MA 3E10 GROUPS AND REPRESENTATIONS

THE UNIVERSITY OF WARWICK

THIRD YEAR EXAMINATION: April 2012

MA3E1 GROUPS AND REPRESENTATIONS

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.
If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. Let G be a finite group. Let $\rho_1, \dots, \rho_s$ be a maximal set of pairwise inequivalent irreducible representations. Let χ_i denote the character of ρ_i. You may use the following if you refer to them:

(1) Every finite-dimensional $\mathbb{C}G$-module is a direct sum of simple submodules;

(2) Orthogonality for irreducible characters

and any earlier results.

(a) State without proof orthogonality for irreducible characters. **[2]**

(b) Prove $s \leq k(G)$ where $k(G)$ is the number of conjugacy classes in G and s is defined in the first line of the question. **[6]**

(c) Let ρ be a representation of G. Prove that ρ is irreducible if and only if $(\chi_\rho, \chi_\rho)_G = 1$. **[6]**

(d) Let ρ, σ be representations of G. Prove $\chi_\rho = \chi_\sigma \Rightarrow \rho \sim \sigma$. **[7]**

(e) Give an example where G is infinite and the result of (d) fails. Justify that it indeed fails. **[4]**

 Question 2 continued overleaf

2. Let $n \geq 4$ and put $A = \{1, 2, \ldots, n\}$. Let V be a complex vector space of dimension $n(n-1)/2$ and with basis $\{v_{ab} \mid a, b \in A,\ a < b\}$. We also write $v_{ba} = v_{ab}$.

The symmetric group $G = S_n$ acts on V by putting $g\,v_{a,b} = v_{ga,gb}$ for all $g \in G$. This makes V into a $\mathbb{C}G$-module whose character will be written χ. The linear map $V \to V\colon v \mapsto gv$ is denoted by t_g.

For $a, b, c, d \in A$ with $a \neq b,\ c \neq d$ put

$$M(a,b,c,d) = \left\{ g \in S_n \,\middle|\, gv_{ab} = v_{ab},\ gv_{cd} = v_{cd} \right\}$$

and $m(a,b,c,d) = \#M(a,b,c,d)$.

(a) Define $(\chi, \chi)_G$. Calculate the trace of t_h if $n = 8$ and $h = (158)(24)$. **[4]**

(b) Prove directly from the definitions **[6]**

$$4 \cdot n! \cdot (\chi, \chi)_G = \sum_{\substack{a,b,c,d \in A \\ a \neq b,\ c \neq d}} m(a,b,c,d).$$

(c) Assume that $a, b, c, d \in A$ are distinct. Prove **[4]**

$$m(a,b,c,d) = 4(n-4)!.$$

In the following you may also use

$$m(a,b,a,b) = 2(n-2)!,$$
$$m(a,b,a,c) = (n-3)!$$

without proving them.

(d) Use the foregoing to prove $(\chi, \chi)_G = 3$. **[7]**

(e) Prove that there are distinct irreducible characters χ_1, χ_2, χ_3 of S_n such that $\chi = \chi_1 + \chi_2 + \chi_3$. State any results that you use. **[4]**

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3. We define

$$G = \langle x, y, z \mid x^2 = z,\ y^2 = z,\ (xy)^2 = z,\ z^2 = 1 \rangle.$$

We consider x, y, z as elements of G. We define $(g_1, \ldots, g_5) = (1, x, y, xy, z)$.

You may use without proof that there exists a unique representation ρ of G such that $\rho(x) = A$, $\rho(y) = B$ where we write

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Hint: Throughout this question, long calculations can be avoided.

(a) Prove that there are precisely 4 linear characters of G, written $\chi_1, \dots, \chi_4$. Clearly state any results that you use. Give a table with the values $\chi_i(g_j)$ whenever $1 \le i \le 4$, $1 \le j \le 5$. **[6]**

(b) Prove that ρ is irreducible. **[5]**

Hint: Proceed directly from the definition of irreducibility.

(c) Prove that $\langle z \rangle$ is a normal subgroup of G. Give an explicit presentation of the quotient group $G/\langle z \rangle$. **[4]**

(d) Prove $\#G = 8$. **[5]**

(e) Prove that $g_1, \dots, g_5$ is a maximal set of pairwise non-conjugate elements of G. In other words, G is the disjoint union $g_1^G \sqcup \cdots \sqcup g_5^G$. Give the character table of G. Clearly state any results that you use. **[5]**

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4. Let G be a finite group and H a subgroup of G. Recall that induction takes a class function q of H to q^G defined by

$$q^G(x) = \frac{1}{\#H} \sum_{g \in G} q(gxg^{-1}) \left[gxg^{-1} \in H\right]$$

where for any assertion P we write

$$\left[P\right] = \begin{cases} 1 & \text{if } P \text{ is true,} \\ 0 & \text{if } P \text{ is false.} \end{cases}$$

(a) Let $p \in \mathrm{CF}(G)$ and $q \in \mathrm{CF}(H)$. Prove $(p_H, q)_H = (p, q^G)_G$. **[8]**

(b) Let $H \le K \le G$ be finite groups and p a class function on H. Prove $(p^K)^G = p^G$. **[6]**

(c) Let R_K denote the regular character of a finite group K. Prove: $(R_H)^G = R_G$. State any results that you use. **[5]**

(d) Let k, m, n be integers with $1 \le m \le n$ and $1 \le k \le n$. Let $G = S_n$ be the symmetric group on $\{1, \dots, n\}$. Let $H = S_m$ be the subgroup of S_n of those elements preserving $\{m+1, \dots, n\}$ pointwise. Let $g \in G$ be a k-cycle. Let 1_H be the trivial linear character of H. Calculate $(1_H)^G(g)$. State any results that you use. **[6]**

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5. Let G be a finite group and $g \in G$. For any representation σ of G write

$$T(\sigma) = \sum_{h \in g^G} \sigma(h).$$

Let ρ be an irreducible representation of G and write $\chi = \chi_\rho$, $n = \chi(1)$.

 Question 5 continued overleaf

(a) State without proof Schur's lemma. **[2]**

(b) Prove that $T(\sigma)$ is an intertwiner $\sigma \to \sigma$. **[4]**

(c) Put $\alpha = \dfrac{\chi(g)}{\chi(1)} \#g^G$. Prove $T(\rho) = \alpha \cdot I_n$ where I_n is the identity matrix. **[6]**

(d) Prove that $\dfrac{\chi(g)}{\chi(1)} \#g^G$ is in the set $\mathbb{I}$ of algebraic integers. **[6]**

(e) Prove that $\chi(1)$ divides $\#G$. Clearly state any results that you use (except those in the following hint). **[7]**

Hint: Use row orthogonality for χ and itself. Use that $\mathbb{Z} = \mathbb{Q} \cap \mathbb{I}$.
