

choosing i with $(c_i, c_{i+p}) = (1, 0)$ and then replacing (c_i, c_{i+p}) with $(0, 1)$.

- (c) Show that if λ has a hook of size divisible by p , then it has a p -hook.
- (d) Start with a partition λ and continually remove border strips of size p until unable to do so. Show that the partition μ that remains is independent of the border strips chosen to be removed. The partition μ is called the p -core of λ , and a partition with no p -hooks is called a p -core.
- (e) Let μ be a p -core. Let $Y_{p,\mu}$ be the set of all partitions whose p -core is μ . Define $\lambda \leq \nu$ in $Y_{p,\mu}$ if λ can be obtained from ν by removing border strips of size p . Show that $Y_{p,\mu} \cong Y^p$ where $Y = Y_{1,\emptyset}$ is the set of partitions, ordered by containment. Deduce that if $f_\mu(n)$ is the number of partitions of n with p -core μ , then

$$\sum_{n \geq 0} f_\mu(n) q^n = q^{|\mu|} \prod_{n \geq 1} (1 - q^{pn})^{-p}.$$

- (f) Let $n \in \mathbb{P}$. Show that the following three numbers are equal.
 - (1). The number of p -cores of size n .
 - (2). The number of solutions $(k_1, \dots, k_{p-1}) \in \mathbb{N}^{p-1}$ to

$$\left[\sum_{i=1}^{p-1} i k_i - p \binom{k_i}{2} \right] - \binom{k_1 + \dots + k_{p-1}}{2} = n.$$

- (3). The coefficient of q^n in $\prod_{n \geq 1} \frac{(1 - q^{pn})^p}{(1 - q^n)}$.

- (g) When $p = 2$, find all partitions satisfying (1) explicitly. What identity results from the equality of (1) and (3)?

- (25) Let $H \subset G$ be finite groups. Let V be a (left) H -module and define

$$W := \{p: G \rightarrow V \mid p(xy) = x \cdot p(y) \text{ for all } x \in H, y \in G\}.$$

- (a) Prove that $\dim W = [G : H] \dim V$.
- (b) For $p \in W, g \in G$, define $gp = q: G \rightarrow V$ by $q(x) = p(xg)$. Prove that $qp \in W$.
- (c) Prove that the map $G \times W \rightarrow W, (g, p) \mapsto gp$ makes W into a left G -module.
- (d) Define $G \times V \rightarrow W, (g, v) \mapsto g \boxtimes v$ by $(g \boxtimes v)(x) = xg(v)$ for all $x \in G$. We extend this by linearity to $KG \times V \rightarrow W$. Prove that the pair (W, \boxtimes) is a tensor product $KG \otimes_{KH} V$. So W is the induced module $V \uparrow G$.
- (e) Prove Frobenius reciprocity.

- (26) In this exercise you will prove the *branching rules* for the symmetric group.
- (a). Let $\lambda \vdash n$. Prove:

$$\chi^\lambda \uparrow S_{n+1} = \sum_{\lambda \subseteq \mu \vdash n+1} \chi^\mu$$

where we use the usual inclusion $S_n = S_n \times S_1 \subset S_{n+1}$. Hint: Use Pieri's rule and the fact that the Frobenius map is a ring homomorphism.

(b). Let $\mu \vdash n + 1$. Prove

$$\chi^\mu \downarrow S_{n+1} = \sum_{\mu \supseteq \lambda \vdash n} \chi^\lambda.$$

(27) Let $\mathbb{A}_n = \mathbb{A}$ be the alternating S_n -module and let $\chi_{\mathbb{A}}$ be its character. Recall the involution ω of Λ . We define a linear map denoted by the same letter $\omega: \mathbb{C}F^n \rightarrow \mathbb{C}F^n$ by $\omega(f) = \chi_{\mathbb{A}} f$ (pointwise multiplication). In particular, ω takes a character χ_V to $\chi_{V \otimes \mathbb{A}_n}$.

(a) Prove that $\omega(p_\mu) = \chi_{\mathbb{A}}(\mu) p_\mu$ for all $\mu \vdash n$.

(b) Prove that the isomorphism $\text{ch}: \mathbb{C}F \rightarrow \Lambda$ commutes with the ω maps, that is, $\text{ch}(\omega f) = \omega(\text{ch} f)$ for all $f \in \mathbb{C}F^n$.

(c) Prove that $\omega: \mathbb{C}F \rightarrow \mathbb{C}F$ is a ring isomorphism.

(d) Prove that $\omega(\chi^\lambda) = \chi^{\lambda'}$ for all $\lambda \in \text{Par}$.

(28) Prove that the elements v_T , as T varies over the standard tableaux on λ , form a basis for the Specht module S^λ . Hint: prove independence directly; then use the identity $\sum_{\lambda} (f^\lambda)^2 = n!$.

(29) Prove the claim in the proof of theorem 38 on characters of $\text{GL}(V)$ that $\text{tr}(w, A) = p_{\rho(w)}(\theta)$.

(30) Let V, W be finite-dimensional vector spaces. Prove that there exists an isomorphism of $\text{GL}(V) \times \text{GL}(W)$ -modules

$$F^\lambda(V \otimes W) \cong \bigoplus_{\mu, \nu} c_{\mu\nu}^\lambda F^\mu(V) \otimes F^\nu(W).$$

(31) Let V be a finite-dimensional vector space and $\mu = (\mu_1, \mu_2, \dots)$ a partition. Prove that there exists an isomorphism of $\text{GL}(V)$ -modules

$$\bigotimes_{i \geq 1} \left(\bigwedge^{\mu_i} V \right) \cong \bigoplus_{\lambda} K_{\lambda\mu} F^{\lambda'}(V).$$

(32) Let V be an n -dimensional vector space and put $G = \text{GL}(V)$. For $k \in \mathbb{Z}$, let $\mathbb{D}^{\otimes k}$ be the k -th power of the determinant, that is, the 1-dimensional G -module $A \mapsto \det(A)^k$. Note that $\mathbb{D}^{\otimes k} \otimes \mathbb{D}^{\otimes \ell} \cong \mathbb{D}^{\otimes k+\ell}$. Write \mathbb{D} instead of $\mathbb{D}^{\otimes 1}$.

(a) Prove $\mathbb{D} \cong F^{(1^n)}(V)$ (isomorphism of G -modules).

(b) Prove that $F^\lambda(V) \otimes \mathbb{D} \cong F^\mu(V)$ whenever $\lambda \in \text{Par}$, $\ell(\lambda) \leq n$ and μ is obtained from λ by adding one column of size n .

(c) If $\ell(\lambda) < n$ and $k \in \mathbb{Z}$ then $F^\lambda(V) \otimes \mathbb{D}^{\otimes k}$ is an irreducible rational G -module. Conversely, every irreducible rational G -module is isomorphic to precisely one of them.