Braid groups are linear

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Abstract

In a previous work [11], the author considered a representation of the braid group \( B_n \rightarrow GL_m(\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]) \) \( (m = n(n-1)/2) \), and proved it to be faithful for \( n = 4 \). Bigelow [3] then proved the same representation to be faithful for all \( n \) by a beautiful topological argument. The present paper gives a different proof of the faithfulness for all \( n \). We establish a relation between the Charney length in the braid group and exponents of \( t \). A certain \( B_n \)-invariant subset of the module is constructed whose properties resemble those of convex cones. We relate line segments in this set with the Thurston normal form of a braid.

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1. Introduction

Statement and history of the problem. A group is said to be linear if it is isomorphic to a subgroup of \( GL(n, K) \) for some natural number \( n \) and some field \( K \). An interesting question asks whether the braid group is linear.

One of the most famous representations of the braid group is the Burau representation \( B_n \rightarrow GL_{n-1}(\mathbb{Z}[q^{\pm 1}]) \). It is easily shown to be faithful for \( n \leq 3 \). Moody [15] proved the Burau representation to be unfaithful for \( n \geq 9 \). This bound was improved to \( n \geq 6 \) by Long and Paton [13] and to \( n \geq 5 \) by Bigelow [2]. It is still unknown whether the Burau representation of \( B_4 \) is faithful.

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One of the braid group representations, previously studied by Lawrence [12], was proved to be faithful by the author [11] in the case of $B_4$. Shortly thereafter, Bigelow [3] found a proof that the same representation is faithful for all $n$ by a beautiful topological argument. The present article deals with again the same representation.

More on the history of the linearity problem for braid groups can be found in Birman’s review [4].

The representation. The representation of our interest will be denoted $\rho: B_n \to \text{GL}(V)$, where $V$ is an $m$-dimensional free module over some ring $R$, with $m = n(n-1)/2$. It depends on two invertible elements $q, t \in R$. There are many definitions of this representation. This paper follows an elementary route by exhibiting the entries of the involved matrices, and completely avoids topological arguments. Other definitions include a second homology group [11], [12] and a pictorial approach [3], [11]. Zinno [19] recently showed the representation to be a summand of the Birman-Wenzl algebra [6].

Combinatorial preliminaries. Our linearity proof for the braid group involves a solution to the word problem in the braid group. Among the solutions to the word problem we mention Artin’s one [1] ($B_n$ is isomorphic to a subgroup of $\text{Aut}(F_n)$) and a solution based on Thurston’s boundary of Teichmüller space [17]. Neither solution is relevant to this paper. Important for us is a third, again totally different solution due to Garside ([10], see also [9], [7]).

For $1 \leq i < j \leq n$, let $s(i, j) = s_{ij}$ denote the permutation (called a reflection) in the symmetric group $S_n$ interchanging $i$ with $j$ and preserving the rest. The set of reflections in $S_n$ will be denoted by $\text{Ref}$. Let $\ell: S_n \to \mathbb{Z}_{\geq 0}$ denote the length function with respect to $\{s_{12}, s_{23}, \ldots, s_{n-1,n}\} \subset S_n$.

The braid group $B_n$ admits a presentation by generators $\{rx \mid x \in S_n\}$ and relations $r(xy) = (rx)(ry)$ whenever $\ell(xy) = \ell(x) + \ell(y)$. The positive braid monoid $B^n_+$ is by definition the submonoid of $B_n$ generated by $\Omega := r(S_n)$. For $x \in B^n_+$ there exists a unique longest $y \in S_n$ with $x \in (ry)B^n_+$, notation: $ry = \text{LF}(x)$. We will make use of the following proposition, which is implied by Garside’s results.

**Proposition A** (See 2.1). Let $B_n$ act on a set $U$. Suppose we are given nonempty disjoint subsets $C_x \subset U$ ($x \in \Omega$) with $xC_y \subset C_{\text{LF}(xy)}$ for all $x, y \in \Omega$. Then the $B_n$-action on $U$ is faithful.

Later on, we will apply Proposition A by putting $U = V$. The central question is to find $C_x$ satisfying the assumption of Proposition A. The $C_x$ we use will be convex in some sense.
For any $x \in B_n^+$ there is a unique $(x_1, \ldots, x_k) \in \Omega^k$ such that $x_1 \cdots x_k = x$ and $LF(x, x_{i+1}) = x_i$ for all $i$, and $x_k = 1$. It is called the greedy form of $x$ and is due to Garside [10]. Thurston [7] showed that any braid $x \in B_n$ can uniquely be written $x = y^{-1}z$ with $y, z \in B_n^+$ such that there is no $w \in B_n^+ - \{1\}$ with $\{y, z\} \subset wB_n^+$. Writing $(y_1, \ldots, y_k)$ for the greedy form for $y$ and $(z_1, \ldots, z_\ell)$ for the greedy form for $z$ then gives $s(x) = x^{-1}y_1^{-1} \cdots y_\ell^{-1}z_1 \cdots z_\ell$: this is called the Thurston normal form. Closely related is the length function $\ell_{\Omega}$.

**Faithfulness.** We will throughout make use of a certain basis $\{x_s \mid s \in \text{Ref}\}$ of $V$, and will identify an element of $\text{End}(V)$ with its matrix with respect to this basis. Thus, $\text{End}(V)$ is identified with $M_m(R)$, the size $m$ matrix algebra over $R$.

We will observe that $\rho B_n^+ \subset M_m(Z[q, q^{-1}, t])$: i.e., for positive braids $x$, the entries of $\rho x$ do not involve negative powers of $t$.

Henceforth, we assume $R = \mathbb{R}[t^\pm 1]$, $q \in \mathbb{R}$ and $0 < q < 1$. Then for all positive braids $x \in B_n^+$, the entries of $\rho x$ are in $\mathbb{R}_{\geq 0} + t \mathbb{R}[t]$. This observation is the most important step of the faithfulness proof of $\rho$. A faithfulness proof of the braid group seems to be impossible without some kind of inequalities involved (think of convex cones), and the foregoing observation fulfills this need.

Let $M_m(\{0, 1\})$ denote the set of size $m$ square matrices with entries in $\{0, 1\} \subset \mathbb{Z}$. Multiplication in $M_m(\{0, 1\})$ is defined as follows. Given two elements, one first multiplies them in $M_m(\mathbb{Z})$, then replaces all positive entries by one, leaving zero entries untouched. This multiplication makes $M_m(\{0, 1\})$ into a monoid. We have a monoid homomorphism $B_n^+ \to M_m(\{0, 1\})$, the image of $x \in B_n^+$ being obtained from $\rho x$ by setting $t = 0$ and then replacing positive entries by one. Now $M_m(\{0, 1\})$ is finite; the combinatorics of the homomorphism $B_n^+ \to M_m(\{0, 1\})$ are crucial in the correct definition of $C_x$, which is briefly as follows.

Define

\[
\begin{align*}
\text{HP} & \quad = \quad \left\{ A \in \text{Ref} \quad | \quad s_{ij}, s_{jk} \in A \Rightarrow s_{ik} \in A \quad \text{whenever} \quad 1 \leq i < j < k \leq n \right\}, \\
L(x) & \quad = \quad \left\{ s_{ij} \quad | \quad 1 \leq i < j \leq n, \quad x^{-1}i > x^{-1}j \right\}, \quad (x \in S_n).
\end{align*}
\]

We will see that for any $A \in \text{HP}$ there is a (unique) greatest $B \in L(S_n)$ with $B \subset A$. Notation: $B = \text{Pro}(A)$. For $x \in \Omega$, one defines $C_x \subset V$ to be the set of those vectors $\sum_{s \in \text{Ref}} a_s x_s$ with $a_s \in \mathbb{R}_{\geq 0} + t \mathbb{R}[t]$ and such that on putting $A := \{ s \in \text{Ref} \mid a_s \in t \mathbb{R}[t] \}$ one has $A \in \text{HP}$ and $x = r L^{-1} \text{Pro}(A)$. 


Clearly, it is a purely combinatorial issue whether \(xC_y \subset C_{LF(xy)}\) for all \(x, y \in \Omega\) (the condition of Proposition A). It turns out to be correct, whence by Proposition A:

**Theorem B (See 4.6).** The representation \(\rho: B_n \to \text{GL}(V)\) is faithful, even if \(q\) is a real number with \(0 < q < 1\).

Theorems C and D below state two closely related properties of the representation. They are new and will be proved in Section 6.

**Theorem C (See 6.1).** Let \(x \in B_n\), and consider the Laurent expansion of \(\rho x\) with respect to \(t\):

\[
\rho x = \sum_{i=0}^{\ell} A_i(q) t^i, \quad A_i \in M_m(\mathbb{Z}[q^{-1}]), \quad A_k \neq 0, \quad A_{\ell} \neq 0.
\]

(a) Then \(\ell_\Omega(x) = \max(\ell - k, \ell, -k)\).

(b) If, in addition, \(x \in B_n^+ - \Delta B_n^+\), then \(k = 0\) and \(\ell = \ell_\Omega(x)\).

We define an ordering on \(R = \mathbb{R}[t^{\pm 1}]\) by giving a nonzero element of it the same sign as its trailing coefficient (the coefficient for the least occurring exponent of \(t\)). We write \(C_1 = C\) and \(U = \cup_{x \in B_n} xC\). The following result shows that \(U\) has properties resembling those of convex cones in real vector spaces, and moreover connects the Thurston normal form with line segments in \(U\).

**Theorem D (See 6.3).**

(a) The \(xC\) (with \(x \in B_n\)) are disjoint.

(b) Let \((\tilde{y}_1, \ldots, \tilde{y}_k)\) be a Thurston normal form; i.e., there are greedy \((u_1, \ldots, u_s), (v_1, \ldots, v_t)\) with \((u_1^{-1}, \ldots, u_s^{-1}, v_1, \ldots, v_t) = (\tilde{y}_1, \ldots, \tilde{y}_k)\), and \(u_s, v_t \neq 1\), and there is no \(w \in B_n^+ - \{1\}\) such that \(\{u_1, v_1\} \subset wB_n^+\). Let \(\tilde{x}_0, \ldots, \tilde{x}_k \in B_n\) be such that \(\tilde{x}_i = \tilde{x}_{i-1} \tilde{y}_i\) \((1 \leq i \leq k)\). Then

\[
\frac{t^i \tilde{x}_0 C + \tilde{x}_k C}{t^i + 1} \subset \begin{cases} \tilde{x}_0 C, & i \leq -s; \\ \tilde{x}_{i+s} C, & -s \leq i \leq t; \\ \tilde{x}_k C, & t \leq i. \end{cases}
\]

(c) The set \(U\) is closed under addition and scalar multiplication by positive elements of \(R\).

**Comparison of three methods.** In a previous paper [11], the representation \(\rho\) is proved to be faithful for \(n = 4\) by a somewhat different method. I do not know whether this method works for \(n > 4\). The differences and similarities between this method and the method of the present paper are as follows. Briefly, the roles (not the meanings) of \(q\) and \(t\) are interchanged.
One of our results, Theorem C, relates the exponents of $t$ with the Charney length function. In [11] one finds a (for $n > 4$ conjectural) relation between the exponents of $q$ and the length function with respect to some other generating subset $Q \subset B_n$ with cardinality

$$|Q| = \frac{1}{n + 1} \binom{2n}{n}.$$ 

A basic reference to $Q$, which is also known as the set of band generators, is [5]. The present paper assumes $q$ to be a real number with $0 < q < 1$; in [11], $t$ is a real number with $0 < t < 1$.

The present paper studies the set

$$\bigoplus_{s \in \text{Ref}} \left( \mathbb{R}_{\geq 0} + t \mathbb{R}[t] \right) x_s \subset V,$$

which is essentially a simplicial cone. If $t = 1$, then the $B_n$-module $V$ can be shown to be the symmetric square of the Burau module, so that ‘the cone of positive semi-definite elements’ makes sense. In [11], a generalization of the cone of positive semi-definite elements is studied. This convex cone is not simplicial at all; rather, it is given by finitely many nonlinear algebraic inequalities.

A third method of proof was found by Bigelow [3]. His beautiful and strikingly short proof involves neither a solution to the word problem, nor a basis of the module. In Bigelow’s proof, both $q$ and $t$ are variables. The total ordering on $\langle q, t \rangle$ he uses makes $q$ “more important” than $t$, so that his method is closer to having $t$ constant than to having $q$ constant.

It seems to be interesting to combine the three approaches into one theory, which presumably involves both generating sets $Q$ and $\Omega$.

**Overview.** The paper is built as follows. There are two sets of combinatorial results. The first set is mainly due to Garside, Thurston and Charney and is collected in Section 2. The second set might be new and is treated in Section 5. In Section 3, we define the representation and establish a few identities. An overview of the faithfulness proof (but more detailed than in the introduction) can be found in Section 4. Section 6 is devoted to proving Theorems C and D.

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2. Combinatorial preliminaries

This section collects some combinatorial properties of braid groups mainly
due to Garside, Thurston and Charney. For proofs, we refer to [10], [9], [7],
[8], [14]; remaining statements are left to the reader to prove.

The braid group $B_n$ is defined to be the fundamental group of $\{X \subset \mathbb{C}: |X| = n\}$, the set of n-element subsets of $\mathbb{C}$, with its obvious topology.
Artin proved that the braid group $B_n$ admits a finite presentation (called the
Artin presentation) with generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations
\begin{align*}
(1) \quad \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad (1 \leq i \leq n-1), \\
(2) \quad \sigma_i \sigma_j &= \sigma_j \sigma_i, \quad (|i-j| > 1).
\end{align*}
(We will view $\sigma_i$ as an element of the braid group.)

Let $S_n$ denote the symmetric group on $I_n = \{1, 2, \ldots, n\}$ (action from the
left). For $1 \leq i < j \leq n$, let $s_{ij} = s(i,j) \in S_n$ denote the permutation (called a reflection) interchanging $i$ with $j$ and fixing the other elements of $I_n$. Put
$s_i = s_{i,i+1}$ and $S = \{s_1, \ldots, s_{n-1}\}$. (The pair $(S_n, S)$ is known as a Coxeter
system of type $A_{n-1}$.) By Ref we will denote the set of reflections in $S_n$.

Let $\ell: S_n \to \mathbb{Z}_{\geq 0}$ denote the length function with respect to $S$; i.e., $\ell(x)$
is the smallest natural number $k$ such that there exist $x_1, \ldots, x_k \in S$ with
$x = x_1 \cdots x_k$. The symmetric group $S_n$ contains a unique longest element $w_0$, given by $w_0(i) = n + 1 - i$.

The braid group $B_n$ admits a presentation with generators $\{r_x \mid x \in S_n\}$
and relations $r(xy) = (rx)(ry)$ whenever $\ell(xy) = \ell(x) + \ell(y)$. We will view $rx$ as an element of $B_n$, and we denote the image of $r: S_n \to B_n$ by $\Omega$. There
exists a well-known homomorphism $B_n \to S_n$ defined by $rx \mapsto x$ ($x \in S_n$).
One can identify $r(s_i)$ with $\sigma_i$ in the Artin presentation of the braid group.
The element $\Delta := r(w_0)$ is known as the half-twist.

The submonoid of $B_n$ generated by $\Omega$ will be denoted $B_n^+$ (this includes 1). Elements of the braid group $B_n$ are called braids and elements of $B_n^+$ are called positive braids. Recall the length function $\ell: S_n \to \mathbb{Z}_{\geq 0}$. By the same symbol, we
will denote the length function $\ell: B_n^+ \to \mathbb{Z}_{\geq 0}$, which is the (unique) monoid
homomorphism with $\ell(rx) = \ell(x)$ for all $x \in S_n$. Let $\Omega_k$ denote the set of elements of $\Omega$ of length $k$.

A smallest (respectively, greatest) element of a (partially) ordered set is
an element which is smaller (respectively, greater) than any other element. A
smallest or greatest element does not necessarily exist, but if it exists, then it
is unique.

Define an ordering on $B_n^+$ by $x \leq y$ if and only if $\ell(x) + \ell(y) = \ell(xy)$; it is known
as the weak Bruhat ordering. The ordered set $S_n$ has a smallest element 1 and a greatest element $w_0$. The smallest element of $\Omega$ is also denoted 1, and its greatest element is $\Delta$.

It can be shown that for any $x \in B_n^+$, the set $\{y \in \Omega \mid y \leq x\}$ has a greatest element. It will be denoted by $\text{LF}(x)$ (Left most Factor). A sequence $(x_1, \ldots, x_k) \in \Omega^k$ is said to be (left) greedy if $\text{LF}(x_i, x_{i+1}) = x_i$ for all $i = 1, \ldots, k - 1$. For any $x \in B_n^+$, there is a unique greedy sequence $(x_1, \ldots, x_k)$ with $x_1 \cdots x_k = x$ and $x_k \neq 1$. It is called the (left) greedy form for $x$.

An important identity reads

$$(3) \quad \text{LF}(xy) = \text{LF}(x \text{LF}(y))$$

for all $x, y \in B_n^+$. It implies that the map $B_n^+ \times \Omega \to \Omega$ defined by $(x, y) \mapsto \text{LF}(xy)$ is an action of the monoid $B_n^+$ on $\Omega$.

The following proposition singles out an aspect of the word problem which will be used in the present paper. A similar result can be found in [11]. Its proof is a simple application of Garside’s results described above. The result gives a sufficient condition on a $B_n$-action on any set to be faithful. Later on, the set will be chosen to be a module.

**Proposition 2.1.** Let $B_n$ act on a set $U$. Suppose we are given subsets $C_x \subset U$ ($x \in \Omega$).

(a) If the inclusion $xC_y \subset C_{\text{LF}(xy)}$ holds for all pairs $(x, y) \in \Omega_1 \times \Omega$, then it holds for all pairs in $B_n^+ \times \Omega$.

(b) Assume the following:

(1) The $C_x$ are nonempty and (pairwise) disjoint.

(2) The properties of (a) hold.

Then the $B_n$-action on $U$ is faithful.

**Proof.** (a) We will show the desired result by induction on $\ell(x)$. If $\ell(x) \leq 1$, there is nothing to prove. Now let $\ell(x) > 1$, say $x = uv$, $u, v \in B_n^+ - \{1\}$. Then $xC_y = u(vC_y) \subset u(C_{\text{LF}(vy)}) \subset C_{\text{LF}(u, \text{LF}(vy))} = C_{\text{LF}(uvy)} = C_{\text{LF}(xy)}$. (The two inclusions follow from the induction hypothesis. The middle equality follows from (3).) This proves the induction step and thereby part (a).

(b) Let $\text{Sym}(U)$ denote the group of permutations of $U$, and let $\pi: B_n \to \text{Sym}(U)$ denote the action. Write $xu$ instead of $(\pi x)u$ ($x \in B_n$, $u \in U$). It is known that for any $z \in B_n$, there are $x, y \in B_n^+$ with $z = xy^{-1}$. Our proposition will therefore be proved if we show that for any $x, y \in B_n^+$, if $\pi(x) = \pi(y)$ then $x = y$. We will show this by induction on $\ell(x) + \ell(y)$.
Suppose $x, y \in B^+_n$ with $\pi(x) = \pi(y)$. If $\ell(x) + \ell(y) = 0$ then $x = 1$ and $y = 1$, so certainly $x = y$. Consider now the case $\ell(x) + \ell(y) > 0$. It is given that $C_1$ is nonempty; choose any $u \in C_1$. By (a), we have $xu \in xC_1 \subset C_{\text{LF}(x)}$ and similarly $yu \in C_{\text{LF}(y)}$. We have $\pi x = \pi y$, whence $xu = yu$. It follows that $xu \in C_{\text{LF}(x)} \cap C_{\text{LF}(y)}$. By assumption (1), all $C_z$ are disjoint however. It follows that $\text{LF}(x) = \text{LF}(y)$. Write $z = \text{LF}(x)$, and define $x', y' \in B^+_n$ by $x = zx'$, $y = zy'$. Note $z \neq 1$, because otherwise $x = y = 1$, contradicting the fact that $\ell(x) + \ell(y) > 0$. It follows that $\ell(x') + \ell(y') < \ell(x) + \ell(y)$. The induction assumption thus yields $x' = y'$ and hence $x = y$. This proves the induction step and thereby part (b) of the proposition. \hfill $\Box$

The results in this section so far suffice to understand the faithfulness proof in Sections 4, 5. We now turn to some more combinatorial results which will be used in the proof of 6.1.

The Charney length function is the length function $\ell_{\Omega}: B_n \to \mathbb{Z}_{\geq 0}$ with respect to $\Omega$; i.e., $\ell_{\Omega}(x)$ is the smallest natural number $k$ such that there exist $x_1, \ldots, x_k \in \Omega \cup \Omega^{-1}$ with $x = x_1 \cdots x_k$.

The center of $B_n$ is isomorphic to $\mathbb{Z}$ and, if $n \geq 3$, generated by $\Delta^2$. We have a bijection $\mathbb{Z} \times (B^+_n - \Delta B^+_n) \to B_n$ defined by $(k, x) \mapsto \Delta^k x$.

From the Artin presentation of the braid group, it follows that there exists an automorphism of $B_n$ which takes any $\sigma_i$ to its inverse. We will denote this automorphism by $x \mapsto \pi$.

The following theorem collects some combinatorial results.

**Theorem 2.2** (Garside, Thurston, Charney).

(a) Let $(x_1, \ldots, x_k)$ denote the greedy form of some positive braid $x \in B^+_n$. Then $\ell_{\Omega}(x) = k$.

(b) Let $x \in B_n$. Then there are unique $y = y_x$ and $z = z_x$ both in $B^+_n$ with $x = y^{-1}z$ such that there is no $w \in \Omega_1$ with $\{y, z\} \subset wB^+_n$. They satisfy $\ell_{\Omega}(x) = \ell_{\Omega}(y) + \ell_{\Omega}(z)$.

(c) Let $x \in B^+_n - \Delta B^+_n$ with $\ell_{\Omega}(x) = k$. Then $\ell_{\Omega}(\Delta^{\ell} x) = \max(k + \ell, k, -\ell)$ for all $\ell \in \mathbb{Z}$.

(d) Let $x \in B^+_n - \Delta B^+_n$ with $\ell_{\Omega}(x) = k$. Then $\Delta^k \pi \in B^+_n - \Delta B^+_n$ and $\ell_{\Omega}(\Delta^k \pi) = k$.

(e) The growth function

$$\sum_{x \in B^+_n} z^{\ell_{\Omega}(x)} \in \mathbb{Z}[[z]]$$

is rational.
(f) There exists an algorithm that on input $n \in \mathbb{Z}_{\geq 0}$ and $x \in B_n$ computes (the greedy forms of) $y_x, z_x$ (as defined in (b)) and $\ell_x(x)$. The time the algorithm takes is bounded by a polynomial in $n + \ell_x(x)$.

Charney’s result Theorem 2.2(e) becomes even more remarkable if one knows that for most other finite generating subsets of $B_n$ (including the Artin generating set $\{\sigma_1, \ldots, \sigma_{n-1}\} = \Omega_1$) it is still unknown whether the growth function with respect to it is rational. This should not be confused with Deligne’s result [9] that the growth function of positive braids $X^x \in B_n^+$ is rational (see also [18]).

In contrast to Theorem 2.2(f) Paterson and Razborov [16] proved that computing the length of a braid in $B_n$ with respect to $\Omega_1$ (with $n$ variable) is an NP-complete problem.

3. The representation

Let $R$ denote a commutative ring and $q, t \in R$ two invertible elements. Let $V$ denote the free $R$-module with basis $\{x_s \mid s \in \text{Ref}\}$. Thus, the dimension of $V$ is $m := |\text{Ref}| = n(n - 1)/2$. We will also write $x_{ij}$ instead of $x_{s(i,j)}$ where $1 \leq i < j \leq n$. We define a representation $\rho: B_n \rightarrow \text{GL}(V)$ as follows (action of $\text{GL}(V)$ on $V$ from the left; instead of $(\rho x)v$, we use the simpler notation $xv$, $x \in B_n$, $v \in V$):

$$\sigma_k x_{k,k+1} = tq^2 x_{k,k+1};$$
$$\sigma_k x_{i,k} = (1 - q) x_{i,k} + q x_{i,k+1}, \quad i < k;$$
$$\sigma_k x_{i,k+1} = x_{i,k} + tq^{k-i+1}(q - 1) x_{k,k+1}, \quad i < k;$$
$$\sigma_k x_{kj} = tq(q - 1) x_{k,k+1} + q x_{k+1,j}, \quad k + 1 < j;$$
$$\sigma_k x_{k+1,j} = x_{kj} + (1 - q) x_{k+1,j}, \quad k + 1 < j;$$
$$\sigma_k x_{ij} = x_{ij}, \quad i < j < k \quad \text{or} \quad k + 1 < i < j;$$
$$\sigma_k x_{ij} = x_{ij} + tq^{k-i}(q - 1)^2 x_{k,k+1}, \quad i < k < k + 1 < j.$$

It should be proved here that these formulas do indeed define a representation, i.e., that they respect relations (1) and (2) in the Artin presentation of the braid group, and that $\rho \sigma_k$ is invertible. This is a straightforward though tedious task which we leave to the reader.
Remark. In [11] the author uses another basis \( \{ v_{ij} \mid 1 \leq i < j \leq n \} \) of the same module \( V \). Its relation with \( \{ x_{ij} \}_{ij} \) is given by

\[
(4) \quad v_{ij} = x_{ij} + (1 - q) \sum_{i < k < j} x_{kj}, \quad x_{ij} = v_{ij} + (q - 1) \sum_{i < k < j} q^{k-1-i} v_{kj}.
\]

In [11] one can also find a topological interpretation of \( v_{ij} \). Combination with (4) then results in a topological interpretation of \( x_{ij} \). In the present paper, we will not consider any other bases than \( \{ x_{ij} \}_{ij} \) and its dual. A quicker proof of our formulas defining a representation is obtained if one is willing to accept the formulas with respect to \( \{ v_{ij} \} \) in [11], by combining with (4).

Let \( V^* \) denote the dual of \( V \) and let \( \langle \cdot, \cdot \rangle : V^* \times V \to R \) denote the pairing. Let \( \{ x_{ij}^* \mid s \in \text{Ref} \} \) denote the basis dual to \( \{ x_s \} \), and write \( x_{ij}^* = x_{s(i,j)}^* \). In formula:

\[
V^* = \bigoplus_{s \in \text{Ref}} Rx_s^*, \quad \langle x_r^*, x_s \rangle = \delta_{rs}.
\]

Let \( \text{End}(V) \) act on \( V^* \) on the right by \( \langle uA, v \rangle = \langle u, Av \rangle \) for all \( A \in \text{End}(V) \), \( (u, v) \in V^* \times V \). Again, we will write \( vx \) instead of \( v(\rho x) \) (\( x \in B_n \), \( v \in V^* \)).

The braid group action on \( V^* \) is given by

\[
(5) \quad x_{ik,k+1}^* \sigma_k = t \left[ q^2 x_{ik,k+1}^* + q(q - 1) \sum_{k+1 < b} x_{kb}^* \right. \\
+ (q - 1) \sum_{a < k} q^{k-a+1} x_{a,k+1}^* + (q - 1)^2 \sum_{a < k < k+1 < b} q^{k-a} x_{ab}^* \right]
\]

and

\[
(6) \quad x_{ij}^* \sigma_k = \begin{cases} 
(1 - q) x_{ik}^* + x_{i,k+1}^*, & i < k, j = k; \\
q x_{ik}^*, & i < k, j = k + 1; \\
x_{ik,j}^*, & i = k, j > k + 1; \\
(1 - q) x_{ik+1,j}^* + q x_{kj}^*, & i = k + 1, j > k + 1; \\
x_{ij}^*, & \{i, j\} \cap \{k, k + 1\} = \emptyset.
\end{cases}
\]

The results in this section so far are sufficient background for reading the faithfulness proof in the next two sections. The remainder of this section deals with a few identities which will be used in Section 6 to prove some more properties of the representation.

We define a linear map \( T(u) : V \to V \) depending on a parameter \( u \) by

\[
T(u) : x_{ij} \mapsto \sum_{i < k < \ell < j} (1 - u)^2 u^{i+\ell} x_{k\ell} + \sum_{i = k < \ell < j} (1 - u) u^{i+\ell} x_{k\ell} \bigg. \\
+ \sum_{i < k < \ell = j} (1 - u) u^{i+\ell} x_{k\ell} + \sum_{i = k < \ell = j} u^{i+\ell} x_{k\ell},
\]

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each sum ranging over those \((k, \ell)\) with \(1 \leq k < \ell \leq n\) satisfying the indicated inequalities and identities.

An obvious total ordering on the basis \(\{x_{ij}\}\) makes \(T(u)\) into a triangle matrix with powers of \(u\) on the diagonal. In particular, \(T(u)\) is invertible if \(u\) is. (A fact which we will not need is that \(T(u)T(u^{-1}) = 1\).)

In order to indicate the dependence of \(\rho\) on \(q, t\), we write \(T(u)\) into a triangle matrix with powers of \(u\) on the diagonal. In particular, \(T(u)\) is invertible if \(u\) is. (A fact which we will not need is that \(T(u)T(u^{-1}) = 1\).)

In order to indicate the dependence of \(\Delta\) on \(q, t\), we write \(\Delta = (\sigma_1 \cdots \sigma_n)(\sigma_1 \cdots \sigma_{n-1}) \cdots (\sigma_1 \sigma_2)\). We reduce the amount of calculations as follows.

Define \(\Delta \in \text{GL}(V)\) by the expected formula for \(\Delta\): \(\Delta x_{n+1-j, n+1-i} = t q^{i-j} x_{ij}\). Our goal is then to prove \(\Delta = \rho \Delta\).

Claim 1. \(A(\rho \sigma)A^{-1} = \rho \sigma\) whenever \(1 \leq k \leq n - 1\). This is a straightforward computation, which will be left to the reader.

Claim 2. The centralizer in \(\text{GL}(V)\) of \(\rho B_n\) consists of the scalar matrices only. (This is closely related to the irreducibility of \(V\), which was established by Zinno [19].) In order to prove Claim 2, let \(B \in \text{GL}(V)\) commute with each element of \(\rho B_n\). We must then show \(B\) to be a scalar matrix. Note that \(x_{k, k+1}\) is an eigenvector of \(\sigma_k\) with eigenvalue \(tq^2\). From our formula for the
\( \sigma_k \)-action on \( V^* \), it readily follows that the eigenvalue \( t_\alpha^2 \) is simple; indeed, all remaining eigenvalues depend on \( q \) only. Since \( v_{12} \) is an eigenvector of \( \sigma_1 \) with simple eigenvalue, and \( B \) commutes with \( \sigma_1 \), one has \( Bx_{12} = \lambda x_{12} \) for some invertible \( \lambda \in R \). After multiplying \( B \) with a scalar matrix, we may assume \( Bx_{12} = x_{12} \); our task is then to show \( B = 1 \). Using the identities
\[
\sigma_k x_{1k} = (1 - q)x_{1k} + qx_{1,k+1} \quad (1 < k)
\]
and the identities
\[
\sigma_k x_{k+1,j} = x_{kj} + (1 - q)x_{k+1,j} \quad (k < j)
\]
one inductively finds \( Bx_{ij} = x_{ij} \). This shows that \( B = 1 \), thus proving Claim 2.

Note \( \Delta_k \Delta^{-1} = \sigma_{n-k} \) for all \( k \). Thus, the property of \( A \) formulated in Claim 1 is also satisfied by \( \rho \Delta \). In other words, \( A^{-1}(\rho \Delta) \) is in the centralizer of \( \rho B_n \). By Claim 2, \( A^{-1}(\rho \Delta) \) is scalar, so it suffices to show that \( Ax_{n-1,n} = \Delta x_{n-1,n} \). We use the following factorization of \( \Delta \):
\[
\Delta = \sigma_1(\sigma_2 \sigma_1)(\sigma_3 \sigma_2) \cdots (\sigma_{n-1} \sigma_{n-2}) \Delta_{n-2}
\]
where
\[
\Delta_{n-2} = (\sigma_1 \cdots \sigma_{n-2})(\sigma_1 \cdots \sigma_{n-3}) \cdots (\sigma_1 \sigma_2) \sigma_1.
\]
First of all, \( \Delta_{n-2} x_{n-1,n} = x_{n-1,n} \). Moreover, for \( 3 \leq k \leq n \), one has
\[
\begin{align*}
\sigma_{k-1} \sigma_{k-2} x_{k-1,k} &= \sigma_{k-1} (x_{k-2,k} + (1 - q)x_{k-1,k}) \\
&= (x_{k-2,k} + (1 - q)t_\alpha^2(q - 1)x_{k-1,k}) \\
&\quad + (1 - q)t_\alpha^2 x_{k-1,k} = x_{k-2,k-1}.
\end{align*}
\]
It follows that \( \Delta x_{n-1,n} = \sigma_1(\sigma_2 \sigma_1)(\sigma_3 \sigma_2) \cdots (\sigma_{n-1} \sigma_{n-2}) \Delta_{n-2} x_{n-1,n} = \sigma_1 x_{12} = t_\alpha^2 x_{12} = Ax_{n-1,n} \). This finishes the proof.

\[ \square \]

4. Faithfulness

Recall our representation \( \rho : B_n \to GL(V) \). We often tacitly identify \( \rho x \) with its matrix with respect to the basis \( \{ x_{ij} \}_{ij} \).

Observe that for any positive braid \( x \), the entries of the matrix of \( \rho x \) are in \( \mathbb{Z}[q, q^{-1}, t] \). This follows from the matrices given in Section 3 and the fact that \( B_n^+ \) is generated by \( \Omega_1 = \{ \sigma_1, \ldots, \sigma_{n-1} \} \).

From now on, we take the one-variable Laurent polynomial ring \( R = \mathbb{R}[t^\pm 1] \) for base ring, and \( q \in \mathbb{R} \subset R \) with \( 0 < q < 1 \). Put
\[
V_1 := \bigoplus_{s \in \text{Ref}} \mathbb{R}[t] x_s \subset V = \bigoplus_{s \in \text{Ref}} \mathbb{R}[t^\pm 1] x_s.
\]
We thus have \( B_n^+ V_1 \subset V_1 \).

Note that all entries of \( \rho \sigma_k \) are in \( \{ 0, 1, q, 1 - q \} + t \mathbb{Z}[q, q^{-1}, t] \). By our assumption that \( q \) is a real number with \( 0 < q < 1 \), they are in \( \mathbb{R}_{\geq 0} + t \mathbb{R}[t] \).
On putting

\[ V_2 = \bigoplus_{s \in \text{Ref}} \left( \mathbb{R}_{\geq 0} + t \mathbb{R}[t] \right) x_s = \left( \bigoplus_{s \in \text{Ref}} \mathbb{R}_{\geq 0} x_s \right) \oplus tV_1 \]

we have \( B_n^+ V_2 \subset V_2 \).

**Definition.** For \( A \subset \text{Ref} \), define

\[
D_A = \left\{ v \in V_2 \middle| \text{for all } s \in \text{Ref}: \langle x_s^*, v \rangle \in \mathbb{R}_{\geq 0} + t \mathbb{R}[t] \Leftrightarrow s \in A \right\}.
\]

Thus, \( V_2 \) is the disjoint union of the \( D_A \) (\( A \subset \text{Ref} \)).

Let \( x \in B_n^+, A \subset \text{Ref} \). Then there is a unique \( B \subset \text{Ref} \) with \( xD_A \subset DB \). (A formula for \( B \) is given in 4.1.) Notation: \( B = xA \). Let \( ^2\text{Ref} \) denote the power set of \( \text{Ref} \). The map \( B_n^+ \times ^2\text{Ref} \rightarrow ^2\text{Ref} \), \((x, A) \mapsto xA \) defines an action of \( B_n^+ \) on \( ^2\text{Ref} \). (This follows from the facts that \( \rho \) is a representation and that \( \rho B_n^+ \) preserves \( V_2 \).) An explicit formula for the \( B_n^+ \)-action on \( ^2\text{Ref} \) is as follows.

**Lemma 4.1.** Let \( A \subset \text{Ref} \) and \( 1 \leq k \leq n - 1 \). Then \( \sigma_k A \) is the set of those \( s(i, j) \) with \( 1 \leq i < j \leq n \) and

\[
\begin{align*}
\{ s(i, k), s(i, k + 1) \} & \subset A, & i = k, \ j = k + 1; \\
\{ s(i, k), s(k, k + 1) \} & \subset A, & i < k, \ j = k; \\
s(i, k) & \in A, & i < k, \ j = k + 1; \\
s(k + 1, j) & \in A, & i = k, \ j > k + 1; \\
\{ s(k + 1, j), s(k, j) \} & \subset A, & i = k + 1, \ j > k + 1; \\
s(i, j) & \in A, & \{ i, j \} \cap \{ k, k + 1 \} = \emptyset.
\end{align*}
\]

**Proof.** This is readily obtained from the formulas for the \( \sigma_k \)-action on \( V^* \), (5) and (6).

If one were given the formula of Lemma 4.1 only, it would not be obvious that it defines a \( B_n^+ \)-action on \( ^2\text{Ref} \); but we get the proof of this fact for free as a consequence of our representation.

The notation \( xA \) (\( A \subset \text{Ref} \)) can have rather different meanings according to whether \( x \in S_n \) or \( x \in B_n^+ \). If \( x \in S_n \) then \( xA = \{ xa \mid a \in A \} \), involving multiplication in the symmetric group. For \( x \in B_n^+ \), the notation refers to the \( B_n^+ \)-action on \( ^2\text{Ref} \).

Note that the \( B_n^+ \)-action on \( ^2\text{Ref} \) preserves inclusions, i.e., for \( A \subset B \subset \text{Ref} \) and \( x \in B_n^+ \) one has \( xA \subset xB \).
Definition. We define a map $L: S_n \to 2^{\text{Ref}}$ by

$$L(x) = \left\{ s(i, j) \mid 1 \leq i < j \leq n, \; x^{-1}i > x^{-1}j \right\}.$$ 

Note: $\ell(x) = |L(x)|$. Moreover, for all $x, y \in S_n$, we have

(7) $$x \leq xy \iff \ell(xy) = \ell(x) + \ell(y) \iff L(xy) = L(x) \cup xL(y)x^{-1} \iff L(x) \subseteq L(xy).$$

The image of $L$ will be denoted by $L(S_n)$. As $L$ is injective, one may identify $S_n$ with $L(S_n)$. We will however distinguish them in our notation, because otherwise it would cause confusion.

It can be shown that the $B_n^+$-action on $2^{\text{Ref}}$ does not preserve $L(S_n)$. (See 4.7 for more details.) In particular, the obvious definition $C_x = D_{Lx^{-1}}(?)$ does not satisfy the conditions of 2.1(b). Therefore we need a new idea, which is as follows.

Definition. A set $A \subseteq \text{Ref}$ is said to be a half-permutation\(^1\) if, whenever $1 \leq i < j < k \leq n$, one has

$$s(i, j), \; s(j, k) \in A \implies s(i, k) \in A.$$ 

We will denote the set of half-permutations by $\text{HP}$. Every element of $L(S_n)$ is a half-permutation. A fact which we shall not need is that a subset $A \subseteq \text{Ref}$ is in $L(S_n)$ if and only if both $A$ and $(\text{Ref} - A)$ are half-permutations. This explains the terminology of half-permutations. There is another interpretation of half-permutations as follows. There is a bijection from the set of (partial) orderings $<_0$ on $I_n = \{1, \ldots, n\}$ with $(i <_0 j) \Rightarrow (i < j)$, to $\text{HP}$, which takes $<_0$ to $\{s(i, j) \mid i <_0 j\}$.

We next record a few combinatorial results whose proofs are deferred to the next section for the sake of readability.

**Lemma 4.2.** Let $x \in B_n^+$, $A \in \text{HP}$. Then $xA \in \text{HP}$.

**Proof.** For a proof, see 5.2.

Recall that a greatest element in a (partially) ordered set is an element greater than all other elements.

**Lemma/Definition 4.3.** For every half-permutation $A$ there is a greatest (with respect to inclusion) $B \in L(S_n)$ with $B \subseteq A$. Notation: $B = \text{Pro}(A)$.

**Proof.** See 5.4.

---

\(^1\)Half-permutations are more commonly called closed sets.
Definition. Let GB (Greatest Braid) denote the map
\[ GB = r L^{-1} \text{Pro}: HP \to \Omega. \]
Moreover, for \( x \in \Omega \), define
\[ C_x = \cup \{ D_A \mid A \in HP, \ GB(A) = x \}. \]
Recall the \( B^+_n \)-action on \( \Omega \) defined by \( B^+_n \times \Omega \to \Omega, (x, y) \mapsto LF(xy) \).

Lemma 4.4. The map \( GB: HP \to \Omega \) is \( B^+_n \)-equivariant. In formula, if \( x \in B^+_n, A \in HP, y = GB(A) \), then \( GB(xA) = LF(xy) \).

Proof. See 5.6.

Since \( GB: HP \to \Omega \) is surjective (indeed, \( GB(L(r^{-1}x)) = x \) for all \( x \in \Omega \)), one may call \( HP \) a refinement of \( \Omega \).

Lemma 4.5.
(a) The \( C_x \) are disjoint and nonempty.
(b) We have \( xC_y \subset C_{LF(xy)} \) for all \( (x, y) \in B^+_n \times \Omega \).

Proof. (a) The set \( C_x \) is nonempty because \( \emptyset \neq D_{L(r^{-1}x)} \subset C_x \). That the \( C_x \) are disjoint is trivial.
(b) The definition of \( C_y \) reads \( C_y = \cup \{ D_A \mid A \in HP, GB(A) = y \} \). We must therefore show \( xD_A \subset C_{LF(xy)} \) whenever \( A \in HP, GB(A) = y \). Our proof is summarized by the following chain:
\[ xD_A \subset (1) \subset (2) \subset C_{GB(xA)} \subset (3) \subset C_{LF(xy)}. \]
Here, (1) follows from the definition of the \( B^+_n \)-action on \( 2^{Ref} \). In order to justify (2), note that \( xA \in HP \) by Lemma 4.2. By 4.3 then, \( \text{Pro}(xA) \) is defined and hence so are \( GB(xA) \) and the right-hand side of (2). Inclusion (2) follows by definition of \( C_x \). Identity (3) is Lemma 4.4. This finishes the proof of (b).

Theorem 4.6. The representation \( \rho: B_n \to \text{GL}(V) \) is faithful.

Proof. This is an immediate consequence of 4.5 and 2.1(b) (with \( U = V \)).

The considerations of this section can be illustrated by the commutative diagram of \( B^+_n \)-equivariant maps in Figure 1 below. The arrows pointing to the left are inclusions.
Remark 4.7. The (easy) proof of 4.5 shows a more general statement as follows. Let $HP' \subset 2^{\text{Ref}}$ denote a $B_n^+$-invariant subset, and $GB': HP' \to \Omega$ a surjective $B_n^+$-equivariant map. Then the sets $C'_x := \cup \{D_A \mid A \in HP', \ GB'(A) = x\}$ satisfy the same conclusion of 4.5, thus proving once more that $\rho$ is faithful. In this section, we have constructed one such a pair $(HP', GB')$, namely, $(HP, GB)$. The following questions arise. Are there more such pairs $(HP', GB')$? Is there a best pair, whatever that means? Any pair with the desired properties cannot involve $HP'_0 = L(S_n)$, because $L(S_n)$ is not $B_n^+$-invariant, as is proved by the following counterexample: If $n = 4$, then $A := \{s_{13}, s_{14}, s_{23}, s_{24}\} \in L(S_4)$, but $\sigma_2 A = \{s_{23}, s_{14}\} \notin L(S_4)$. Another solution is given by $HP' = HP_0$, the smallest $B_n^+$-invariant subset of $HP$ containing $L(S_n)$, and $GB' = GB_0$, the appropriate restriction of $GB$. For example, if $n = 4$, then $|HP_0| = 25$ and $HP_0 = L(S_4) \cup \{s_{23}, s_{14}\}$. The sets $HP_0$ seem to be rather messy, and $(HP, GB)$ is a comfortable solution after all.

5. Half-permutations

The aim of this section is to prove some combinatorial results among which are the promised lemmas of the previous section.

Remark 5.1. There exists an involutory automorphism of the system $(S_n, S, B_n^+, r: S_n \to B_n^+, \text{Ref}, B_n^+ \times 2^{\text{Ref}} \to 2^{\text{Ref}}, HP, L, \text{Pro}, GB)$, defined by conjugation by $w_0 \in S_n$, or by $\Delta$ in $B_n$. (Note: $\Delta^2$ is central in $B_n$.) The easy proof is left to the reader. For example, this involution maps $s_k$ to $s_{n-k}$. Especially the fact that the involution preserves the $B_n^+$-action on $2^{\text{Ref}}$ is remarkable. This symmetry will prove useful as it can be used to reduce the number of cases in a few case-by-case proofs.

Lemma 5.2. Let $x \in B_n^+$, $A \in HP$. Then $xA \in HP$.

Proof. One may suppose $x \in \Omega_1$, say $x = \sigma_k$. Let $1 \leq p < q < r \leq n$. We must prove (H): $s(p, q), s(q, r) \in xA \Rightarrow s(p, r) \in xA$. Modulo the symmetry of 5.1, there are five cases to consider, as shown in the first two columns of
Figure 2. For each of these cases, the table in Figure 2 gives a statement in terms of \( A \) which is equivalent to a given one among \( s_{pq} \in xA \), \( s_{qr} \in xA \), \( s_{pr} \in xA \). This table is a consequence of 4.1. Using the table, one readily verifies (H). As an example, we do Case 4:

\[
\begin{align*}
\text{Case 4:} & \quad s(p, q), s(q, r) \in xA \quad \Rightarrow \quad s(p, q), s(p, k + 1), s(k + 1, r) \in A \\
& \quad \Rightarrow \quad s(p, k + 1), s(k + 1, r) \in A \\
& \quad \Rightarrow \quad s(p, r) \in A \Rightarrow s(p, r) \in xA,
\end{align*}
\]

which proves case 4.

\[
\begin{array}{|c|c|c|c|}
\hline
& s(p, q) \in xA & s(q, r) \in xA & s(p, r) \in xA \\
\hline
1 & \{p, q, r\} \cap \{k, k + 1\} = 0 & s(p, q) \in A & s(q, r) \in A & s(p, r) \in A \\
\hline
2 & r = k & s(p, q) \in A & s(q, r) \in A & s(p, r) \in A \\
& & & s(q, k + 1) \in A & s(p, k + 1) \in A \\
\hline
3 & q < k, \ r = k + 1 & s(p, q) \in A & s(q, k) \in A & s(p, k) \in A \\
\hline
4 & q = k, \ r > k + 1 & s(p, q) \in A & s(p, k + 1) \in A \\
& & s(k + 1, r) \in A & s(p, r) \in A \\
\hline
5 & q = k, \ r = k + 1 & s(p, q) \in A & s(p, r) \in A \\
& & & \text{true} & s(p, q) \in A \\
\hline
\end{array}
\]

Figure 2. To the proof of 5.2

**Lemma 5.3.** Let \( A \in \text{HP} \), \( x \in S_n \), \( L(x) \subseteq A \), \( B = x^{-1}(A - L(x))x \). Then \( B \in \text{HP} \).

**Proof.** First we prove the lemma for the case \( \ell(x) = 1 \), say \( x = s_k \). Notice that \( x^2 = 1 \). Let \( 1 \leq p < q < r \leq n \). We must show \((H)\): \( s_{pq}, s_{qr} \in B \Rightarrow s_{pr} \in B \). First consider the case where \( \{p, q, r\} \cap \{k, k + 1\} \) consists of at most one element. Write \( (p', q', r') = (xp, xq, xr) \). Then \( p' < q' < r' \). From \( s(p, q) \in B \) we find \( s(p', q') = x s(p, q) x \in A \); similarly \( s(q, r) \in B \) implies \( s(q', r') = x s(q, r) x \in A \). As \( A \) is a half-permutation and \( p' < q' < r' \), it follows that \( s(p', r') \in A \). Hence \( s(p, r) = x s(p', r') x \in B \), thus proving \((H)\) if \( |\{p, q, r\} \cap \{k, k + 1\}| \leq 1 \). Because of the symmetry of 5.1, it remains only to consider the case \( q = k, \ r = k + 1 \). Then the left-hand side of \((H)\) implies \( s_{qr} \in B \), whence \( x \in Bx = A - L(x) = A - \{x\} \), a contradiction. This proves \((H)\) in the case \( q = k, r = k + 1 \), thus establishing the lemma for \( \ell(x) = 1 \).
We finish the proof of the lemma by induction on \( \ell(x) \). For \( \ell(x) \leq 1 \) there is nothing left to prove. Suppose \( u \leq uv = x \) with \( u, v \in S_n - \{1\} \). Recall (7) that \( L(x) \) is the disjoint union of \( L(u) \) with \( uL(v)u^{-1} \). Since \( L(x) \subset A \), we have \( L(u) \subset A \). Applying the induction hypothesis to \((A, u)\) shows that \( C := u^{-1}(A - L(u))u \) is a half-permutation. From \( L(x) \subset A \) we find \( L(v) \subset C \). Applying the induction hypothesis to \((C, v)\) then yields

\[
\text{HP} \ni v^{-1}(C - L(v))v = v^{-1}(u^{-1}(A - L(u))u - L(v))v \\
= v^{-1}u^{-1}(A - L(u) - uL(v)u^{-1})uv \\
= x^{-1}(A - L(x))x = B.
\]

This proves the induction step and hence the lemma. \( \square \)

**Lemma/Definition 5.4.** For every half-permutation \( A \) there is a greatest (with respect to inclusion) \( B \in L(S_n) \) with \( B \subset A \). Notation: \( B = \text{Pro}(A) \).

**Proof.** Recall (7) that for \( x, y \in S_n \) we have \( x \leq y \iff L(x) \subset L(y) \). So an equivalent formulation of the lemma is that \( P := \{ y \in S_n \mid L(y) \subset A \} \) contains a greatest element. This is the formulation which we will prove.

Note that the ordering on \( P \) is generated by \( x \leq xs \) whenever true, with \( x, xs \in P \), \( s \in S \). Let \( x, xs, xt \in P \) with \( x \leq xs, x \leq xt, s, t \in S \) \( (s \neq t) \). Since \( P \) is finite and has a smallest element, it suffices (by a well-known elementary result on partial ordered sets) to show that there exists then \( y \in P \) with \( xs, xt \leq y \). Let \( m_{st} \in \{2, 3\} \) denote the order of \( st \), and put \( y = xst \) if \( m_{st} = 2 \), and \( y = xsts \) if \( m_{st} = 3 \). It is well-known that \( xs, xt \leq y \). We claim that \( y \in P \). The lemma would clearly follow from this claim. We consider two cases according to the value of \( m_{st} \).

**Case 1.** \( m_{st} = 2 \). Then \( L(y) = L(xs) \cup L(xt) \subset A \) whence \( y \in P \).

**Case 2.** \( m_{st} = 3 \). Write \( s = s_k, t = s_{k+1} \). Define \( C = x^{-1}(A - L(x))x \). Let \( II \) denote disjoint union. For any \( u \in S_n \) with \( x \leq xu \) we have \( xu \in P \iff L(xu) \subset A \iff L(x) \cup xL(u)x^{-1} \subset A \iff xL(u)x^{-1} \subset A - L(x) \iff L(u) \subset x^{-1}(A - L(x))x = C \), thus showing (for any \( u \in S_n \) with \( x \leq xu \)):

\[
(8) \quad xu \in P \iff L(u) \subset C.
\]

Applying (8) to \( u = s, t \) gives \( s, t \in C \); i.e., \( s(k, k + 1), s(k + 1, k + 2) \in C \). By 5.3, we have \( C \in \text{HP} \), which means that we may conclude \( s(k, k + 2) \in C \). Hence \( L(sts) = \{s(k, k + 1), s(k + 1, k + 2), s(k, k + 2)\} \subset C \). Applying (8) in the reverse direction to \( u = sts \) we find \( y = xsts \in P \). This finishes case 2 and thereby the proof of the lemma. \( \square \)

Notice that \( \text{Pro} \) is a projection; i.e., \( \text{Pro}^2 = \text{Pro} \). Moreover, \( \text{Pro} L = L \).
As to the following lemma, we will only make use of the special case of (a) where \( \ell(x) = 1 \). We prove the entire lemma because it appears to have interest of its own. Recall the \( B_n^+ \)-action on \( 2^{\Ref} \) defined in the previous section (or by 4.1), and which preserves \( \HP \) by 5.2.

**Lemma 5.5.**

(a) *Let \( A \in \HP \), \( x \in S_n \). Then \((rx)A \) equals the greatest (with respect to inclusion) half-permutation \( B \) with*

\[
L(x) \subset B \subset L(x) \cup xAx^{-1}.
\]

*(In particular, a greatest such half-permutation exists.)*

(b) *For \( x, y \in S_n \) with \( x \leq xy \) we have \((ry)L(y) = L(xy) \). In particular (for \( y = 1 \)), \((rx)\emptyset = L(x)\).*

**Proof.** We start by proving (a) if \( \ell(x) = 1 \). Write \( x = s_k \), and note \( L(x) = \{x\} \). By 5.2, we have \((rx)A \in \HP \). From the definition of \((rx)A \) one readily finds \( \{x\} \subset (rx)A \subset \{x\} \cup xAx \).

It remains to show, for any half-permutation \( B \), that (9) implies \( B \subset (rx)A \). Suppose \( s_{ij} \in B \), \( 1 \leq i < j \leq n \). We must prove \( s_{ij} \in (rx)A \). We consider four cases.

- **Case 1.** \( i = k \), \( j = k + 1 \). Then \( s_{ij} \in (rx)A \) by 4.1.
- **Case 2.** \( i < k \), \( j = k + 1 \). We have \( x \neq s_{ij} \in B \subset \{x\} \cup xAx \), whence \( s_{ij} \in xAx \), whence \( s_{ik} = xs_{ij}x \in A \), whence \( s_{ij} = s_{i,k+1} \in (rx)A \) by 4.1.
- **Case 3.** \( i < k \), \( j = k \). Then similarly to Case 2, we have \( x \neq s_{ij} \in B \subset \{x\} \cup xAx \), whence \( s_{ij} \in xAx \) and \( s_{i,k+1} = xs_{ij}x \in A \). Moreover, as \( s_{ik} \), \( s_{k,k+1} \in B \) and \( B \in \HP \), we also have \( s_{i,k+1} \in B \). In Case 2 we already saw that \( s_{i,k+1} \in B \) implies \( s_{ik} \in A \). Summarizing, we have \( s_{ik}, s_{i,k+1} \in A \) whence \( s_{ij} = s_{ik} \in (rx)A \) by 4.1. This finishes Case 3.
- **Case 4.** \( \{i,j\} \cap \{k,k+1\} = \emptyset \). Then \( s_{ij} \in B \) readily implies \( s_{ij} \in A \) and hence \( s_{ij} \in (rx)A \).

By the symmetry of 5.1, it suffices to do Cases 1–4. The proof of (a) with \( \ell(x) = 1 \) is thus finished.

We will now prove (a) by induction on \( \ell(x) \). For \( \ell(x) \leq 1 \), we have seen it before. Suppose \( u \leq uv = x \), \( u, v \in S_n - \{1\} \). Recall (7): \( L(x) = L(u) \amalg u L(v) u^{-1} \) where \( \amalg \) denotes disjoint union.

By the induction hypothesis applied to \( (A, v) \) (instead of \( (A, x) \)), we have \( L(v) \subset (vu)A \). Hence \( (ru)L(v) \subset (ru)(rv)A = (rx)A \). But the induc-
tion hypothesis for \((L(v), u)\) implies that \((ru)L(v)\) equals the greatest half-permutation \(B\) with \(L(u) \subset B \subset L(u) \cup uL(v)u^{-1} = L(x)\). As \(L(x)\) is itself a half-permutation, we find \((ru)L(v) = L(x)\). We have thus shown:

\[
L(x) \subset (rx)A.
\]

Applying the induction hypothesis to \((A, v)\), we see that \((rv)A \subset L(v) \cup vAv^{-1}\).

Combining with the induction hypothesis on \(((rv)A, u)\), we find \((rx)A = (ru)(rv)A \subset L(u) \cup u(rv)Au^{-1} \subset L(u) \cup u(L(v) \cup vAv^{-1})u^{-1} = (L(u) \cup uL(v)u^{-1}) \cup uvAv^{-1}u^{-1} = L(x) \cup xAx^{-1}\). We have shown:

\[
(rx)A \subset L(x) \cup xAx^{-1}.
\]

In view of (10) and (11), it remains to show that for any half-permutation \(B\) with (9) one has \(B \subset (rx)A\). Let \(B \in HP\) have the property (9). We have \(L(u) \subset L(x) \subset B\), which shows

\[
L(u) \subset B.
\]

Define \(C = u^{-1}(B - L(u))u\). By 5.3 and (12), we have

\[
C \in HP.
\]

We have \(L(u) \Pi uL(v)u^{-1} = L(x) \subset B = L(u) \Pi uCu^{-1}\), which shows

\[
L(v) \subset C.
\]

We also have \(L(u) \Pi uCu^{-1} = B \subset L(x) \cup xAx^{-1} = L(u) \cup uL(v)u^{-1} \cup uvAv^{-1}u^{-1}\), which shows

\[
C \subset L(v) \cup vAv^{-1}.
\]

Applying the induction hypothesis to \((A, v)\) and invoking (13), (14), (15), one finds \(C \subset (rv)A\). Hence \(B = L(u) \cup uCu^{-1} \subset L(u) \cup u(rv)Au^{-1}\). Combining with (12), we have \(L(u) \subset B \subset L(u) \cup u(rv)Au^{-1}\). By the induction hypothesis applied to \(((rv)A, u)\), it follows that \(B \subset (ru)(rv)A = (rx)A\). This finishes the induction step and hence the proof of (a).

We turn to (b). By (a), \((rx)L(y)\) is the greatest half-permutation \(B\) with \(L(x) \subset B \subset L(x) \cup xL(y)x^{-1} = L(xy)\), the last identity being (7). But \(L(xy)\) is itself a half-permutation. This proves that \((rx)L(y) = L(xy)\).

Recall the map \(GB = rL^{-1}\) Pro: \(HP \to \Omega\).

**Lemma 5.6.** The map \(GB: HP \to \Omega\) is \(B_n^+\)-equivariant. In formula, if \(x \in B_n^+, A \in HP, y = GB(A)\) then \(GB(xA) = LF(xy)\).

**Proof.** It suffices to give a proof for \(\ell(x) = 1\), so we will henceforth assume this is the case. Write \(x = ru, y = rv\) \((u, v \in S_n)\), and note \(L(u) = \{u\}\). We know:
\[ \text{Pro}((ru)A) \text{ equals the greatest } B \in L(S_n) \text{ with } \{u\} \subset B \subset (ru)A \text{ (by Lemma 5.4 and the observation } L(S_n) \ni \{u\} \subset (ru)A). \]

\[ (ru)A \text{ is the greatest } C \in \text{HP with } \{u\} \subset C \subset \{u\} \cup uAu \text{ (by Lemma 5.5(a)).} \]

Combining these observations and recalling that \( L(S_n) \subset \text{HP}, \) we immediately find that \( \text{Pro}((ru)A) \) is the greatest \( B \in L(S_n) \) with
\[ \{u\} \subset B \subset \{u\} \cup uAu. \]

Write \( B = L(uw), w \in S_n. \) Assume the left-hand inclusion of (16) to hold: \( \{u\} \subset B \) or, equivalently, \( u \leq uw. \) We have
\[ \text{right-hand inclusion of (16)} \iff L(uw) \subset \{u\} \cup uAu \]
\[ \iff \{u\} \cup uL(w)u \subset \{u\} \cup uAu \]
\[ \iff L(w) \subset A \]
\[ \iff L(w) \subset \text{Pro}(A) = L(v) \]
\[ \iff w \leq v \]
\[ \iff uw \leq r^{-1} LF(xy), \]
the last equivalence following from the assumption that \( u \leq uw. \) The greatest \( B \) satisfying these properties is given by \( uw = r^{-1} LF(xy) \). This shows \( \text{Pro}((ru)A) = L r^{-1} LF(xy) \) and the lemma follows. \( \square \)

6. Two more properties of the representation

Let \( M_m(R) \) denote the algebra of size \( m \) square matrices over \( R. \) We identify \( M_m(R) \) with \( \text{End}(V). \)

\text{Theorem 6.1. \ Suppose } R = \mathbb{Z}[q^{\pm 1}, t^{\pm 1}], \text{ the Laurent polynomial ring in two variables. Let } x \in B_n, \text{ and consider the Laurent expansion of } \rho x \text{ with respect to } t:\]
\[ \rho x = \sum_{i=k}^{\ell} A_i(q) t^i, \quad A_i \in M_m(\mathbb{Z}[q^{\pm 1}]), \quad A_k \neq 0, \quad A_\ell \neq 0. \]

(a) Then \( \ell_\Omega(x) = \max(\ell - k, \ell, -k). \]
(b) If in addition \( x \in B_n^+ - \Delta B_n^+, \) then \( k = 0 \) and \( \ell = \ell_\Omega(x). \)

\text{Proof. \ First we prove (H1): If } x \in B_n^+ - \Delta B_n^+, \text{ then } k = 0. \text{ While we are primarily interested in the case where } R \text{ is a Laurent polynomial ring, it obviously suffices to prove (H1) in the case where } R = \mathbb{R}[t^{\pm 1}] \text{ and } q \in \mathbb{R} \text{ with} \]
0 < q < 1. During the proof of (H1), we will assume this is the case. Since $x \not\in \Delta B_n^+$ we have $\operatorname{LF}(x) \neq \Delta$; hence $C_{\operatorname{LF}(x)} \cap t V_1 = \emptyset$. Choose any $v \in C_1$. Then $xv \in C_{\operatorname{LF}(x)}$ whence $xv \not\in t V_1$.

Recall that $\rho B_n^+ \subset M_n(\mathbb{Z}[q, q^{-1}, t])$. Since $x \in B_n^+$, we thus have $k \geq 0$. Assume now $k > 0$. Then all entries of $px$ are in $t \mathbb{R}[t]$; hence $xv \in t V_1$. This is a contradiction. This proves that $k = 0$; i.e., (H1) has been proved.

Next, we will show (H2): If $x \in B_n^+ - \Delta B_n^+$ then $\ell = \ell(x)$. Write $p = \ell(x)$. Define $\Gamma \in \operatorname{GL}(V)$ by $\rho \Delta = t \Gamma$. By 3.1 we have

$$\begin{align*}
(17) \quad T(q) \rho(x, q^{-1}, t^{-1}) T(q)^{-1} &= \rho(\tau, q, t) = \rho(\Delta^{-\tau}, q, t) \rho(\Delta^\tau, q, t) \\
&= \Gamma^{-\ell} t^{-\ell} \rho(\Delta^{\ell}, q, t).
\end{align*}$$

Note that $T(q)$ does not involve $t$. Neither does $\Gamma$, by 3.2. We now compare the least exponents of $t$ occurring on either side of (17). (For a matrix $A \in \operatorname{GL}(V)$, the least exponent of $t$ is by definition the greatest integer $a$ such that $A \in t^a M_n(\mathbb{Z}[q, q^{-1}, t])$.) By Theorem 2.2(d), we have $\Delta^{\ell} \tau \in B_n^+ - \Delta B_n^+$. Applying (H1) to $\Delta^{\ell} \tau$ then shows that the least exponent of $t$ on the right-hand side of (17) equals $-\ell$. The least exponent on the left-hand side equals $\ell - \ell$. It follows that $\ell = p = \ell(x)$. This finishes the proof of (H2), and hence of (b).

Finally, we prove (a). Recall the bijection $\mathbb{Z} \times (B_n^+ - \Delta B_n^+) \to B_n, (a, y) \mapsto \Delta^a y$. Write $x = \Delta^a y, a \in \mathbb{Z}, y \in B_n^+ - \Delta B_n^+, \ell(x) = b$. Then $k = a$, $\ell = a + b$ by (b) and 3.2. Using Theorem 2.2(c), one finds $\ell(x) = \ell(\Delta^a y) = \max(a + b, b, -a) = \max(\ell, \ell - k, -k)$. This proves (a).

An immediate consequence of Theorem 6.1 is another proof of the faithfulness of $\rho: B_n \to \operatorname{GL}(V)$ (Theorem 4.6). Indeed, if $x \in B_n^+$ is in the kernel of $\rho$, then in the notation of 6.1, we have $k = \ell = 0$, whence $\ell(x) = 0$. It follows that $x = 1$.

We return to our assumption $R = \mathbb{R}[t^\pm 1], q \in \mathbb{R} \subset R, 0 < q < 1$. Before proving our next theorem, we establish a simple lemma. The results of Section 2 (or see [8]) imply that any two positive braids $x, y$ have a greatest common lower bound, notation $x \wedge y$. For any two subsets $X, Y$ of some additive abelian group, we write $X + Y = \{x + y \mid x \in X, y \in Y\}$.

**Lemma 6.2.**

(a) Let $s \in S, A \in \Pi, x = \text{GB}(A)$. Then $rs \leq x \Leftrightarrow s \in A$.

(b) Let $x, y \in \Omega, x \wedge y = 1$. Then $C_x + C_y \subset C_1$.

**Proof.** (a) We have $rs \leq x \Leftrightarrow s \leq r^{-1}x \Leftrightarrow \{s\} \subset L(r^{-1}x) \Leftrightarrow \{s\} \subset \text{Pro}(A) \Leftrightarrow \{s\} \subset A$. The last equivalence holds because $\text{Pro}(A)$ is the greatest element of $L(S_n)$ contained in $A$ by 5.4, and $\{s\} \in L(S_n)$. This proves (a).
Let $A \in GB^{-1}(x)$, $B \in GB^{-1}(y)$. We must show $D_A + D_B \subseteq C_1$.

The intersection of any two half-permutations is again a half-permutation, so $A \cap B \in HP$. Note $D_A + D_B = D_{A \cap B} \subseteq C_z$ where $z = GB(A \cap B)$. We must therefore show $z = 1$. Suppose $z \neq 1$, say $s \in S$, $rs \leq z$. By (a), we have $s \in A \cap B$. By the other direction of (a) and the fact that $s \in A$, we have $rs \leq x$. Similarly, $rs \leq y$; hence $rs \leq x \wedge y = 1$. This contradiction shows $z = 1$ and thus finishes the proof.

**Definition.** We define a (total) ordering on $R = \mathbb{R}[t^{\pm 1}]$ as follows. Let $a \in R - \{0\}$, and write $a = \sum_{i=k}^{\ell} a_i t^i$, $a_i \in \mathbb{R}$, $a_k \neq 0$. Then the sign of $a$ is defined to be the sign of $a_k$. (This is the only ordering of the ring $R$ which restricts to the usual ordering on $\mathbb{R}$ and with $0 < t < b$ for all positive real numbers $b$.) We also define a map $TP: R \to t^\mathbb{Z} \cup \{0\}$ (Trailing Power) which in the above notation takes $a$ to $t^k$, and with $TP(0) = 0$.

**Definition.** We write $C$ instead of $C_1$. The union of all $xC$ (with $x \in B_n$) will be denoted by $U$.

Obviously, $C$ is closed under addition and scalar multiplication by elements of $\{a \in R \mid TP(a) = 1 \text{ and } a > 0\}$. We also have

$$aC = TP(a)C \quad \text{for all } a \in R_{>0}.$$

The following theorem shows that $U$ has properties resembling those of convex cones in real vector spaces, and moreover relates the greedy form with line segments in $U$ defined over $\mathbb{R}[t^{\pm 1}]$.

**Theorem 6.3.**

(a) $\Delta C = tC$.

(b) The $xC$ (with $x \in B_n$) are disjoint.

(b) Let $(y_1, \ldots, y_k) \in \Omega^k$ be greedy; i.e., $LF(y_i y_{i+1}) = y_i$ (1 \leq i < k). Let $x_0, \ldots, x_k \in B_n$ be such that $x_i = x_{i-1} y_i$ (1 \leq i \leq k). Then

$$t^i x_0 C + x_k C \subset \begin{cases} t^i x_0 C, & \text{if } i \leq 0 \\ x_i C, & 0 \leq i \leq k \\ x_k C, & k \leq i \end{cases}$$

(c) Let $(\tilde{y}_1, \ldots, \tilde{y}_k)$ be a Thurston normal form; i.e., there are greedy $(u_1, \ldots, u_s), (v_1, \ldots, v_l)$ with $(u_s^{-1}, \ldots, u_1^{-1}, v_1, \ldots, v_l) = (\tilde{y}_1, \ldots, \tilde{y}_k)$, and $u_s, v_l \neq 1$, and there is no $w \in B_n^+ - \{1\}$ such that $\{u_1, v_1\} \subset wB_n^+$. Let $\tilde{x}_0, \ldots, \tilde{x}_k \in B_n$ be such that $\tilde{x}_i = \tilde{x}_{i-1} \tilde{y}_i$ (1 \leq i \leq k). Then

$$t^i \tilde{x}_0 C + \tilde{x}_k C \subset \begin{cases} \tilde{x}_0 C, & i \leq -s; \\ \tilde{x}_{i+s} C, & -s \leq i \leq t; \\ \tilde{x}_k C, & t \leq i. \end{cases}$$
(e) The set $U$ is closed under addition and scalar multiplication by positive elements of $R$.

Proof. (a) This follows from 3.2: $\Delta x_{n+1-j, n+1-i} = tq^{i+j-1}x_{ij}$ whenever $1 \leq i < j \leq n$, and the involution of 5.1.

(b) Let $x \in B_n$, $x \neq 1$. We must show that $C$ and $xC$ are disjoint. Write $x = y\Delta^k$, $y \in B_n^+ - \Delta B_n^+$, $k \in \mathbb{Z}$. Note:

\begin{equation}
C \subset V_1 - tV_1.
\end{equation}

Similarly, we have $yC = yC_1 \subset C_{LF(y)} \subset V_1 - tV_1$ (because $y \notin \Delta B_n^+$) whence

\begin{equation}
xC = y\Delta^kC = t^k yC \subset t^k V_1 - t^{k+1} V_1.
\end{equation}

Suppose now $k > 0$. Then (20) shows $xC \subset t^k V_1 \subset tV_1$. Combining with (19), one finds that $C$ and $xC$ are disjoint. Next, suppose $k < 0$. Then (19) shows $C \subset V_1 \subset t^{k+1}V_1$, which cannot meet $xC$ by (20). It remains to consider the case $k = 0$, i.e., $y = x$. Suppose $C \cap xC \neq \emptyset$. We have $xC \subset C_{LF(x)}$ and $C = C_1$, so that $C_1 \cap C_{LF(x)} \neq \emptyset$. Since all $C_1$ are disjoint by 4.5(a), it follows that $LF(x) = 1$, whence $x = 1$. This finishes the proof in the case $k = 0$, and thereby proves (b).

(c1) Let $i \leq 0$. Then $t^ix_0C + x_kC = t^ix_0(C + t^{-i}(y_1 \cdots y_k)C) \subset t^ix_0(C + t^{-i}V_2) \subset t^ix_0(C + V_2) = t^ix_0C$. This proves (c1).

(c2) First, we consider the case $i = 1$. Note that $y_1^{-1}\Delta \in \Omega$. Moreover, the fact that $(y_1, y_2)$ is greedy (i.e., $LF(y_1, y_2) = y_1$) is equivalent to $(y_1^{-1}\Delta) \wedge y_2 = 1$. Using (a), we find

\[
x_1^{-1}(tx_0C + x_kC) \overset{(a)}{=} (y_1^{-1}\Delta)C_1 + (y_2 \cdots y_k)C_1 \subset C_{y_1^{-1}\Delta} + C_{y_2} \subset C_1 = C.
\]

Here, the first inclusion follows from 4.5(b), and the second inclusion from 6.2(b). This finishes the proof in the case $i = 1$. We now give a proof of (c2) by induction on $i$. For $i = 0$, it follows from (c1). The induction step is shown as follows. Notice $C = C + tC$. Hence

\[
t^ix_0C + x_kC = t^ix_0C + (tx_kC + x_kC)
= t(t^{i-1}x_0C + x_kC) + x_kC \subset tx_{i-1}C + x_kC \subset x_iC.
\]

Here, the first inclusion is the induction hypothesis and the second inclusion is a shifted version of the $i = 1$ case. This finishes the proof of (c2).

(c3) Notice that in (c2), it is not excluded that some $y_i$ is 1. By extending the sequence $(y_1, \ldots, y_k)$ in (c1) far enough to the right by ones, one can assume some new $k$ to be at least $i$. Then (c2) applies and shows (c3).
(d) Define $x_i \in B_n$ by

$$x_i = \begin{cases} \bar{x}_i \Delta^{i-s}, & 0 \leq i \leq s; \\ \bar{x}_i, & s < i \leq k. \end{cases}$$

Define $y_i \in \Omega$ by $x_i = x_{i-1}y_i$ ($1 \leq i \leq k$). Then $x_i$ and $y_i$ are as in (c).

Observe:

$$x_i C = t^{i-s} \bar{x}_i C \quad (0 \leq i \leq s).$$

We have

$$\frac{t^{i+s} x_0 C + x_k C}{t^i + 1} = \frac{t^{i+s} x_0 C + x_k C}{t^i + 1}.$$

The inclusions in the sequel will be consequences of (c). If $i \leq -s$ then

$$\frac{t^{i+s} x_0 C + x_k C}{t^i + 1} \subset \frac{t^{i+s} x_0 C}{t^i + 1} = t^s x_0 C = \bar{x}_0 C.$$

If $-s \leq i \leq 0$ then

$$\frac{t^{i+s} x_0 C + x_k C}{t^i + 1} \subset \frac{x_{i+s} C}{t^i + 1} = t^{i+s} \bar{x}_i C = \bar{x}_{i+s} C.$$

If $0 \leq i \leq t$ then

$$\frac{t^{i+s} x_0 C + x_k C}{t^i + 1} \subset \frac{x_{i+s} C}{t^i + 1} = x_{i+s} C = \bar{x}_{i+s} C.$$

Finally, if $t \leq i$ then

$$\frac{t^{i+s} x_0 C + x_k C}{t^i + 1} \subset \frac{x_k C}{t^i + 1} = x_k C = \bar{x}_k C.$$

This proves (d).

Part (e) is an easy consequence of (a), (c) and (18). \[\square\]

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