

MA5P2 The Symmetric Group – Exercises

October 30, 2008

(1) Prove: $\sum_{\lambda \in \text{Par}} q^{|\lambda|} = \prod_{n \geq 1} (1 - q^n)^{-1}$.

(2) Prove Proposition 7.5.3 on page 296.

(3) Prove

$$\sum_{\mu \vdash n} q^{\ell(\mu)-1} m_\mu = \sum_{j=0}^{n-1} (q-1)^j s_{n-j, 1^j}.$$

Here, $(n-j, 1^j)$ is the *hook* partition, for example

$$(5, 1^3) = \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & & & & \\ \hline \square & & & & \\ \hline \square & & & & \\ \hline \end{array}.$$

(4) Prove the following lemma on inverse bumping, used in the proof of Theorem 7.11.5 on page 319. **Lemma.** Let Q be a tableau of shape μ . Let λ be a partition such that $\lambda \subset \mu$ and $|\mu/\lambda| = 1$. Then there exists a unique tableau P of shape λ and a unique $k > 0$ such that $Q = P \leftarrow k$.

(5) Let $d_\lambda(k)$ denote the number of tableaux of shape λ whose entries are taken from $\{1, \dots, k\}$.

(a) Prove: $k^n = \sum_{\lambda \vdash n} d_\lambda(k) f^\lambda$.

(b) Prove: $\binom{n+k\ell-1}{n} = \sum_{\lambda \vdash n} d_\lambda(k) d_\lambda(\ell)$.

(6) (a). Lemma 7.11.2 compares the insertion paths $I(P, j)$ and $I(P \leftarrow j, k)$ if $j \leq k$. State and prove a similar result if $j > k$.

(b). Every permutation $w = w_1 \cdots w_n$ gives rise to an *up-down sequence*, a sequence of $n-1$ signs, the i -th one being $+$ if $w_i < w_{i+1}$ and $-$ otherwise. Prove that if $w \xrightarrow{\text{RSK}} (P, Q)$ then the up-down sequence of w is determined by Q .

(7) Fill in the details of the proof of Corollary 7.12.3 on page 323.

(8) Prove the following claims in the proof of Theorem 7.13.1, all made on page 328.

(a) $|\mu/\lambda| = 1$ in case (L2).

(b) $\lambda = \mu = \nu$ in case (L4).

(c) If a corner p is labelled by λ then $|\lambda|$ is the number of X 's to the left and below p .

(9) On the last line of page 324 the book claims "... [in order to prove Theorem 7.13.1] we may assume u and v have no repeated elements". Justify this in full detail.

- (10) In the proof of Theorem 7.13.5 we claimed that $T(i, j) = U(i, j)$. Prove this. ($U(i, j)$ is our notation for what the book writes as $((\emptyset \leftarrow i_1) \leftarrow i_2) \cdots i_k$).
- (11) (a). Let $w \in S_n$ be an involution and let $\nu(i, j)$ be the partition at the (i, j) corner of the growth diagram of w (so $\nu(i, j) = \nu(j, i)$). Prove that, for $0 \leq i \leq n$, the number of columns of $\nu(i, i)$ of odd length is the number of fixed points k of w satisfying $k \leq i$.
- (b). Suppose that $A \xrightarrow{\text{RSK}} (P, Q)$ and that A is symmetric (so $P = Q$). Show that $\text{tr}(A)$ is the number of columns of P of odd length.
- (c). Prove

$$\prod_i (1 - qx_i)^{-1} \prod_{i < j} (1 - x_i x_j)^{-1} = \sum_{\lambda} q^{c(\lambda)} s_{\lambda}(x) \quad (1)$$

where $c(\lambda)$ denotes the number of odd parts of λ' .

(d). Deduce that

$$\prod_{i < j} (1 - x_i x_j)^{-1} = \sum_{\mu} s_{(2\mu)'}(x)$$

where $2\mu = (2\mu_1, 2\mu_2, \dots)$.

(e). Fix $k \geq 0$. Evaluate the sum $a(n, k) = \sum_{\lambda} f^{\lambda}$ where λ ranges over all partitions of n of k odd parts (and any number of even parts).

(f). What identity results when we apply ω to (1)?

- (12) Let a_k denote the number of tableaux whose entries sum to k . Prove:

$$\sum_{k \geq 0} a_k q^k = \prod_{0 < i} (1 - q^{2i-1})^{-i} (1 - q^{2i})^{-i}.$$

- (13) Let $\ell(\lambda) \leq m$ and $\ell(\lambda') \leq n$. Define

$$\bar{\lambda} := (n - \lambda_m, n - \lambda_{m-1}, \dots, n - \lambda_1). \quad (2)$$

Use the classical definition of the Schur function to prove that

$$(x_1 x_2 \cdots x_m)^n s_{\lambda}(x_1^{-1}, \dots, x_m^{-1}) = s_{\bar{\lambda}}(x_1, \dots, x_m). \quad (3)$$

- (14) Prove that

$$\prod_{i=1}^m \prod_{j=1}^n (x_i + y_j) = \sum_{\lambda} s_{\lambda}(x) s_{\bar{\lambda}}(y)$$

summed over all partitions λ with $\ell(\lambda) \leq m$ and $\ell(\lambda') \leq n$ and where $\bar{\lambda}$ is defined by (2).

- (15) For $m \geq n \geq 0$, let $s(m, n)$ denote the Schur function of the partition $(m, n) = (m, n, 0, 0, \dots)$ of at most two parts. Use Pieri's rule to prove $s(m, n) = s_m s_n - s_{m+1} s_{n-1}$.

- (16) For $1 \leq m \leq n$, let $s\langle 1^{n-m}, m \rangle$ denote the Schur function of the hook partition $\langle 1^{n-m}, m \rangle$ of m boxes in the first row, then $n - m$ rows of one box each. Use Pieri's rule to express $s\langle 1^{n-m}, m \rangle$ as a linear combination of $\{e_i h_j \mid i, j \in \mathbb{N}\}$.

(17) Prove corollary 7.15.9: $s_{\lambda/n} = \sum_{\nu} s_{\nu}$ where the sum is over all partitions $\nu \subseteq \lambda$ such that λ/ν is a horizontal strip of size n .

(18) (a). Prove that

$$s_{\lambda}(1, q, q^2, \dots, q^{m-1}) = q^r \prod_{1 \leq i < j \leq m} \frac{q^{\lambda_i - \lambda_j + j - i} - 1}{q^{j-i} - 1}$$

where $r = \sum_i (i-1)\lambda_i$.

(b). Prove that

$$s_{\lambda}(1^m) = \prod_{1 \leq i < j \leq m} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

(19) In this exercise we prove the hook length formula which is a formula for f^{λ} , the number of standard tableaux of n boxes.

Let $\lambda \vdash n$, $\ell(\lambda) \leq k$. Write $\ell_i := \lambda_i + k - i$. Define

$$\Delta(x_1, \dots, x_k) := \prod_{1 \leq i < j \leq k} (x_i - x_j), \quad F(\ell_1, \dots, \ell_k) := \frac{n! \Delta(\ell_1, \dots, \ell_k)}{\ell_1! \cdots \ell_k!}.$$

(a). Prove:

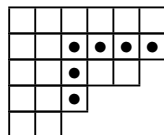
$$\sum_{i=1}^k x_i \Delta(x_1, \dots, x_i + t, \dots, x_k) = \left[x_1 + \cdots + x_k + \binom{k}{2} t \right] \Delta(x_1, \dots, x_k). \quad (4)$$

(b). The numbers $f(\lambda) = f^{\lambda}$ satisfy a recurrence formula

$$f(\lambda_1, \dots, \lambda_k) = \sum_{i=1}^k f(\lambda_1, \dots, \lambda_i - 1, \dots, \lambda_k) \quad (5)$$

where it is understood that $f(\mu_1, \dots, \mu_k)$ is zero if (μ_1, \dots, μ_k) is not a partition, that is, if $\mu_i < \mu_{i+1}$ or $\mu_i < 0$ for some i . Prove that $f^{\lambda} = F(\ell_1, \dots, \ell_k)$, by showing that the right hand side satisfies a recurrence analogous to (5).

(c). By $(i, j) \in \lambda$ we mean that (i, j) is a box in the diagram of λ , that is, $1 \leq j \leq \lambda_i$. For $(i, j) \in \lambda$, let the *hook length* $h(i, j; \lambda)$ be the number of $(u, v) \in \lambda$ in the same row or column as (i, j) and such that $u + v \geq i + j$:



$$h(2, 3; \lambda) = 6.$$

Prove the *hook length formula*:

$$f^{\lambda} = n! \prod_{(i, j) \in \lambda} h(i, j; \lambda)^{-1}. \quad (6)$$