ij} \in (rx)A$.

By the symmetry of 5.1, it suffices to do Cases 1–4. The proof of (a) with $\ell(x) = 1$ is thus finished.

We will now prove (a) by induction on $\ell(x)$. For $\ell(x) \le 1$, we have seen it before. Suppose $u \le uv = x$, $u, v \in S_n - \{1\}$. Recall (7): $L(x) = L(u) \amalg u\,L(v)\,u^{-1}$ where $\amalg$ denotes disjoint union.

By the induction hypothesis applied to (A, v) (instead of (A, x)), we have $L(v) \subset (rv)A$. Hence $(ru)L(v) \subset (ru)(rv)A = (rx)A$. But the induction hypothesis for $(L(v), u)$ implies that $(ru)L(v)$ equals the greatest half-permutation B with $L(u) \subset B \subset L(u) \cup u\,L(v)\,u^{-1} = L(x)$. As $L(x)$ is itself a half-permutation, we find $(ru)L(v) = L(x)$. We have thus shown:

$$L(x) \subset (rx)A. \tag{10}$$

Applying the induction hypothesis to (A, v), we see that $(rv)A \subset L(v) \cup vAv^{-1}$. Combining with the induction hypothesis on $((rv)A, u)$, we find $(rx)A = (ru)(rv)A \subset L(u) \cup u(rv)Au^{-1} \subset L(u) \cup u(L(v) \cup vAv^{-1})u^{-1} = (L(u) \cup u\,L(v)\,u^{-1}) \cup uvAv^{-1}u^{-1} = L(x) \cup xAx^{-1}$. We have shown:

$$(rx)A \subset L(x) \cup xAx^{-1}. \tag{11}$$

In view of (10) and (11), it remains to show that for any half-permutation B with (9) one has $B \subset (rx)A$. Let $B \in \mathrm{HP}$ have the property (9). We have $L(u) \subset L(x) \subset B$, which shows

$$L(u) \subset B. \tag{12}$$

Define $C = u^{-1}(B - L(u))u$. By 5.3 and (12), we have

$$C \in \mathrm{HP}. \tag{13}$$

We have $L(u) \amalg u\,L(v)\,u^{-1} = L(x) \subset B = L(u) \amalg uCu^{-1}$, which shows

$$L(v) \subset C. \tag{14}$$

We also have $L(u) \amalg uCu^{-1} = B \subset L(x) \cup xAx^{-1} = L(u) \cup u\,L(v)\,u^{-1} \cup uvAv^{-1}u^{-1}$, which shows

$$C \subset L(v) \cup vAv^{-1}. \tag{15}$$

Applying the induction hypothesis to (A, v) and invoking (13), (14), (15), one finds $C \subset (rv)A$. Hence $B = L(u) \cup uCu^{-1} \subset L(u) \cup u(rv)Au^{-1}$. Combining with (12), we have $L(u) \subset B \subset L(u) \cup u(rv)Au^{-1}$. By the induction hypothesis applied to $((rv)A, u)$, it follows that $B \subset (ru)(rv)A = (rx)A$. This finishes the induction step and hence the proof of (a).

We turn to (b). By (a), $(rx)L(y)$ is the greatest half-permutation B with $L(x) \subset B \subset L(x) \cup x\,L(y)\,x^{-1} = L(xy)$, the last identity being (7). But $L(xy)$ is itself a half-permutation. This proves that $(rx)L(y) = L(xy)$. □

Recall the map $\mathrm{GB} = r\,L^{-1}\,\mathrm{Pro}\colon \mathrm{HP} \to \Omega$.

LEMMA 5.6. *The map* $\mathrm{GB}\colon \mathrm{HP} \to \Omega$ *is* B_n^+*-equivariant. In formula, if* $x \in B_n^+$, $A \in \mathrm{HP}$, $y = \mathrm{GB}(A)$ *then* $\mathrm{GB}(xA) = \mathrm{LF}(xy)$.

Proof. It suffices to give a proof for $\ell(x) = 1$, so we will henceforth assume this is the case. Write $x = ru$, $y = rv$ $(u, v \in S_n)$, and note $L(u) = \{u\}$. We know:

- $\mathrm{Pro}((ru)A)$ equals the greatest $B \in L(S_n)$ with $\{u\} \subset B \subset (ru)A$ (by Lemma 5.4 and the observation $L(S_n) \ni \{u\} \subset (ru)A$).
- $(ru)A$ is the greatest $C \in \mathrm{HP}$ with $\{u\} \subset C \subset \{u\} \cup uAu$ (by Lemma 5.5(a)).

Combining these observations and recalling that $L(S_n) \subset \mathrm{HP}$, we immediately find that $\mathrm{Pro}((ru)A)$ is the greatest $B \in L(S_n)$ with

$$\{u\} \subset B \subset \{u\} \cup uAu. \tag{16}$$

Write $B = L(uw)$, $w \in S_n$. Assume the left-hand inclusion of (16) to hold: $\{u\} \subset B$ or, equivalently, $u \leq uw$. We have

$$\begin{aligned}
\text{right-hand inclusion of (16)} \quad &\Longleftrightarrow \quad L(uw) \subset \{u\} \cup uAu \\
&\Longleftrightarrow \quad \{u\} \amalg u\, L(w)\, u \subset \{u\} \cup uAu \\
&\Longleftrightarrow \quad L(w) \subset A \\
&\overset{(5.4)}{\Longleftrightarrow} \quad L(w) \subset \mathrm{Pro}(A) = L(v) \\
&\Longleftrightarrow \quad w \leq v \\
&\Longleftrightarrow \quad uw \leq r^{-1}\,\mathrm{LF}(xy),
\end{aligned}$$

the last equivalence following from the assumption that $u \leq uw$. The greatest B satisfying these properties is given by $uw = r^{-1}\,\mathrm{LF}(xy)$. This shows $\mathrm{Pro}((ru)A) = L\,r^{-1}\,\mathrm{LF}(xy)$ and the lemma follows. □

6. Two more properties of the representation

Let $M_m(R)$ denote the algebra of size m square matrices over R. We identify $M_m(R)$ with $\mathrm{End}(V)$.

THEOREM 6.1. *Suppose $R = \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$, the Laurent polynomial ring in two variables. Let $x \in B_n$, and consider the Laurent expansion of ρx with respect to t:*

$$\rho x = \sum_{i=k}^{\ell} A_i(q)\, t^i, \quad A_i \in M_m(\mathbb{Z}[q^{\pm 1}]), \quad A_k \neq 0, \quad A_\ell \neq 0.$$

(a) *Then $\ell_\Omega(x) = \max(\ell - k, \ell, -k)$.*

(b) *If in addition $x \in B_n^+ - \Delta B_n^+$, then $k = 0$ and $\ell = \ell_\Omega(x)$.*

Proof. First we prove (H1): If $x \in B_n^+ - \Delta B_n^+$ then $k = 0$. While we are primarily interested in the case where R is a Laurent polynomial ring, it obviously suffices to prove (H1) in the case where $R = \mathbb{R}[t^{\pm 1}]$ and $q \in \mathbb{R}$ with

$0 < q < 1$. During the proof of (H1), we will assume this is the case. Since $x \notin \Delta B_n^+$ we have $\mathrm{LF}(x) \neq \Delta$; hence $C_{\mathrm{LF}(x)} \cap t\,V_1 = \emptyset$. Choose any $v \in C_1$. Then $xv \in C_{\mathrm{LF}(x)}$ whence $xv \notin t\,V_1$.

Recall that $\rho B_n^+ \subset M_m(\mathbb{Z}[q, q^{-1}, t])$. Since $x \in B_n^+$, we thus have $k \geq 0$. Assume now $k > 0$. Then all entries of ρx are in $t\,\mathbb{R}[t]$; hence $xv \in t\,V_1$. This is a contradiction. This proves that $k = 0$; i.e., (H1) has been proved.

Next, we will show (H2): If $x \in B_n^+ - \Delta B_n^+$ then $\ell = \ell_\Omega(x)$. Write $p = \ell_\Omega(x)$. Define $\Gamma \in \mathrm{GL}(V)$ by $\rho\Delta = t\Gamma$. By 3.1 we have

$$\begin{aligned} (17) \quad T(q)\,\rho(x, q^{-1}, t^{-1})\,T(q)^{-1} &= \rho(\overline{x}, q, t) = \rho(\Delta^{-p}, q, t)\,\rho(\Delta^p\overline{x}, q, t) \\ &= \Gamma^{-p} t^{-p}\,\rho(\Delta^p \overline{x}, q, t). \end{aligned}$$

Note that $T(q)$ does not involve t. Neither does Γ, by 3.2. We now compare the least exponents of t occurring on either side of (17). (For a matrix $A \in \mathrm{GL}(V)$, the least exponent of t is by definition the greatest integer a such that $A \in t^a M_m(\mathbb{Z}[q, q^{-1}, t])$.) By Theorem 2.2(d), we have $\Delta^p\overline{x} \in B_n^+ - \Delta B_n^+$. Applying (H1) to $\Delta^p\overline{x}$ then shows that the least exponent of t on the right-hand side of (17) equals $-p$. The least exponent on the left-hand side equals $-\ell$. It follows that $\ell = p = \ell_\Omega(x)$. This finishes the proof of (H2), and hence of (b).

Finally, we prove (a). Recall the bijection $\mathbb{Z} \times (B_n^+ - \Delta B_n^+) \to B_n$, $(a, y) \mapsto \Delta^a y$. Write $x = \Delta^a y$, $a \in \mathbb{Z}$, $y \in B_n^+ - \Delta B_n^+$, $\ell_\Omega(y) = b$. Then $k = a$, $\ell = a + b$ by (b) and 3.2. Using Theorem 2.2(c), one finds $\ell_\Omega(x) = \ell_\Omega(\Delta^a y) = \max(a + b, b, -a) = \max(\ell, \ell - k, -k)$. This proves (a). □

An immediate consequence of Theorem 6.1 is another proof of the faithfulness of ρ: $B_n \to \mathrm{GL}(V)$ (Theorem 4.6). Indeed, if $x \in B_n$ is in the kernel of ρ, then in the notation of 6.1, we have $k = \ell = 0$, whence $\ell_\Omega(x) = 0$. It follows that $x = 1$.

We return to our assumption $R = \mathbb{R}[t^{\pm 1}]$, $q \in \mathbb{R} \subset R$, $0 < q < 1$. Before proving our next theorem, we establish a simple lemma. The results of Section 2 (or see [8]) imply that any two positive braids x, y have a greatest common lower bound, notation $x \wedge y$. For any two subsets X, Y of some additive abelian group, we write $X + Y = \{x + y \mid x \in X,\ y \in Y\}$.

Lemma 6.2.

(a) *Let* $s \in S$, $A \in \mathrm{HP}$, $x = \mathrm{GB}(A)$. *Then* $rs \leq x \Leftrightarrow s \in A$.

(b) *Let* $x, y \in \Omega$, $x \wedge y = 1$. *Then* $C_x + C_y \subset C_1$.

Proof. (a) We have $rs \leq x \Leftrightarrow s \leq r^{-1}x \Leftrightarrow \{s\} \subset L(r^{-1}x) \Leftrightarrow \{s\} \subset \mathrm{Pro}(A) \Leftrightarrow \{s\} \subset A$. The last equivalence holds because $\mathrm{Pro}(A)$ is the greatest element of $L(S_n)$ contained in A by 5.4, and $\{s\} \in L(S_n)$. This proves (a).

(b) Let $A \in \mathrm{GB}^{-1}(x)$, $B \in \mathrm{GB}^{-1}(y)$. We must show $D_A + D_B \subset C_1$. The intersection of any two half-permutations is again a half-permutation, so $A \cap B \in \mathrm{HP}$. Note $D_A + D_B = D_{A\cap B} \subset C_z$ where $z = \mathrm{GB}(A \cap B)$. We must therefore show $z = 1$. Suppose $z \neq 1$, say $s \in S$, $rs \leq z$. By (a), we have $s \in A \cap B$. By the other direction of (a) and the fact that $s \in A$, we have $rs \leq x$. Similarly, $rs \leq y$; hence $rs \leq x \wedge y = 1$. This contradiction shows $z = 1$ and thus finishes the proof. □

Definition. We define a (total) ordering on $R = \mathbb{R}[t^{\pm 1}]$ as follows. Let $a \in R - \{0\}$, and write $a = \sum_{i=k}^{\ell} a_i t^i$, $a_i \in \mathbb{R}$, $a_k \neq 0$. Then the sign of a is defined to be the sign of a_k. (This is the only ordering of the ring R which restricts to the usual ordering on $\mathbb{R}$ and with $0 < t < b$ for all positive real numbers b.) We also define a map TP: $R \to t^{\mathbb{Z}} \cup \{0\}$ (Trailing Power) which in the above notation takes a to t^k, and with $\mathrm{TP}(0) = 0$.

Definition. We write C instead of C_1. The union of all xC (with $x \in B_n$) will be denoted by U.

Obviously, C is closed under addition and scalar multiplication by elements of $\{a \in R \mid \mathrm{TP}(a) = 1 \text{ and } a > 0\}$. We also have

$$aC = \mathrm{TP}(a)C \quad \text{for all } a \in R_{>0}. \tag{18}$$

The following theorem shows that U has properties resembling those of convex cones in real vector spaces, and moreover relates the greedy form with line segments in U defined over $\mathbb{R}[t^{\pm 1}]$.

THEOREM 6.3.

(a) $\Delta C = tC$.

(b) *The xC (with $x \in B_n$) are disjoint.*

(b) *Let $(y_1, \ldots, y_k) \in \Omega^k$ be greedy; i.e., $\mathrm{LF}(y_i y_{i+1}) = y_i$ $(1 \leq i < k)$. Let $x_0, \ldots, x_k \in B_n$ be such that $x_i = x_{i-1} y_i$ $(1 \leq i \leq k)$. Then*

$$t^i x_0 C + x_k C \subset \begin{cases} t^i x_0 C, & i \leq 0 \qquad \text{(c1)}; \\ x_i C, & 0 \leq i \leq k \qquad \text{(c2)}; \\ x_k C, & k \leq i \qquad \text{(c3)}. \end{cases}$$

(d) *Let $(\tilde{y}_1, \ldots, \tilde{y}_k)$ be a Thurston normal form; i.e., there are greedy $(u_1, \ldots, u_s)$, $(v_1, \ldots, v_t)$ with $(u_s^{-1}, \ldots, u_1^{-1}, v_1, \ldots, v_t) = (\tilde{y}_1, \ldots, \tilde{y}_k)$, and $u_s, v_t \neq 1$, and there is no $w \in B_n^+ - \{1\}$ such that $\{u_1, v_1\} \subset w B_n^+$. Let $\tilde{x}_0, \ldots, \tilde{x}_k \in B_n$ be such that $\tilde{x}_i = \tilde{x}_{i-1} \tilde{y}_i$ $(1 \leq i \leq k)$. Then*

$$\frac{t^i \tilde{x}_0 C + \tilde{x}_k C}{t^i + 1} \subset \begin{cases} \tilde{x}_0 C, & i \leq -s; \\ \tilde{x}_{i+s} C, & -s \leq i \leq t; \\ \tilde{x}_k C, & t \leq i. \end{cases}$$

(e) *The set U is closed under addition and scalar multiplication by positive elements of R.*

Proof. (a) This follows from 3.2: $\Delta\, x_{n+1-j,n+1-i} = tq^{i+j-1}\, x_{ij}$ whenever $1 \le i < j \le n$, and the involution of 5.1.

(b) Let $x \in B_n$, $x \neq 1$. We must show that C and xC are disjoint. Write $x = y\Delta^k$, $y \in B_n^+ - \Delta B_n^+$, $k \in \mathbb{Z}$. Note:

$$C \subset V_1 - tV_1. \tag{19}$$

Similarly, we have $yC = yC_1 \subset C_{\mathrm{LF}(y)} \subset V_1 - tV_1$ (because $y \notin \Delta B_n^+$) whence

$$xC = y\Delta^k C = t^k yC \subset t^k V_1 - t^{k+1} V_1. \tag{20}$$

Suppose now $k > 0$. Then (20) shows $xC \subset t^k V_1 \subset tV_1$. Combining with (19), one finds that C and xC are disjoint. Next, suppose $k < 0$. Then (19) shows $C \subset V_1 \subset t^{k+1} V_1$, which cannot meet xC by (20). It remains to consider the case $k = 0$, i.e., $y = x$. Suppose $C \cap xC \neq \emptyset$. We have $xC \subset C_{\mathrm{LF}(x)}$ and $C = C_1$, so that $C_1 \cap C_{\mathrm{LF}(x)} \neq \emptyset$. Since all C_z are disjoint by 4.5(a), it follows that $\mathrm{LF}(x) = 1$, whence $x = 1$. This finishes the proof in the case $k = 0$, and thereby proves (b).

(c1) Let $i \le 0$. Then $t^i x_0 C + x_k C = t^i x_0 (C + t^{-i}(y_1 \cdots y_k)C) \subset t^i x_0 (C + t^{-i} V_2) \subset t^i x_0 (C + V_2) = t^i x_0 C$. This proves (c1).

(c2) First, we consider the case $i = 1$. Note that $y_1^{-1}\Delta \in \Omega$. Moreover, the fact that (y_1, y_2) is greedy (i.e., $\mathrm{LF}(y_1 y_2) = y_1$) is equivalent to $(y_1^{-1}\Delta) \wedge y_2 = 1$. Using (a), we find

$$x_1^{-1}(tx_0 C + x_k C) \overset{\text{(a)}}{=} (y_1^{-1}\Delta)C_1 + (y_2 \cdots y_k)C_1 \subset C_{y_1^{-1}\Delta} + C_{y_2} \subset C_1 = C.$$

Here, the first inclusion follows from 4.5(b), and the second inclusion from 6.2(b). This finishes the proof in the case $i = 1$. We now give a proof of (c2) by induction on i. For $i = 0$, it follows from (c1). The induction step is shown as follows. Notice $C = C + tC$. Hence

$$\begin{aligned} t^i x_0 C + x_k C &= t^i x_0 C + (t x_k C + x_k C) \\ &= t(t^{i-1} x_0 C + x_k C) + x_k C \subset t x_{i-1} C + x_k C \subset x_i C. \end{aligned}$$

Here, the first inclusion is the induction hypothesis and the second inclusion is a shifted version of the $i = 1$ case. This finishes the proof of (c2).

(c3) Notice that in (c2), it is not excluded that some y_i is 1. By extending the sequence $(y_1, \ldots, y_k)$ in (c1) far enough to the right by ones, one can assume some new k to be at least i. Then (c2) applies and shows (c3).

(d) Define $x_i \in B_n$ by

$$x_i = \begin{cases} \tilde{x}_i \Delta^{i-s}, & 0 \leq i \leq s; \\ \tilde{x}_i, & s \leq i \leq k. \end{cases}$$

Define $y_i \in \Omega$ by $x_i = x_{i-1} y_i$ $(1 \leq i \leq k)$. Then x_i and y_i are as in (c). Observe:

$$x_i C = t^{i-s} \tilde{x}_i C \quad (0 \leq i \leq s).$$

We have

$$\frac{t^i \tilde{x}_0 C + \tilde{x}_k C}{t^i + 1} = \frac{t^{i+s} x_0 C + x_k C}{t^i + 1}.$$

The inclusions in the sequel will be consequences of (c). If $i \leq -s$ then

$$\frac{t^{i+s} x_0 C + x_k C}{t^i + 1} \subset \frac{t^{i+s} x_0 C}{t^i + 1} = t^s x_0 C = \tilde{x}_0 C.$$

If $-s \leq i \leq 0$ then

$$\frac{t^{i+s} x_0 C + x_k C}{t^i + 1} \subset \frac{x_{i+s} C}{t^i + 1} = \frac{t^i \tilde{x}_{i+s} C}{t^i + 1} = \tilde{x}_{i+s} C.$$

If $0 \leq i \leq t$ then

$$\frac{t^{i+s} x_0 C + x_k C}{t^i + 1} \subset \frac{x_{i+s} C}{t^i + 1} = x_{i+s} C = \tilde{x}_{i+s} C.$$

Finally, if $t \leq i$ then

$$\frac{t^{i+s} x_0 C + x_k C}{t^i + 1} \subset \frac{x_k C}{t^i + 1} = x_k C = \tilde{x}_k C.$$

This proves (d).

Part (e) is an easy consequence of (a), (c) and (18). □

University of Basel, Basel, Switzerland
E-mail address: Daan.Krammer@unibas.ch

References

[1] E. Artin, Theorie der Zöpfe, *Hamburg Abh.* **4** (1925), 47–72.

[2] S. Bigelow, The Burau representation is not faithful for $n \geqq 5$, *Geom. Topol.* **3** (1999), 397–404.

[3] ———, Braid groups are linear, *J. Amer. Math. Soc.* **14** (2001), 471–486.

[4] J. Birman, Review of “Braid groups are linear groups” by S. Bachmuth, MR 98h:20061 (1998).

[5] J. Birman, K. H. Ko, and S. J. Lee, A new approach to the word and conjugacy problems in the braid groups, *Adv. Math.* **139** (1998), 322–353.

[6] J. Birman and H. Wenzl, Braids, link polynomials and a new algebra, *Trans. Amer. Math. Soc.* **313** (1989), 249–273.

[7] J. W. Cannon, D. B. A. Epstein, D. F. Holt, S. V. F. Levy, M. S. Paterson, and W. P. Thurston, *Word Processing in Groups*, Jones and Bartlett Publishers, Boston, MA, 1992.

[8] R. Charney, Geodesic automation and growth functions for Artin groups of finite type, *Math. Ann.* **301** (1995), 307–324.
[9] P. Deligne, Les immeubles des groupes de tresses généralisés, *Invent. Math.* **17** (1972), 273–302.
[10] F. A. Garside, The braid group and other groups, *Quart. J. Math. Oxford Ser. (2)* **20** (1969), 235–254.
[11] D. Krammer, The braid group B_4 is linear, *Invent. Math.* **142** (2000), 451–486.
[12] R. J. Lawrence, Homological reprsentations of the Hecke algebra, *Comm. Math. Phys.* **135** (1990), 141–191.
[13] D. D. Long and M. Paton, The Burau representation is not faithful for $n \geq 6$, *Topology* **32** (1993), 439–447.
[14] J. Michel, A note on words in braid monoids, *J. Algebra* **215** (1999), 366–377.
[15] J. A. Moody, The Burau representation of the braid group B_n is unfaithful for large n, *Bull. Amer. Math. Soc.* **25** (1991), 379–384.
[16] M. S. Paterson and A. A. Razborov, The set of minimal braids is co-NP-complete, *J. Algorithms* **12** (1991), 393–408.
[17] R. C. Penner, The action of the mapping class group on curves in surfaces *Enseign. Math.* **30** (1984), 39–55.
[18] P. Xu, Growth of the positive braid semigroups, *J. Pure Appl. Algebra* **80** (1992), 197–215.
[19] M. G. Zinno, On Krammer's representation of the braid group, *Math. Ann.* **321** (2001), 197–211.

(Received March 11, 2000)