MA 3E10 GROUPS AND REPRESENTATIONS

THE UNIVERSITY OF WARWICK

THIRD YEAR EXAMINATION: April 2011

MA3E1 GROUPS AND REPRESENTATIONS

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.
If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. Put $G = \langle x, y \mid x^2, y^3, (xy)^3 \rangle$ and consider the elements $a = (12)(34)$ and $b = (123)$ of the alternating group A_4.

(a) Assuming no more than the definition of free groups, state in three lines what $\langle x, y \mid x^2, y^3, (xy)^3 \rangle$ means. **[2]**

(b) Prove that there exists a unique homomorphism $f\colon G \to A_4$ such that $f(x) = a$, $f(y) = b$. Briefly state any theorems you use. **[5]**

(c) Consider the subgroup $H = \langle y \rangle$ of G and the set of cosets **[8]**

$$A = \{H,\ xH,\ yxH,\ y^2xH\}.$$

Prove $xyxH \in A$. Justify that $zC \in A$ for all $z \in \{x, y\}$ and $C \in A$ by writing down, without proof, a table which says, for all $z \in \{x, y\}$ and $C \in A$, which of the four elements of A equals zC.

(d) Prove $gH \in A$ for all $g \in G$. **[5]**

(e) Prove that $f\colon G \to A_4$ is an isomorphism. You may assume that it is surjective. **[5]**

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2. (a) Define "inner product" on a complex vector space V. **[3]**

(b) Let V be a finite-dimensional complex vector space equipped with an inner product $\langle\cdot,\cdot\rangle$. Let $W \subset V$ be a linear subspace. Prove $V = W \oplus W^{\perp}$. **[7]**

(c) Define "$\mathbb{C}G$-module". **[3]**

(d) Let G be a finite group and V a finite-dimensional $\mathbb{C}G$-module. Let W be a $\mathbb{C}G$-submodule of V. Using the fact (which you needn't prove) that there exists a G-invariant inner product on V, prove that there exists a $\mathbb{C}G$-submodule $X \subset V$ such that $V = W \oplus X$. **[6]**

(e) Let G be the free group on x, y. Let ρ be the representation of G defined by **[6]**

$$\rho(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho(y) = \frac{1}{2}\begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}.$$

Suppose that ρ is afforded by (V, A) where V is a $\mathbb{C}G$-module and A is a basis (v_1, v_2) of V. Find an explicit G-invariant inner product $\langle\cdot,\cdot\rangle$ on V for example by giving the 2×2 matrix $(\langle v_i, v_j\rangle)_{ij}$.

You don't need to show how you found your answer but you should prove that it has the required properties.

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3. Let ρ, σ be irreducible representations of a finite group G. Let $E(i,j)$ denote the $(\deg\sigma) \times (\deg\rho)$ matrix with 1 in position (i,j) and zeroes elsewhere. Put

$$T(i,j) := \sum_{g \in G} \sigma(g^{-1}) \cdot E(i,j) \cdot \rho(g).$$

(a) Define "intertwining matrix $\rho \to \sigma$". **[3]**

(b) Prove from first principles that $T(i,j)$ is an intertwiner $\rho \to \sigma$. **[5]**

(c) Prove **[7]**

$$\sum_{i,j} T(i,j)_{ij} = \#G \cdot (\chi_\rho, \chi_\sigma)_G$$

where $D_{k\ell}$ denotes the entry in position (k,ℓ) of a matrix D.

Hint: if A, B, C are matrices such that the product ABC is defined then

$$(ABC)_{ij} = \sum_{s,t} A_{is}\, B_{st}\, C_{tj}. \qquad (1)$$

(d) Assume $\rho \not\sim \sigma$. Prove $(\chi_\rho, \chi_\sigma)_G = 0$. State any results that you use. **[5]**

(e) Let $\mathrm{Rep}_n(G)$ denote the set of equivalence classes of n-dimensional representations of G. Prove that $\mathrm{Rep}_n(G)$ is finite. You may use any results from the lectures if you state them clearly. **[5]**

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4. (a) Let χ be a character of a finite group G and $g \in G$. Prove $|\chi(g)| \le \chi(1)$ clearly stating any results that you use. **[4]**

(b) Let χ be a character of a finite group G. Define $\ker \chi$ to be $\ker \rho$ if ρ is a representation of G with character χ. Prove **[7]**

$$\ker \chi = \{g \in G \mid \chi(g) = \chi(1)\}$$

clearly stating any results that you use.

(c) Let $H \le G$ be finite groups. Let p a nonzero character of H and recall **[8]**

$$p^G(g) = \frac{1}{\#H} \sum_{x \in G} [xgx^{-1} \in H]\, p(xgx^{-1})$$

where

$$[P] = \begin{cases} 1 & \text{if } P \text{ is true,} \\ 0 & \text{if } P \text{ is false.} \end{cases}$$

Prove that

$$\ker(p^G) = \bigcap_{x \in G} x^{-1} (\ker p) x.$$

Where does your proof break down if $p = 0$?

(d) Let $H \le G$ be finite groups. Let $p \in \mathrm{CF}(H)$ and $q \in \mathrm{CF}(G)$ be class functions. Prove that $(p\, q_H)^G = p^G q$ where multiplication is pointwise: $(fg)(x) := f(x)\, g(x)$. **[6]**

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5. We define the group $G = \langle a, b \mid a^3, b^7, a\, b\, a^{-1}\, b^{-2} \rangle$. Then:

Every element of G can uniquely be written in the form $a^k\, b^\ell$ with $0 \le k < 3$ and $0 \le \ell < 7$. (2)

There exists a unique homomorphism $f\colon G \to C_3 := \langle c \mid c^3 \rangle$ such that $f(a) = c$, $f(b) = 1$. (3)

You may assume these without proof.

(a) Assuming no more than the definition of "group" define "conjugacy class". **[3]**

(b) Put **[8]**

$$D_1 = \{1\}, \qquad D_2 = \{b, b^2, b^4\}, \qquad D_3 = \{b^3, b^5, b^6\},$$
$$D_4 = \{a\, b^\ell \mid \ell \in \mathbb{Z}\}, \qquad D_5 = \{a^2\, b^\ell \mid \ell \in \mathbb{Z}\}.$$

Prove that $D_1, \ldots, D_5$ are the conjugacy classes of G.

(c) Find three linear characters χ_1, χ_2, χ_3 of G explicitly. **[3]**

(d) Write $H = \langle b \rangle \leq G$ and $\varepsilon = \exp(2\pi i/7)$. Let λ, μ be the linear characters of H defined by $\lambda(b^\ell) = \varepsilon^\ell$ and $\mu(b^\ell) = \varepsilon^{3\ell}$ for all ℓ. Calculate the induced characters $\chi_4 := \lambda^G$ and $\chi_5 := \mu^G$. These should be expressed in terms of ε. The definition of induced characters can be found in question 4(c). **[6]**

(e) Prove that χ_4 and χ_5 are irreducible. You don't need to prove that they are distinct. **[5]**
