RACKS AND LINKS IN CODIMENSION TWO

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ABSTRACT

A rack, which is the algebraic distillation of two of the Reidemeister moves, is a set with a binary operation such that right multiplication is an automorphism. Any codimension two link has a fundamental rack which contains more information than the fundamental group. Racks provide an elegant and complete algebraic framework in which to study links and knots in 3–manifolds, and also for the 3–manifolds themselves. Racks have been studied by several previous authors and have been called a variety of names. In this first paper of a series we consolidate the algebra of racks and show that the fundamental rack is a complete invariant for irreducible framed links in a 3–manifold and for the 3–manifold itself. We give some examples of computable link invariants derived from the fundamental rack and explain the connection of the theory of racks with that of braids.

Keywords: Racks, knots, links, 3–manifolds, invariants, braids, quandle, crystal, codimension two embeddings, self-distributive axiom, automorphic set.

This is the first of a series of papers by the authors. More papers, some in collaboration with Brian Sanderson, are in preparation. In these papers we shall study a natural algebraic theory, strongly connected with the theories of groups, group presentations and crossed modules. This is the theory of **racks**. A **rack** is a set with a binary operation satisfying two simple laws which are the algebraic distillation of two of the Reidemeister moves (the Ω_2 and Ω_3 moves). Racks have been variously studied by previous authors under a variety of names (including rack) and using a variety of different notations and terminology. We shall give a summary of this previous work shortly. One of the aims of this paper is to attempt to establish a uniform set of conventions for notation and terminology in this subject.

Racks provide an elegant and complete algebraic framework in which to study links and knots in 3–manifolds, and also for the 3–manifolds themselves. Included in this framework are complete algebraic invariants for both framed links in 3–manifolds and for 3–manifolds. The theory of racks opens the practical possibility of finding a complete sequence of computable invariants for framed links and for 3–manifolds.

A codimension two submanifold has a **fundamental rack**, which is a complete invariant for irreducible links in any 3–manifold. Moreover there is a notion of a Grothendieck-style K–theory invariant which we call the **Goeritz equivalence class** of the rack, and this invariant, applied to the fundamental rack of the framed link in S^3 which is used to specify a 3–manifold by surgery, is a complete invariant for the 3–manifold. We consider this in a later paper.

In this first paper of the series we shall review and extend some of the basic algebra of racks and give the complete invariant for framed links and some examples of computable link invariants derived from this invariant. In future papers we shall examine invariants in greater detail and shall combine the results of this paper with the Kirby calculus (Kirby [19], Fenn-Rourke [9]) to construct the complete invariant for 3–manifolds mentioned above. In another paper of the series [11] (joint work with Brian Sanderson) we shall construct the **space** of a rack which classifies cobordism classes of link representations on the rack, and which has strong connections with classical cobordism theories. For an introduction to the rack space see [10].

Previous work

The earliest work on racks (known to us) is due to Conway and Wraith [5], and we are indebted to these authors for a copy of their (unpublished) correspondence. They used the name **wrack** for the concept and we have adopted this name, not merely because it is the oldest name, but also because it is a simple English word which (to our knowledge) has no other mathematical meaning. We have however chosen the more common spelling. Rack is used in the same sense as in the phrase "rack and ruin". The context of Conway and Wraith's work is the conjugacy operation in a group and they regarded a rack as the wreckage of a group left behind after the group operation is discarded and only the notion of conjugacy remains. They studied the basic algebra of racks in a special case (the quandle case) but also were aware of the general case and the main topological application (the fundamental rack of a knot in a 3–manifold).

The most comprehensive published study of racks in a toplogical context is due to Joyce [16]. He studied a particular special case and used the name **quandle**. Because his work has been widely quoted and the word quandle is now well known, we have with some reservations continued to use the name quandle for this special case. Joyce establishes the basic algebra of quandles, giving several examples, and defines augmented quandles and the associated group (which he calls *Adconj*). He defines the fundamental quandle of a knot in S^3 giving both the topological definition in terms of "nooses" and the definition in terms of the presentation which can be read from a diagram of the knot. He proves the equivalence of the two definitions. His main result is that the fundamental quandle classifies the knot. Joyce's work was largely duplicated independently by Matveev [24] who used the name **distributive groupoid**. But note that a rack is not a groupoid in the usually accepted sense.

Kauffman [17] defines racks in full generality using the name **crystal** for the concept. We have not adopted this name because of the historical precedence of the name rack and because of the strong connection of the theory with groups in which context crystal might suggest crystallographic groups. Kauffman defines the fundamental rack of a knot in S^3 and applies Joyce's theorem to prove that it is a classifying invariant. He also extends Joyce's work on the Alexander quandle to racks and defines an associated R–matrix.

The most extensive algebraic survey of racks is given by Brieskorn [3], who is particularly interested in the context of braids and singularities. In his introduction Brieskorn writes:

> *"Whilst preparing this survey, I found an extremely simple concept unifying many investigations on this subject as well as classical results of E.Artin, A.Hurwitz and W.Magnus. This is the notion of an automorphic set."*

The definition of an automorphic set coincides with that of a rack or crystal. Although "automorphic set" is the mathematically correct terminology for the concept, we have not adopted it because it is too unwieldy in context. A short single word makes phrases such as augmented rack, quandle rack, fundamental rack usable. Brieskorn's paper contains a wealth of algebraic material about racks, most of which is not relevant to the topological context of our work, except for the connection with braid groups, which we shall examine in section 7.

Winker [31] extends Joyce's work and defines an analogue of the Cayley graph for a quandle and Krüger [20], independently and simultaneously with our work, defines free products of racks and investigates the automorphism group of the free rack, cf. our section 7 and appendix. There is an interesting connection of racks with computer theory. Roscoe [27] studies an algebraic object which satisfies just one of the two rack laws (the rack identity) in the context of computer information updates. Finally there is also a connection with some problems in logic, see Dehorney, Jech and Laver [7,14,22].

The content of this paper

This paper reviews and consolidates much of the previous work on the subject and also contains many new results and new formulations of old results. An outline detailing these new results and the previous work now follows.

Section 1 contains the basic definition and some of the examples of racks that we shall need. Most of these examples come from Joyce [16] and Brieskorn [3]. The new material consists of a careful treatment of the operator group (which is in general distinct from the associated group) and new notation (exponential notation) for the rack operation, which to our knowledge, has not been introduced in any previous work. This notation makes the algebra very easy to handle, for example the "fully left associated" form of a repeated operation proved by Winker and Kauffman is almost an observation using our notation.

Section 2 contains a review of some of the basic algebra of racks. Most of this section is a reworking for racks of algebra in Joyce's paper. However the section also includes the new concept of the free product of two racks and the connection with crossed modules (the associated crossed module). Further basic algebra can be found in Brieskorn [3] and in Ryder's Ph.D. thesis [28].

In section 3 we consider the fundamental rack of a codimension 2 embedding and its properties. This material contains a reworking in the rack context of material of Matveev and Joyce. It also contains several new results including the identification of the associated crossed module as the relative second homotopy group and a calculation of the operator group for the fundamental rack of a classical link.

Section 4 is about presentations; we prove a new "Tietze" theorem for rack presentations and, in a reworking of material of Joyce and Kauffman, show how the fundamental rack of a classical link (in S^3) has a finite presentation which can be read in a natural way from the diagram. This section also contains new material on presentations of augmented racks and the new result that the fundamental rack of a link in a homotopy 3–sphere also has a finite presentation.

Section 5 contains the main classification theorem which is a generalisation to arbitrary 3–manifolds of the results of Joyce and Kauffman. This section also contains an interesting new result on homotopy 3–spheres: the fundamental rack classifies both the link and the homotopy 3–sphere. This result opens the possibility of using a rack invariant to detect a homotopy 3–sphere. The idea of using rack invariants (of which there are myriad examples) as 3–manifold invariants will be explored further in a later paper.

In section 6 we make a start on the invariants that can be read from the fundamental rack; this latter subject will be explored more fully in subsequent papers, and see also the theses of Devine [8], Azcan [1], Kelly [18], Lambropoulou [21] and Ryder [28]. New material in this section includes the invariants derived from the (t, s)–rack and matrix racks (examples 6 and 7 of 6.1 and example 3 of 6.3).

Finally in section 7 we explain the connection of the theory of racks with that of braids and in an appendix we give the analogue of Nielsen theory for automorphisms of the free rack. These results allow us to give a criterion for a rack to be a **classical** rack (i.e. isomorphic to the fundamental rack of a framed link in S^3). This has strong connections both with the Poincaré conjecture and the homeomorphism problem for S^3, see the remarks at the end of the paper. The material in this section is largely new although there is a strong connection with Brieskorn's work in our material on invariants.

1. Definitions and Examples

We consider sets X with a binary operation which we shall write exponentially

$$(a, b) \mapsto a^b.$$

There are several reasons for writing the operation exponentially.

(1) The operation is unbalanced and should be thought of as an action, i.e. think of a^b as meaning *the result of* b *acting or operating on* a.

(2) In group contexts exponentiation signifies conjugation. A group with conjugation is one of the principal examples of a rack — indeed this was the source for one strand of the earlier work on racks [5]. A rack is an algebraic object which has just *some* of the properties of a group with conjugacy as the operation.

(3) Finally, and most conveniently, exponential notation allows brackets to be dispensed with, because there are standard conventions for association with exponents. In particular

$$a^{bc} \text{ means } (a^b)^c \text{ and } a^{b^c} \text{ means } a^{(b^c)}.$$

1.1 Definition *Racks*

A **rack** is a non-empty set X with a binary operation satisfying the following two axioms:

Axiom 1 Given $a, b \in X$ there is a unique $c \in X$ such that $a = c^b$.

Axiom 2 Given $a, b, c \in X$ the formula

$$a^{bc} = a^{cb^c}$$

holds. We call this formula the **rack identity (first form).**

Several consequences flow from these axioms.

The first axiom implies that, for each $b \in X$ the function $f_b(x) := x^b$ is a bijection of X to itself, and this fits with the idea that the operation is a (right) action of X on itself.

We shall write $a^{\overline{b}} = f_b^{-1}(a)$ for the element c given by axiom 1, but notice that $a^{\overline{b}}$ is *a single symbol for an element of* X. It is not suggested that $\overline{b}$ is itself an element of X; however the notation is suggestive (and intended to be) because now $a^{b\overline{b}} = a^{\overline{b}b} = a$ for all $a, b \in X$. Thus if we identify $\overline{b}$ with b^{-1} then we can give a meaning to any expression of the form x^w where $w = w(a, b, \ldots)$ is a word in $F(X)$ the free group on X, namely the result of repeatedly acting on x by $f_a, f_a^{-1}, f_b, f_b^{-1}$ etc. The word w is again *not to be regarded as an element of* X, but as an **operator** on X. Shortly, we shall formalise this by introducing the **operator group**.

The rack identity is a right self-distributive law as can be seen if we temporarily use the notation $a \cdot b$ for a^b:

$$(a \cdot b) \cdot c = (a \cdot c) \cdot (b \cdot c).$$

Both axioms together are equivalent to the statement that *right multiplication is an automorphism.* The rack identity can be restated in more elegant and mnemonic form if we use the notation introduced above.

Substituting $d = a^c$ in the rack identity and then changing d back to a gives the alternative form:

Axiom 2′ Given $a, b, c \in X$ the formula

$$a^{b^c} = a^{\overline{c}bc}$$

holds. This is the **rack identity (second form).**

In other words b^c operates like $\overline{c}bc$, which makes clear the connection between the rack operation and conjugacy in a group.

The Operator Group

In expressions such as $a^{b\overline{c}}$ we refer to a as being at **primary** level and $b, \overline{c}$ as at **operator** level. The second form of the rack identity makes clear that we do not need any "higher" level operators. Expressions involving repeated operations can always be resolved into one of the form a^w where $a \in X$ is at the primary level and w, lying in the free group $F(X)$ on X, is at the operator level.

In this way we have an action by the group $F(X)$ on X. In general if G acts on X, written $(a, g) \mapsto a \cdot g$ and if $\partial : X \to G$ is a map satisfying $\partial(a \cdot g) = g^{-1}\partial(a)g$ then X has the structure of a rack given by $a^b := a \cdot \partial(b)$. In many situations this is the most convenient method of describing the rack operation. The similarity with crossed modules should be clear.

We shall pursue the notion of a rack with a group G operating in section 2, when we introduce the formal notion of an augmented rack. We shall then be able to formalise the connection with crossed modules.

To make operators precise we define **operator equivalence** by:

$$w \equiv z \iff a^w = a^z \text{ for all } a \in X$$

where $w, z \in F(X)$.

The equivalence classes form the **Operator Group** $Op(X)$ which could also be defined as $F(X)/N$ where N is the normal subgroup

$$N = \{w \in F(X) \mid w \equiv 1\}.$$

1.2 Examples of operator equivalence

Since $b^{a^a} = b^{\overline{a}aa}$ (by the rack identity) $= b^a$ for all $a, b \in X$. We have

$$a^a \equiv a \text{ for all } a \in X.$$

More generally if a^{a^n} means $a^{aa\ldots a}$ (n repeats) then $a \equiv a^{a^n}$.

In terms of operator equivalence, the rack identity can again be restated:

Axiom 2″ Given $a, b \in X$ we have

$$a^b \equiv \bar{b}ab.$$

This is the **rack identity (third form).**

Orbits and stabilizers

We can now see that a rack is a set X with an action of $F(X)$ (or its quotient $Op(X)$) on X satisfying the rack identity. In section 2 we shall see that there is another group naturally associated to a rack, lying between $F(X)$ and $Op(X)$, called the **associated group**, which therefore also acts on X. The associated group is particularly important because it has a universal property not shared by either $F(X)$ or $Op(X)$.

Since X is a set with a group action we can use all the language of group actions in the context of racks. In particular X splits into disjoint **orbits** and each element has a **stabilizer** (in $F(X)$ or $Op(X)$) associated with it.

1.3 Examples of Racks

Example 1 *The Conjugation Rack*

Let G be a group, then conjugation in G i.e. $g^h := h^{-1}gh$ defines a rack operation on G. This makes G into the **conjugation rack** written $\text{conj}(G)$ or alternatively G_{conj}.

The operator group in this rack is the group of inner automorphisms of G and the orbits are the conjugacy classes. Given $g, h \in G$ then $g \equiv h$ if and only if gh^{-1} is in the centre of G.

Example 2 *The Dihedral Rack*

Any union of conjugacy classes in a group forms a rack with conjugation as operation. In particular let R_n be the set of reflections in the dihedral group D_{2n} of order $2n$ (which we regard as the symmetry group of the regular n–gon). Then R_n forms a rack of order n, with operator group D_{2n}, called the **dihedral** rack of order n.

Example 3 *The Core Rack*

The rule $g^h := hg^{-1}h$ also defines a rack operation in a group G called the **core rack**, $\text{core}(G)$, cf. Joyce [16].

Great care is needed working with this rack because composition in the operator group does not correspond to composition in G (g^{hj} has two meanings according as the product hj is taken in G or the operator group).

This is an example of an **involutive rack**: where $a^2 \equiv 1$ for all $a \in X$, since:

$$g^{h^2} = h(hg^{-1}h)^{-1}h = g, \text{ for all } g \in G,$$

here h^2 means composition in the operator group and the other products are taken in G.

Example 4 *The Reflection Rack*

Let P, Q be points of the plane and define P^Q to be P reflected in Q (i.e. $2Q - P$ in vector notation).

It is elementary to show that this is a rack operation. This example can be generalised by replacing the plane by any geometry with point symmetries satisfying certain general conditions (see Joyce [16] for details). Examples include the natural geometries of S^n and $\mathbb{R}P^n$. Interesting subracks of these latter racks are given by the action of Coxeter groups on root systems, cf. example 10 below.

Example 5 *The Alexander Rack*

Let Λ be the ring of Laurent polynomials $\mathbb{Z}[t, t^{-1}]$ in the variable t. Any $\Lambda-$module M has the structure of a rack with the rule $a^b := ta + (1-t)b$.

For example, letting M be the plane and the action of t multiplication by -1, yields the reflection rack of example 4.

The Quandle Condition

All the above examples have satisfied the identity

$$a^a = a \text{ for all } a \in X,$$

which we call the **quandle** condition. We shall call a rack satisfying the quandle condition a **quandle rack** or **quandle**. The term quandle is due to Joyce [16].

Example 5 can be generalised to yield a non-quandle rack:

Example 6 *The* $(t, s)-$*Rack*

Let Λ_s be the ring $\mathbb{Z}[t, t^{-1}, s]$ modulo the ideal generated by $s(t + s - 1)$. Any Λ_s-module M has the structure of a rack by the rule

$$a^b := ta + sb.$$

This operation satisfies the **Abelian entropy condition**:

$$u^{vw^x} = u^{wv^x}.$$

For explicit representations in terms of matrices see section 6 (6.1 example 6). When s acts like $1 - t$ this rack reverts to the Alexander rack, discussed above.

Example 7 *The Cyclic Rack*

Here is a finite rack which is also not a quandle:

The **cyclic rack of order** n, is given by $C_n = \{0, 1, 2, \ldots n-1\}$, the residues modulo n, with operation $i^j := i + 1 \bmod n$ for all $i, j \in C_n$.

This example can be generalised: Let X be any G–set and choose a fixed element $g \in G$, then $a^b := a \cdot g$ for all $a, b \in X$ defines a rack structure on X.

Example 8 *There are six different isomorphism classes of racks of order 3:*

1. The **trivial rack** $\{a, b, c \mid x^y = x \text{ for all } x, y\}$.
2. The cyclic rack C_3.
3. The dihedral rack R_3.
4. $\{a, b, c \mid f_a = f_b = f_c = (b, c)\}$ where (b, c) means the symmetry which interchanges b and c and leaves a fixed.
5. $\{a, b, c \mid f_b = f_c = (b, c), f_a = \text{id.}\}$
6. $\{a, b, c \mid f_a = (b, c), f_b = f_c = \text{id.}\}$

Classes 1,3,6 are quandles, whilst 2,4,5 are not.

The last example gives some idea of the rich and varied structure of racks as compared with groups, cf. Ryder [28].

Example 9 *The Free Rack*

The **free rack** $\text{FR}(S)$ on a given set S is defined, as a set, to be $S \times F(S)$. We write the pair (a, w) as a^w, i.e.

$$\text{FR}(S) = \{a^w \mid a \in S, w \in F(S)\}.$$

The rack operation is defined by

$$(a^w)^{(b^z)} = a^{w\bar{z}bz}.$$

Axiom 1 of definition 1.1 is easy to check whilst for the rack identity notice

$$(a^w)^{b^z} = a^{w\bar{z}bz} \equiv \overline{w\bar{z}bz}\ a\ w\bar{z}bz = \overline{\bar{z}bz}\ \overline{w}aw\ \bar{z}bz \equiv \overline{b^z}a^w b^z$$

which is the third form of the identity (axiom $2''$).

The operator group is $F(S)$ whilst the set of orbits is in bijective correspondence with the elements of S and all stabilizers are trivial.

The free rack has the universal property that any function $S \to X$, where X is a given rack, extends uniquely to a rack homomorphism $\text{FR}(S) \to X$.

Example 10. *Coxeter racks.*

Let $(\ ,\ )$ be a symmetric bilinear form on $\mathbb{R}^n$. Then, if S is the subset of $\mathbb{R}^n$ consisting of vectors $\mathbf{v}$ satisfying $\mathbf{v}.\mathbf{v} \neq 0$, there is a rack structure defined on S by the formula

$$\mathbf{u}^{\mathbf{v}} := \mathbf{u} - \frac{2(\mathbf{u}, \mathbf{v})}{(\mathbf{v}, \mathbf{v})}\mathbf{v}.$$

Geometrically, this is the result of reflecting $\mathbf{u}$ in the hyperplane $\{\mathbf{w} | (\mathbf{w}, \mathbf{v}) = 0\}$.

If we multiply the right-hand side of the above formula by -1, then the result geometrically is reflection in the line containing $\mathbf{v}$. In this case the formula

$$\mathbf{u}^{\mathbf{v}} := \frac{2(\mathbf{u}, \mathbf{v})}{(\mathbf{v}, \mathbf{v})}\mathbf{v} - \mathbf{u}$$

defines a quandle structure on S.

Now a **root system** is precisely a finite subrack of S which is closed under multiplication by -1 (i.e. closed under *both* rack operations), and then the operator group is the corresponding Coxeter group. For details see Humphreys [13], and for more information on the rack structures see Azcan [1], Brieskorn [3].

We can generalise this example in the following way.

Example 11 *Racks defined by Hermitian Forms.*

Let R be a commutative ring with identity and an involutive automorphism $r \mapsto \overline{r}$ called **conjugation.** Let A be an $R-$module with a **Hermitian form**

$$(\, , \,) : A \times A \to R.$$

In other words $(\, , \,)$ is linear in the first variable and $(b,a) = \overline{(a,b)}$.

Let A^* denote those elements of A for which (a,a) is a unit of the ring R.

Let λ, μ be elements of R, such that μ is a unit and λ satisfies $\lambda\overline{\lambda} = 1$. Define the rack operation on A^* by the formula

$$a^b := \mu(a + (\lambda - 1)\frac{(a,b)}{(b,b)}b).$$

Specialisation yields the following examples:

(a) Let $R = \mathbb{R}$ be the reals, let $A = V$ be a vector space over $\mathbb{R}$ and let conjugation be the identity. Let $(\, , \,)$ be a symmetric bilinear real form defined on V and let $\lambda = -1$ and $\mu = 1$. Then this is the rack structure considered in the last example.

(b) The obvious specialisation of the above to the complex field yields a rack in which the action is complex reflection, see Coxeter [6].

The application of this rack to links in S^3 has strong connections with the Jones polynomial [15], and this will be investigated in a future paper.

(c) Take $R = \mathbb{Z}[t, t^{-1}]$ to be the ring of Laurent polynomials with integer coefficients and conjugation defined by $t \mapsto t^{-1}$. Then λ is of the form $\lambda = t^n$ for some integer n.

We are indebted to Tony Carbery for pointing out the following infinite generalisation of the above example.

(d) Let R denote the ring of complex valued continuous functions defined on the unit circle of the complex plane.

The conjugation operation in R is given by

$$\overline{f}(z) := \overline{f(z)}$$

where $z \mapsto \overline{z}$ is just the usual conjugation of complex numbers. The set of functions λ satisfying $\lambda\overline{\lambda} = 1$ can be identified with the multiplicative subset of functions from the unit circle to itself.

The last two examples are somewhat mysterious and their applications to knot theory are unknown to us.

2. Some Basic Algebra of Racks

In this section we will present some of the algebraic properties of racks needed in the rest of the paper. Further basic algebra will be given in section 4, when we consider presentations of racks and the analogue of the Tietze theorem. See also Brieskorn [3], Ryder [28].

Homomorphisms and congruences

There are obvious notions of rack **homomorphism**, **isomorphism** and **subrack**.

An equivalence relation $\sim$ on X is called a **congruence** if it respects the rack operation, i.e.

$$a \sim b,\ c \sim d \implies a^c \sim b^d.$$

The equivalence classes form a rack $X/\sim$ with operation defined by $[a]^{[b]} := [a^b]$, where $[a]$ denotes the equivalence class of a.

A homomorphism $f : X \to Y$ of racks defines a congruence by $a \sim b \iff f(a) = f(b)$. Then the quotient $X/\sim$ is isomorphic to $f(X)$. This is an analogue of the first isomorphism theorem for groups.

The associated group

We have already met the operator group in the previous section. This is an invariant of racks but is not functorial. If we interpret the operation of a rack as conjugation (i.e. read a^w as $w^{-1}aw$) then we obtain a group $As(X)$ called the **associated group**. More precisely let $As(X) = F(X)/K$ where K is the normal subgroup of $F(X)$ generated by the words $a^b b^{-1} a^{-1} b$ where $a, b \in X$. So $As(X)$ is the biggest quotient of $F(X)$ with the property that, when considered as a rack via conjugation, the natural map from $F(X)$ to $As(X)$ is a rack homomorphism.

Given a rack homomorphism $f : X \to Y$, then there is an induced group homomorphism $f_\sharp : As(X) \to As(Y)$; thus we have an **associated group functor** As from the category of racks to the category of groups.

2.1 Proposition *Universal Property of the Associated Group*

Let X be a rack and let G be a group. Given any rack homomorphism $f : X \to G_{\text{conj}}$ there exists a unique group homomorphism $f_\sharp : As(X) \to G$ which makes the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\ \eta\ } & As(X) \\ \Big\downarrow f & & \Big\downarrow f_\sharp \\ G_{\text{conj}} & \xrightarrow{\ \text{id}\ } & G \end{array}$$

where η is the natural map.

Moreover any group with the same universal property is isomorphic to $As(X)$.

Proof Let $\phi : F(X) \to G$ be the homomorphism defined on the free group on X by f. Then by hypothesis $\phi(a^b b^{-1} a^{-1} b) = 1$ for all $a, b \in X$. It follows that ϕ factors through a unique homomorphism $f_\sharp : As(X) \to G$ of groups and the commutativity of the diagram is clear. Uniqueness of $As(X)$ follows by the usual universal property argument. □

The following corollary is an easy consequence:

2.2 Corollary *The functor As is a left adjoint to the conjugation functor. This means there is a natural identification*

$$Hom(As(X), G) \cong Hom(X, \mathrm{conj}(G))$$

of group homomorphisms with rack homomorphisms. □

Example The cyclic rack C_n has operator group $\mathbb{Z}/n$ and associated group $\mathbb{Z}$.

This example makes it clear that the operator group is in general a non-trivial quotient of the associated group. We will now use this fact to make the following definition.

Definition The Excess of a Rack.

Let X be a rack and let N be the subgroup of $F(X)$ which acts trivially on X. Let K be the normal subgroup of $F(X)$ generated by the elements $a^b b^{-1} a^{-1} b$ for all $a, b \in X$. Define the **excess** of the rack X to be

$$Ex(X) = N/K = \ker\{As(X) \to Op(X)\}.$$

In the example above the excess is a copy of the integers.

The associated quandle

There is a natural inclusion of the category of quandles in the category of racks and there is a functor from racks to quandles $X \mapsto X_q$. Here X_q is called the **associated quandle** defined as follows: Let $\sim$ be the smallest congruence on X satisfying $a^a \sim a$ for all $a \in X$. Then $X_q := X/\sim$.

This functor is a retraction because it is clearly the identity for a rack which is already a quandle. For explicit examples, consider the Coxeter rack X (section 1 example 10):

$$\mathbf{u}^{\mathbf{v}} := \mathbf{u} - \frac{2(\mathbf{u}, \mathbf{v})}{(\mathbf{v}, \mathbf{v})}\mathbf{v},$$

defined on the unit sphere $S = (\mathbf{u}, \mathbf{u}) = 1$. Then the associated quandle is the **projectivisation** of X, defined by quotienting S by ± 1.

For the (t, s)–rack (section 1 example 6) the associated quandle is the Alexander rack (example 5).

Racks with an explicit group (augmented racks)

The concept of rack can be generalised to make the group action explicit.

An **augmented rack** comprises a set X with an action by a group G, which we write

$$(x, g) \mapsto x \cdot g \quad \text{where} \quad x,\ x \cdot g \in X \quad \text{and} \quad g \in G,$$

and a function $\partial : X \to G$ satisfying the **augmentation identity**:

$$\partial(a \cdot g) = g^{-1}(\partial a)g \quad \text{for all} \quad a \in X, g \in G,$$

which is precisely the same as saying that ∂ is a G–map when the action of G on itself is taken to be conjugation.

We can now define an operation of X on itself by defining a^b to be $a \cdot \partial b$. Then the augmentation identity implies

$$\partial(a^b) = (\partial b)^{-1} \partial a \partial b$$

i.e.

$$a^b \equiv \bar{b} a b$$

which is the third form of the usual rack identity (axiom $2''$).

So an augmented rack is an ordinary rack with the extra structure of an explicit operator group.

Note that Joyce [16] used ϵ for the augmentation map ∂. We have chosen to use ∂ instead of ϵ because of the analogy with crossed modules; see the definition of associated crossed module below.

The fundamental rack (to be defined in the next section) has a natural structure as an augmented rack with the fundamental group of the link acting as a group of operators. Here are some further examples.

2.3 Examples

(1) *The Lie rack*

We are indebted to Hyman Bass for pointing out the following important class of augmented racks. Let G be a Lie group and $\mathcal{G}$ the associated Lie algebra. Let $\partial : \mathcal{G} \to G$ be the exponential map and let G act on $\mathcal{G}$ via the adjoint action. Then the augmentation identity follows readily from definitions. Therefore the Lie algebra $\mathcal{G}$ is an augmented rack, with group the corresponding Lie group.

(2) *Gauge transformations*

Let E be a principal G–bundle and f a **gauge transformation** of E, that is an automorphism of E as a G–bundle. Then associated to f is an augmented rack structure on E with group G.

Let $p \in E$ then we can write $f(p) = p \cdot \partial(p)$ where $\partial(p) \in G$. This defines the function $\partial : E \to G$. To check the augmentation identity note that

$$\begin{aligned} p \cdot (g\partial(p \cdot g)) &= (p \cdot g) \cdot \partial(p \cdot g) \\ &= f(p \cdot g) \\ &= f(p) \cdot g \qquad \text{since } f \text{ is equivariant} \\ &= (p \cdot \partial(p)) \cdot g \\ &= p \cdot (\partial(p)g) \end{aligned}$$

which implies $g\partial(p \cdot g) = \partial(p)g$ since G acts freely, i.e.

$$\partial(p \cdot g) = g^{-1}\partial(p)g.$$

An augmented rack is a plain rack if we ignore or forget about the explicit group action. Conversely, there is a natural way to regard a plain rack as an augmented rack by taking as group $G = As(X)$ (the associated group) with ∂ the natural map. Thus we can regard the category of racks as a subcategory of the category of augmented racks and then the forgetful functor is a retraction of the larger category onto the smaller.

Crossed modules

A **crossed module** is an augmented rack in which

(1) X is a group

(2) ∂ is a homomorphism

(3) and we have the **crossed module identity**:

$$a \cdot \partial b = b^{-1}ab \text{ for all } a, b \in X,$$

where the left-hand side is the G action and the right-hand side is multiplication in X.

Note that condition (3) implies that the rack operation in X is conjugation. Thus crossed modules correspond precisely to conjugation racks.

Crossed modules occur naturally in topology: the second homotopy group $\pi_2(X, A)$ of a pair of topological spaces is a crossed module with group $G = \pi_1(A)$ (see Whitehead [30]). We shall use the notation $\widehat{\pi}_2(X, A)$ for this crossed module.

The associated crossed module

The associated group for a plain rack becomes the associated crossed module for an augmented rack. More precisely notice that if X is an augmented rack then its group G acts on $F(X)$ in the obvious way. Moreover

$$\begin{aligned} (a^b) \cdot g = a \cdot (\partial b g) &= a \cdot g(g^{-1}\partial b g) \\ &= a \cdot g \ \partial(b \cdot g) \quad \text{by the augmentation identity} \\ &= (a \cdot g)^{b \cdot g} \end{aligned}$$

Therefore in $F(X)$

$$(a^b b^{-1} a^{-1} b) \cdot g = (a \cdot g)^{(b \cdot g)} (b \cdot g)^{-1} (a \cdot g)^{-1} (b \cdot g)$$

and therefore the action of G on $F(X)$ induces an action on $As(X)$.

Thus $As(X)$ is a G–set and we also have the induced homomorphism $\partial_\sharp : As(X) \to G$. It can be readily checked that this gives $As(X)$ the structure of a crossed module, the **associated crossed module** to the augmented rack X.

Products of Racks

There are many kinds of products which can be defined in the category of racks. We shall only need to consider in detail the following:

The free product Let X, Y be two racks. Define their **free product** $X * Y$ to be the free rack on the disjoint union $X \amalg Y$ quotiented out by the original actions of X and Y.

More precisely $X * Y$ consists of elements of the form x^w or y^w where $x \in X$, $y \in Y$ and $w \in As(X) * As(Y)$ under the equivalence generated by the following:

$$x^{wt} \sim u^t \text{ where } x \in X,\ w \in As(X),\ t \in As(X) * As(Y) \text{ and } x^w = u \text{ in } X$$

and a similar equivalence for Y.

The rack operation on $X * Y$ is defined by the same formula as for the free rack (section 1 example 9). That the operation is well defined follows from the definition of the associated group. For example suppose that $x^w = t$ in X then

$$(z^u)^{x^w} := z^{u\overline{w}xw} = z^{ut} \text{ since } \overline{w}xw = t \text{ in } As(X).$$

Notice that there are natural inclusions of X and Y in $X*Y$ and that the associated group is the free product:

$$As(X * Y) = As(X) * As(Y).$$

The following lemma implies that the free product is the categorical 'sum' in the category of racks:

2.4 Lemma *Let $f : X \to Z$, $g : Y \to Z$ be rack homomorphisms. Then there is a unique extension $f * g : X * Y \to Z$.*

Proof The free product $X * Y$ is generated as a rack by the images of X and Y under the natural inclusions, and the lemma follows. □

Free product of augmented racks

The free product $X * Y$ of augmented racks X, Y with groups G, H is defined in a similar way:

We consider pairs (x, g) where $x \in X$ or Y and $g \in G * H$ with equivalence generated by

$$(x, gt) \sim (u, t) \text{ where } x \in X, g \in G, t \in G * H \text{ and } x \cdot g = u \text{ in } X$$

and a similar equivalence for Y.

The group for $X * Y$ is $G * H$, $\partial_{X*Y} := \partial_X * \partial_Y$ and the action of $G * H$ on $X * Y$ is defined by right multiplication in the second coordinate.

There is again a universal property which we leave the reader to formulate.

Other products

The categorical 'product' for racks is the cartesian product with operation

$$(a, x)^{(b,y)} := (a^b, x^y).$$

There are several other products, for example the disjoint union $X \amalg Y$ where the rack operation is defined by letting Y act trivially on X and vice-versa (Breiskorn [3]). This last product can be generalised by allowing Y to act via any function $Y \to \text{centre}(Op(X))$ and vice versa. Further products are defined by Ryder [28].

The Inverted Rack

Given a rack X there is a (possibly) different rack X^* called the **inverted rack** in which the new binary operation is $a^{\bar{b}}$. Racks and their inverted cousins arise naturally in the geometric context of the fundamental rack, see the remarks near the end of section 5.

3. The Fundamental Rack of a Link.

This is the most important rack of all and is the *raison d'être* of the whole theory. A (codimension two) **link** is defined to be a codimension two embedding $L : M \subset Q$ of one manifold in another. We shall assume that the embedding is proper at the boundary if necessary, that M is non-empty, that Q is connected and that M is **transversely oriented** in Q. In other words we assume that each normal disc to M in Q has an orientation which is locally and globally coherent.

The link is said to be **framed** if there is given a cross section (called a **framing**) $\lambda : M \to \partial N(M)$ of the normal disk bundle. Denote by M^+ the image of M under λ. We call M^+ the **parallel** manifold to M.

We consider homotopy classes Γ of paths in $Q_0 = \text{closure}(Q - N(M))$ from a point in M^+ to a base point. During the homotopy the final point of the path at the base point is kept fixed and the initial point is allowed to wander at will on M^+.†

The set Γ has an action of the fundamental group of Q_0 defined as follows: let γ be a loop in Q_0 representing an element g of the fundamental group. If $a \in \Gamma$ is represented by the path α define $a \cdot g$ to be the class of the composite path $\alpha \circ \gamma$.

† This reverses the more usual dog wagging tail convention where the initial point of a path stays fixed. However the tail wagging dog convention fits in more comfortably with operations on the right.

We can use this action to define a rack structure on Γ. Let $p \in M^+$ be a point on the framing image. Then p lies on a unique meridian circle of the normal circle bundle. Let m_p be the loop based at p which follows round the meridian in a positive direction. Let $a, b \in \Gamma$ be represented by the paths α, β respectively. Let $\partial(b)$ be the element of the fundamental group determined by the loop $\overline{\beta} \circ m_\beta \circ \beta$. (Here $\overline{\beta}$ represents the reverse path to β and m_β is an abbreviation for $m_{\beta(0)}$ the meridian at the initial point of β.) The **fundamental rack** of the framed link L is defined to be the set $\Gamma = \Gamma(L)$ of homotopy classes of paths as above with operation

$$a^b := a \cdot \partial(b) = [\alpha \circ \overline{\beta} \circ m_\beta \circ \beta].$$

If L is an unframed link then we can define its **fundamental quandle.** The definition is very similar. Let $\Gamma_q = \Gamma_q(L)$ be the set of homotopy classes of paths from the boundary of the regular neighbourhood to the base point where the initial point is allowed to wander during the course of the homotopy over the *whole* boundary. The rack structure on Γ_q is similar to that defined on Γ.

There is a convenient halfway-house between framed and unframed links: a link L is **semi-framed** if *some* of the components of M are framed. A semi-framed link has a fundamental rack defined by allowing the initial point to wander on the whole boundary of the neighbourhoods of unframed components and on M^+ otherwise. This gives a common generalisation for the rack of a framed link and the quandle of an unframed link, and allows us to make economical statements of results which apply to all cases.

3.1 Proposition *The fundamental rack of a semi-framed link satisfies the axioms of a rack.*

The fundamental quandle of an unframed link satisfies the axioms of a rack together with the quandle condition.

In the semi-framed case the fundamental quandle of the corresponding unframed link (i.e. ignore framings), is the associated quandle of the fundamental rack.

Proof The axioms are easy to verify. The inverse action is determined by the class of

$$\overline{\alpha} \circ \overline{m}_\alpha \circ \alpha.$$

To check the rack identity we again use the action of the fundamental group. Using the notation above, $\partial(a^b)$ is represented by the loop

$$\overline{\beta} \circ \overline{m}_\beta \circ \beta \circ \overline{\alpha} \circ m_\alpha \circ \alpha \circ \overline{\beta} \circ m_\beta \circ \beta$$

which is the class of

$$\partial(b)^{-1} \partial(a) \partial(b).$$

In the unframed case note that the element a^a is represented by the path

$$\alpha \circ \overline{\alpha} \circ m_\alpha \circ \alpha \quad \simeq \quad m_\alpha \circ \alpha.$$

However in a homotopy in the definition of Γ_q the initial point is allowed to move along the loop m_α and so the path is homotopic to α which represents a.

The last part of the proposition is obvious. □

Note that if G denotes the fundamental group $\pi_1(Q_0)$ then the set Γ is in fact an augmented rack with group G. We shall use the notation $\widehat{\Gamma}$ for this augmented rack in order to distinguish it from the **plain** fundamental rack Γ. Note that Γ is the underlying plain rack to $\widehat{\Gamma}$.

We will now identify the associated group of the fundamental rack of an arbitary codimension two link and the operator group of the fundamental rack of a link of circles in an oriented 3–manifold.

Consider the following fragment of the exact homotopy sequence of the pair (Q, Q_0):

$$\pi_2(Q) \to \pi_2(Q, Q_0) \to \pi_1(Q_0) \to \pi_1(Q).$$

We shall call $\pi_1(Q_0)$ the **fundamental group** of the link and $\ker\{\pi_1(Q_0) \to \pi_1(Q)\} = \mathrm{im}\{\pi_2(Q, Q_0) \to \pi_1(Q_0)\}$ the **kernel** of the link. Further we shall call the relative group $\pi_2(Q, Q_0)$ the **associated group of the link**. Note that if $\pi_2(Q) = 0$ then the associated group and the kernel of the link coincide, and if in addition $\pi_1(Q) = 0$ as in the classical case of links in S^3 then all three groups coincide.

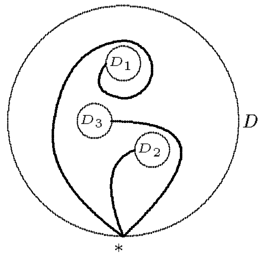
3.2 Proposition *The associated group of the fundamental rack $\Gamma(L)$ of a semi-framed link L can be naturally identified with the associated group of L.*

Moreover the associated crossed module of the fundamental augmented rack $\widehat{\Gamma}(L)$ can be identified with the crossed module $\widehat{\pi}_2(Q, Q_0)$ corresponding to the second relative homotopy group $\pi_2(Q, Q_0)$.

Proof Let $a \in \Gamma$, then ∂a is represented by the path $\overline{\alpha} \circ m_\alpha \circ \alpha$ which bounds an obvious 2–disc, namely the meridinal disc at the initial point of α. Thus there is a map $\Gamma \to \pi_2(Q, Q_0)$. Under this map the rack operation corresponds to conjugacy. Therefore it induces a homomorphism $As(\Gamma) \to \pi_2(Q, Q_0)$. We shall show that this is an isomorphism by constructing an inverse map.

Suppose $g \in \pi_2(Q, Q_0)$ is represented by the disc D. After a homotopy we may assume that D meets the neighbourhood N of the link transversely in a finite number of little discs $D_1, \ldots, D_n$. Assign to D_i the sign ϵ_i where $\epsilon_i = +1$ if the orientation of D agrees with the orientation of D_i and -1 if not.

Pick a base point in each ∂D_i. In the case of a framed link let the base point be the intersection of ∂D_i with M^+. Join each of these n base points to the base point $*$ of Q by n paths $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $D - \cup_i\{D_i\}$ which only meet at $*$ and arrive at $*$ in the order $1, 2, \ldots, n$.



Note that this implies that the paths are uniquely determined up to isotopy and possible initial twists about the little disks. Each path α_i determines an element a_i of Γ and this defines a word $a_1^{\epsilon_1} a_2^{\epsilon_2} \cdots a_n^{\epsilon_n}$ in $F(\Gamma)$ and so an element of its quotient $As(\Gamma)$.

In order to check that this element is well defined it is only necessary to see what happens if we change the number of initial twists or the order of the subdiscs or if we change the choice of the disc D by a homotopy.

Now an initial twist changes a_i to $a_i^{a_i}$ (see the end of the proof of proposition 3.1) and an interchange of two elements a_i and a_j replaces $a_i a_j$ by $a_j a_i^{a_j}$, or a similar replacement with different signs. None of these affects the value of the product in $As(\Gamma)$.

Now suppose D' is a homotopic choice of disc. By making the homotopy transverse to the link we see that the word in $F(\Gamma)$ changes in two ways: either an interchange of order as above or an introduction or deletion of cancelling pairs aa^{-1} or $a^{-1}a$. Both leave the value in the associated group unchanged. The resulting map is the required inverse.

The last part of the proposition is readily checked. □

Remark The proposition shows that the fundamental augmented rack of a link is a *sharpened form* of the crossed module, $\widehat{\pi}_2(Q, Q_0)$.

To see that the rack really does contain more information than the crossed module consider the example of a knot k in S^2 which is the sum of two knots. The elements of $\pi_2(Q, Q_0)$ represented by meridinal discs across the two connecting arcs coincide, however in general the corresponding elements of the *rack* are different. Indeed it can be shown that the two elements of the rack are never the same if the two knots are both non-trivial.

3.3 Corollary *The associated group of the fundamental rack of a link in a 2–connected space can be identified with the fundamental group of the link. In particular the associated group and the fundamental group coincide for classical links in the 3–sphere and for links in a homotopy 3–sphere.* □

Remark The corollary implies that for links in a homotopy 3–sphere the plain fundamental rack Γ and the fundamental augmented rack $\widehat{\Gamma}$ essentially coincide (i.e. coincide under the embedding of racks in augmented racks described in the last section).

Orbits and stabilizers

The orbits in the fundamental augmented rack of a link are in bijective correspondence with the components of the link and the next lemma identifies the corresponding stabilizers.

An element of the fundamental group represented by a loop of the form $\overline{\alpha} \circ \gamma \circ \alpha$ where γ lies in $\partial N(M)$ and α represents the element $a \in \Gamma$ is called a–**peripheral.** The set of a–peripheral elements forms the a–**peripheral subgroup.** If γ lies in the subset M^+ then the class of $\overline{\alpha} \circ \gamma \circ \alpha$ is called a–**longitudinal** and the set of a–longitudinal elements forms the a–**longitudinal** subgroup. If γ lies in the boundary of a normal disc to M then $\overline{\alpha} \circ \gamma \circ \alpha$ is called a–**meridinal.** The set of a–meridinal elements forms the a–**meridinal subgroup.** (The kernel of the link is generated by meridinal elements.)

3.4 Lemma *If the element a of the fundamental rack is represented by a path α from ∂N to the base point and if h is an element of the fundamental group which fixes a then h is a–peripheral. If α starts on the neighbourhood of a framed component of M, then h is a–longitudinal.*

Proof Let h be represented by a loop γ. During the course of the homotopy of $\alpha \circ \gamma$ to α the initial point describes a loop δ in ∂N. This implies that γ is homotopic to the loop $\overline{\alpha} \circ \delta \circ \alpha$ and the result follows. □

3.5 Corollary *With the notation above, the stabilizer of a in the fundamental group is the a–longitudinal subgroup or the a–peripheral subgroup, according as the component where α starts is framed or unframed.* □

Seifert links and the operator group

We shall finish this section by identifying the operator group for knots and links in 3–manifolds.

Let G be the group of $\widehat{\Gamma}$ (the fundamental group of the link) and let K (the kernel of the link) be the image of $As(\Gamma)$ in G. Define the **action kernel** $J \subset G$ to comprise all elements of G which act trivially on Γ. Recall that the operator group $Op(\Gamma)$ is the quotient of $As(\Gamma)$ by the subgroup of elements which act trivially on Γ. Since the action of $As(\Gamma)$ factors via the action of G, $Op(\Gamma)$ is the quotient of K by the subgroup of K of elements which act trivially on Γ. In other words

$$Op(\Gamma) = \frac{K}{J \cap K}. \qquad \textbf{(3.6)}$$

We shall now compute J.

We will need the following definition:

Definition *Seifert Links*

Consider a link in a 3–manifold whose complement has a Seifert circle fibration. An example is a torus link in S^3. We say that a component C of such a link is **framable** if the fibration extends to C and its neighbourhood so that C is a regular fibre. A component is said to be **naturally** framed if it is framable and has the framing given by neighbouring fibres. A **Seifert link** is a link in a 3–manifold whose complement has a Seifert circle fibration such that all framed components are naturally framed. Note that a Seifert link might be unframed and that some framable components might be unframed.

3.7 Proposition *Consider a semi-framed link in a 3–manifold whose complement is P^2 irreducible. The action kernel J of the link is non-trivial if and only if the link is a Seifert link. Moreover J can be described explicitly. There are four cases:*

(1) *The link is a Seifert link with at least one framed component, in which case J is the infinite cyclic subgroup of the fundamental group defined by the regular fibres.*

(2) *The link is unframed and Q_0 is $T \times I$ where T is a torus, in which case $J \cong \mathbb{Z}^2$ is the fundamental group.*

(3) *The link is unframed and Q_0 is $K \tilde{\times} I$ (the twisted I bundle over a Klein bottle), in which case $J \cong \mathbb{Z}^2$ is a subgroup of index 2 in the fundamental group.*

(4) *The link is an unframed Seifert link and Q_0 is neither $T \times I$ nor $K \tilde{\times} I$, in which case J is the same as in case (1).*

Proof: It is convenient to make the following observation about groups and normal subgroups. Let G be a group which has a subgroup H containing a non trivial element h such that any conjugate $g^{-1}hg$ is in H, where g lies in G. Then the group generated by $g^{-1}hg$ for all $g \in G$ is a non trivial normal subgroup of H and G.

Assume that $J \neq \{1\}$ and choose $h \in J$, $h \neq 1$. By lemma 3.4 h is a–peripheral and, for framed components, a–longitudinal *for all* $a \in \Gamma$.

For convenience take the base point $*$ in $\partial N(C)$ where C is a component of the link, and assume that C is framed. Consider elements $a \in \Gamma$ defined by loops α based at $*$. Such an element can be regarded (non-uniquely) as an element $g \in G$, the fundamental group. Now h is a–longitudinal for all such a hence $h = g^{-1}lg$ in G where l is some power of the longitude at $*$. It follows that our observation applies where H is the infinite cyclic subgroup of powers of the longitude. The complementary manifold M therefore has fundamental group containing a normal infinite cyclic subgroup. A result of Waldhausen [29] shows that M is a Seifert manifold and that l the longitude is a fibre.

A similar argument works if C is unframed. In this case the group H is $\mathbb{Z}^2$. If the resulting normal subgroup is infinite cyclic we can apply the previous argument and deduce that the complementary space is Seifert fibred. If the normal subgroup is $\mathbb{Z}^2$ then we know that it is peripheral and the complementary space is therefore either $T \times I$ where T is a torus or the twisted I bundle over a Klein bottle. The relevant details may be found in Hempel's book [12] as was kindly pointed out to us by A. Swarup. In either case both are Seifert fibred spaces.

Conversely if the link is Seifert then it is easy to see that the infinite cyclic subgroup defined by the regular fibres acts trivially. If a larger subgroup acts trivially then, by the argument in the previous paragraph, we are in the unframed case and Q_0 is $T \times I$ or $K \tilde{\times} I$. In the first case all elements act trivially while, in the second case, a subgroup of index 2 acts trivially. □

3.8 Corollary *The operator group of the (plain) fundamental rack of a semi-framed link in a connected orientable 3–manifold is one of: (a) the kernel of the link, (b) the kernel modulo the integers, (c) $\mathbb{Z}/2$, or (d) the trivial group.*

In the last three cases, the link is a Seifert link with possibly some homotopy discs or spheres added by connected sum.

Proof Decompose the complement into irreducible pieces. If two or more pieces are non-simply connected then the fundamental group has no normal subgroup isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$, and hence the action kernel is trivial by the proof of the proposition.

Thus if the action kernel is non-trivial then the complement is irreducible, with possibly some homotopy discs or spheres added by connected sum. Remove these connected summands, then by the proposition the complement is a Siefert link and the action kernel is $\mathbb{Z}$ or $\mathbb{Z}^2$. Hence by equation 3.6 the operator group is the link kernel modulo $\mathbb{Z}$ or $\mathbb{Z}^2$. In the $\mathbb{Z}^2$ case we must be in case (2) or (3) of the proposition and the quotient is either $\mathbb{Z}/2$ or trivial. □

4. Presentations

The main purpose of this section is to explain the natural presentation that can be given to the fundamental rack of a link in a 3–manifold. This will involve explaining several layers of rack presentations. We start with the simplest.

Throughout the section, we shall concentrate on framed links (rather than semi-framed links). There are analogues for semi-framed links of most of the results in the section, which are proved in analogous ways. By and large we leave the reader to formulate these parallel results, contenting ourselves with brief comments.

Primary Rack Presentations

A **primary presentation** for a rack consists of two sets S (the generating set) and R (the set of relators). A typical element of R is an ordered pair (x, y), where $x, y \in \mathrm{FR}(S)$, which we shall usually write as an equation: $x = y$ or $(x = y)$.

The presentation defines a rack $[S : R]$ as follows: Define the congruence $\sim$ on $\mathrm{FR}(S)$ to be the smallest congruence containing R (i.e. such that $x \sim y$ whenever $(x = y) \in R$). Then

$$[S : R] = \frac{\mathrm{FR}(S)}{\sim}.$$

We can describe $\sim$ more constructively as follows.

Consider the following process for generating relators. Start with the given set R of relators and enlarge R by repeating any or all of the following moves:

(a) Add a **trivial** relator $x = x$ for some $x \in \mathrm{FR}(S)$.

(b) If $(x = y) \in R$ then add $y = x$.

(c) If $(x = y), (y = z) \in R$ then add $x = z$.

(d) If $(x = y) \in R$ then add $x^w = y^w$ for some $w \in \mathrm{FR}(S)$.

(e) If $(x = y) \in R$ then add $t^x = t^y$ for some $t \in \mathrm{FR}(S)$.

Define a **consequence** of R to be any statement which can be generated by a finite number of these moves, and define $\langle R \rangle$ to be the set of consequences of R.

Now a congruence is a relation which is: (1) an equivalence relation and (2) respects the rack operation. If we use $=$ instead of $\sim$ for the congruence, then (1) says that the congruence is closed under moves (a), (b) and (c) whilst (2) says it is closed under moves (d) and (e). It follows that the smallest congruence containing R is precisely the set of consequences, $\langle R \rangle$.

Remark The asociated quandle $[S : R]_q$ has a presentation, obtained by adding to R the relators $a^a = a$ for all $a \in S$. If $[S : R]$ is a finite presentation then so is $[S : R]_q$.

Proof Clearly the new relators hold in the associated quandle, but the new rack *is* a quandle because

$$(a^w)^{a^w} = a^{w\overline{w}aw} = a^{aw} = a^w \text{ since } a^a = a.$$

The "Tietze" Theorem

We shall now prove an analogue for racks of the Tietze move theorem for group presentations.

The two basic moves on presentations are the following:

Tietze move 1 Add to R a consequence (or delete from R a consequence) of the other relators.

Tietze move 2 Introduce a new generator x and a new relator $x = a^w$ (where x does not occur in w), or delete such a pair if x occurs nowhere else in the presentation.

There is an equivalent set of moves which are rather more constructive:

4.1 Lemma *Tietze moves 1 and 2 are equivalent to the following moves:*

(1) *Repeat a relator (or delete a repeated relator).*

(2) *Conjugate a relator, i.e. replace for example* $a^t = b^w$ *by* $a^{tz} = b^{wz}$.

(3) *Substitute at primary level, i.e. if* $a = b^w \in R$ *then we can replace* $c^z = a^t$ *by* $c^z = b^{wt}$ *and we can replace* $a^z = c^t$ *by* $b^{wz} = c^t$.

(4) *Substitute at operator level, i.e. if* $a = b^w \in R$ *then we can replace* $c^{taq} = d^z$ *by* $c^{t\overline{w}bwq} = d^z$ *or* $c^z = d^{taq}$ *by* $c^z = d^{t\overline{w}bwq}$.

(5) *Introduce a new generator* x *and a new relator* $x = a^w$ *(where* x *does not occur in* w*), or delete such a pair if* x *occurs nowhere else in the presentation.*

Proof

Moves (1) to (5) are all equivalent to, or special cases of, the two Tietze moves, so it suffices to show that the two Tietze moves can be achieved by moves (1) to (5). Since Tietze move 2 is move (5) we have to show that any consequence can be introduced (Tietze move 1). But by definition any consequence can be constructed by the relator moves (a) to (e) (in the definition of rack presentation). So we shall start by proving that any of these can be achieved by moves (1) to (5).

To achieve move (a) (to introduce a trivial relator) use the following trick:

Introduce a new generator t and relation $t = a$. Repeat the relation $t = a$ and substitute to obtain $a = a$. Now delete t and the redundant copy of $t = a$.

To achieve move (b) (to add $y = x$ where $(x = y) \in R$) use another trick:

Repeat $x = y$ and substitute to get $y = y$ and then again to get $y = x$.

Finally moves (c), (d) and (e) are precisely moves (2), (3) and (4) in a different form.

It follows that we can use moves (1) to (5) to replace $[\,S : R\,]$ by $[\,S : R \cup T\,]$ where T contains the required consequence $x = y$. Repeat $x = y$ and then reverse the moves used to generate T to delete it. □

4.2 Theorem : Tietze move analogue *Suppose that we have two finite presentations* $[\,S : R\,]$ *and* $[\,S' : R'\,]$ *of isomorphic racks then the two presentations are related by a finite sequence of Tietze moves.*

Proof

Identify $[\,S : R\,]$ with $[\,S' : R'\,]$ by the isomorphism. We shall start with the presentation $[\,S : R\,]$ and move it into the other presentation.

To avoid confusion we shall use the letters $a_1, a_2, \ldots$ for elements of S and $b_1, b_2, \ldots$ for elements of S'. We shall also use $w_1, w_2, \ldots$ for words in the a_i and $z_1, z_2, \ldots$ for words in the b_i.

Step 1 Since the elements of S' are in the rack, each can be expressed in terms of the generators S, i.e.

$$b_1 = a_{(1)}^{w(1)},\ b_2 = a_{(2)}^{w(2)}, \ldots \qquad (*)$$

where $a_{(i)} = a_j$ some j. Let Q be the set of statements $(*)$. Use Tietze move 2 to introduce the "new" generators $b_1, b_2, \ldots$ together with the set Q of new relators.

Step 2 Since each statement in R' is true in the rack it is a consequence of R and hence can be introduced by Tietze move 1. Thus we can enlarge the set of relators to $R \cup Q \cup R'$.

Step 3 Since S' generates the rack, we can express each element of S in terms of S':

$$a_1 = b_{(1)}^{z(1)},\ a_2 = b_{(2)}^{z(2)}, \ldots$$

(forming a set of statements Q' dual to Q). Since each of the statements in Q' is true in the rack, it can be introduced as a new relator using move 1 again.

At this point we have reached a symmetrical situation. We have $S \cup S'$ as generating set and $R \cup R' \cup Q \cup Q'$ as relators. We now reverse steps 1 to 3 to delete first Q then R and finally S together with Q'. □

Presentations and the Associated Group

Using the proof of the Tietze theorem we can prove that the associated group of a finitely presented rack $[S : R]$ has a finite presentation as a group — in fact it has the obvious presentation:

Given a rack presentation $[S : R]$ then we obtain a group presentation by interpreting the elements of R as group equations (i.e. read a^w as $w^{-1}aw$) yielding the group $<S : R>$. It follows from the Tietze theorem that $<S : R>$ is independent of the presentation of the rack, since each of moves (1) to (5) leaves the group $<S : R>$ unchanged. This also follows from the following result:

4.3 Lemma *$<S : R>$ is the associated group $As[S : R]$.*

Proof We shall prove that $<S : R>$ has the universal property of the associated group. First of all there is a rack homomorphism

$$\phi : [S : R] \to <S : R>_{\mathrm{conj}}$$

because the congruence which defines $[S : R]$ comprises all (rack) consequences of R. But examining moves (1) to (5) we see that each rack consequence is a group consequence of R (as group relators) and therefore the identity on S extends to a rack homomorphism ϕ.

Now suppose we are given a rack homomorphism $\psi : [S : R] \to G_{\mathrm{conj}}$ where G is any group. Consider a typical relator $a^w = b^z$ in in R then $\psi(a^w) = \psi(b^z)$ in G_{conj}, but since ψ is a rack homomorphism this implies

$$\psi(\overline{w})\psi(a)\psi(w)\psi(\overline{z})\psi(\overline{b})\psi(z) = 1 \text{ in } G.$$

But this says that $\psi(R)$ as a *group* relation is true in G, and therefore ψ factors via ϕ. Uniqueness of this factoring is clear since both $[S : R]$ and $<S : R>$ have the same generating set. □

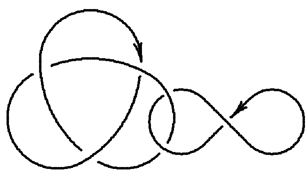
4.4 Remark *Racks and group presentations*

The Tietze theorem gives a way of regarding racks as group theoretic objects, namely equivalence classes of **group presentations of conjugacy type** (presentations with relators all of the form $x^w = y^z$) under our moves (1) to (5) of lemma 4.1, moves which all preserve this class.

Links in S^3

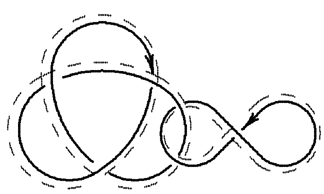
We consider links $L : M \subset S^3$. Since M is a codimension two submanifold of S^3, it is the image of a finite disjoint collection of smooth embeddings of S^1 in S^3. In this case, a framing on a component of M can be 'measured' because the isotopy class of the framing can be regarded, either as (the isotopy class of) a parallel curve (as in §1) or as an integer (the linking number of the component with its parallel curve), cf. [9].

If we project the link in general position onto a plane $\mathbb{R}^3 = S^3 - \text{pt}$ we obtain a **diagram**: a finite collection of arcs and circles, the arcs terminating at crossings, as exemplified in the following figure.



The chosen orientation on each component of the link is indicated by the arrows in the diagram.

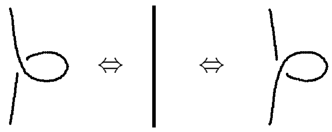
Now a link diagram has a **natural** framing: each component in the diagram has a canonical parallel curve obtained by drawing a curve in the diagram adjacent to the component (indicated as the broken curves in the following figure).



The following lemma implies that we may suppose that the natural framing and the given framing of L coincide.

4.5 Lemma *Given a framed link L in S^3 there is a diagram for L whose natural framing coincides with the given one.*

Proof The Ω_1 moves



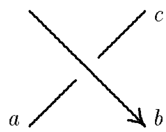
change the natural framing by ± 1. Hence the diagram can be altered by Ω_1 moves to make the two coincide. □

The rack presentation given by a diagram

A link diagram determines a primary rack presentation by:

(1) Label all arcs (or circles) in the diagram by generators $a, b, c, \ldots$ forming the **generating set** S.

(2) At each crossing write down a relator by the following rule.



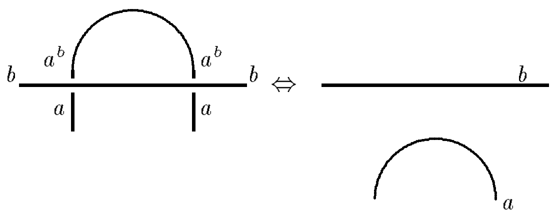
Write $c = a^b$ or $a = c^{\bar{b}}$.

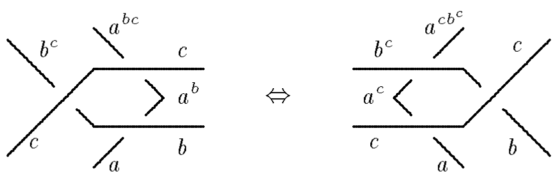
Note that b crosses a *from the right* as a passes under to become a^b. Note also that the orientation of the under-arc is not used in the rule.

The set of relators gives the **relator set** R.

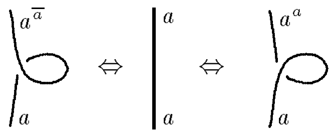
Extended remark We shall prove shortly that $[S : R]$ is $\Gamma(L)$ (the fundamental rack of L in S^3) which implies that $[S : R]$ is independent of the choice of diagram used to represent L. However, it is worth remarking that this can easily be proved directly, and indeed the *definition* of a rack is tailor made to prove this.

The two rack laws (1.1 axioms 1 and 2) correspond to invariance of $[S : R]$ under the Reidemeister Ω_2 and Ω_3 moves respectively, see the following figures.

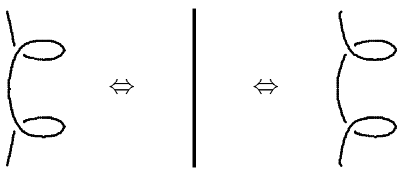




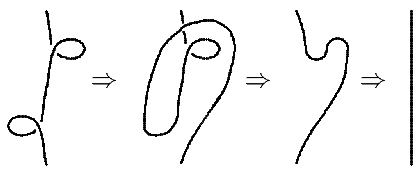
Now for unframed links, the fundamental quandle is invariant under the final Reidemeister move (the Ω_1 move) by the quandle condition; see the figure.



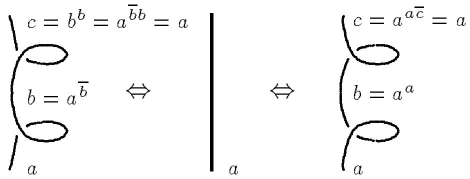
For framed links, we need a modified version of the Reidemeister move theorem: isotopy classes of framed links correspond to equivalence classes of diagrams under the Ω_2 and Ω_3 moves and the following "double" Ω_1 move:



This follows from the Reidemeister theorem: replace all Ω_1 insertions by double insertions and leave all Ω_1 deletions to the end. Collect all the extra twists on one arc of each component (using Ω_2 and Ω_3 moves). Then the fact that the framings are the same, means that there are (algebraically) the same number of extra twists. But excess pairs of opposite twists can be cancelled using the double Ω_1 move or the following sequence of Ω_2 and Ω_3 moves:



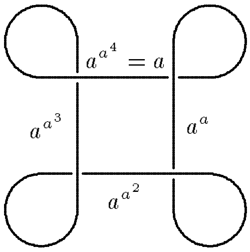
The figure below shows how the rack laws imply that $[S : R]$ is independent of the double Ω_1 move. The critical observation on the right is that $c = a^{a\bar{c}}$ implies $c \equiv a$ and also $a^{a\bar{c}} = a$:



This completes the extended remark.

4.6 Examples

Example 1 *The unknotted circle with framing n. Shown here with $n = 4$.*



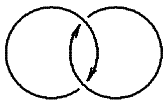
In the diagram, we have simplified the labels, by making obvious substitutions. The presentation gives the rack

$$[\, a \; : \; a^{a^n} = a \,]$$

i.e. the cyclic rack C_n. Notice that the circle with framing $-n$ has isomorphic fundamental rack. Thus extra structure will be needed to cope with orientations (see end of section 5).

Example 2 *The Hopf link.*

With both framings 0



the rack is

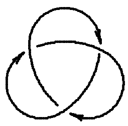
$$[\, a, b \; : \; a^b = a, b^a = b \,].$$

With framings n and m the rack becomes

$$[\, a, b \; : \; a = a^{a^n b}, b = b^{ab^m} \,].$$

Imposing the two quandle relations: $a^a = a, b^b = b$, makes these two racks identical (the fundamental quandle of the unframed Hopf link).

Example 3 *The left hand trefoil knot (framing -3).*



The fundamental rack is

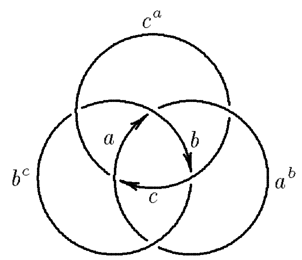
$$[\, a, b, c \; : \; a^b = c, \; c^a = b, \; b^c = a \,]$$

or

$$[\, a, b \; : \; a^{ba} = b, \; b^{\bar{b}ab} = a \,].$$

Remark Note that the presentation of the fundamental rack has deficiency zero in contrast to the deficiency of the fundamental group which is one.

Example 4 *The Borromean rings.*



The fundamental rack is

$$[\, a, b, c \; : \; c^{a\bar{b}\bar{a}b} = c, \; a^{b\bar{c}\bar{b}c} = a, \; b^{c\bar{a}\bar{c}a} = b \,].$$

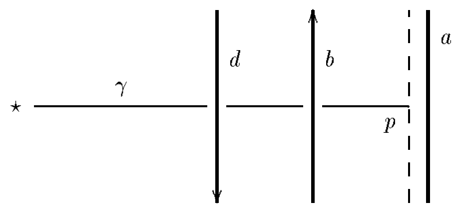
4.7 Theorem *Let D be a diagram for a framed link L in S^3 and $X = [\, S : R \,]$ the rack presented by D, then $X = \Gamma(L)$, in particular $\Gamma(L)$ has a finite presentation.*

Proof We shall define rack homomorphisms

$$\lambda : \Gamma(L) \to X \qquad \mu : X \to \Gamma(L)$$

such that $\lambda \circ \mu = \mu \circ \lambda = \mathrm{id}$.

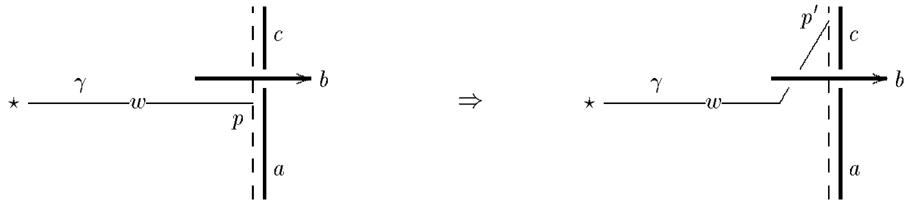
Definition of λ An element of $\Gamma(L)$ is represented by a path γ from a point $p \in M^+$ to $*$. Project γ in general position onto the plane of the diagram D and then read from γ an element of $\mathrm{FR}(S)$ as follows. Suppose that the initial point of γ lies on the arc labelled by the generator a and suppose that γ subsequently passes *under* arcs labelled $b, c, d \ldots$ then associate to γ the element $a^{b^\epsilon c^\epsilon d^\epsilon} \ldots$ where $\epsilon = +1$ if the arc labelled b crosses γ in the right-hand sense and $\epsilon = -1$ otherwise. For an illustration see the following figure.



Read $\lambda(\gamma) = a^{b\bar{d}}$.

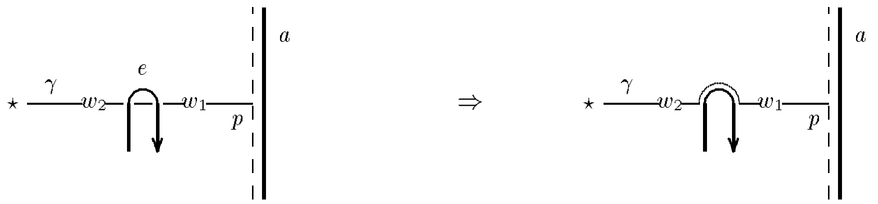
To prove that λ is well defined we have to check that if we change γ by a homotopy then we get the same element of X.

There are three types of critical stages during the homotopy where the expression for $\lambda(\gamma)$ changes. These are illustrated in the following three typical pictures:



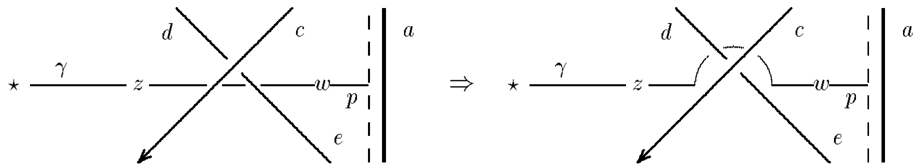
Critical stage type 1

In this picture $c = a^b$ in X. Moving p to p' the value of λ changes to $c^{\overline{b}w} = a^{b\overline{b}w} = a^w$ i.e. we get the same element of X.



Critical stage type 2

In the left-hand picture we read $a^{w_1\overline{e}ew_2}$ whilst in the right-hand picture we read $a^{w_1w_2}$, which is the same element of X.

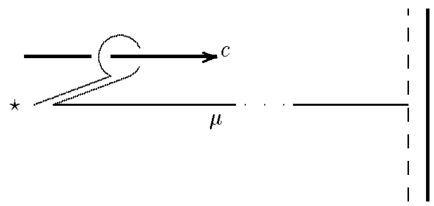


Critical stage type 3

In the left-hand picture we read $a^{we\overline{c}z}$ whilst in the right- hand picture we read $a^{w\overline{c}dz}$. But $e = d^c$ which implies $e \equiv \overline{c}dc$ i.e. $\overline{c}d \equiv e\overline{c}$. These are the same element of X.

Thus $\lambda : \Gamma \to X$ is well defined.

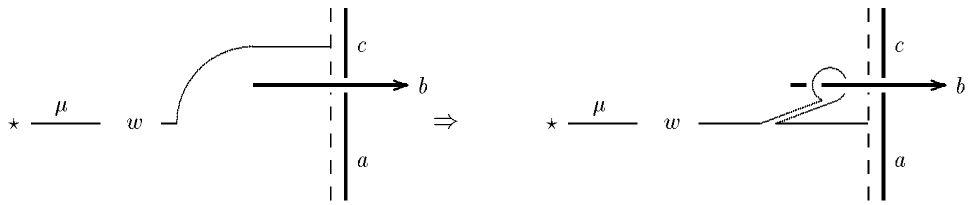
Definition of μ. We start by defining μ on the free rack $\mathrm{FR}(S)$. In what follows we shall misuse notation and write $\mu(a)$ for both the class and the path which represents it. First define $\mu(a)$ where $a \in S$ to be any path from the parallel curve to the arc labelled a to the base-point *over* all other arcs of the diagram. Next suppose that $\mu(x)$ is defined; we will define $\mu(x^c)$ and $\mu(x^{\overline{c}})$ where $c \in S$. This is done by post-composing the path for x with a loop that starts at the base-point, goes over the other arcs to near the arc labelled c, once around this arc in the positive sense for x^c and negative for $x^{\bar{c}}$ and back to the base-point over the other arcs:



This defines μ on $\mathrm{FR}(S)$. To prove that μ is well-defined we have to check that if $a^w \sim b^z$ in the congruence generated by R then $\mu(a^w)$ is homotopic to $\mu(b^z)$.

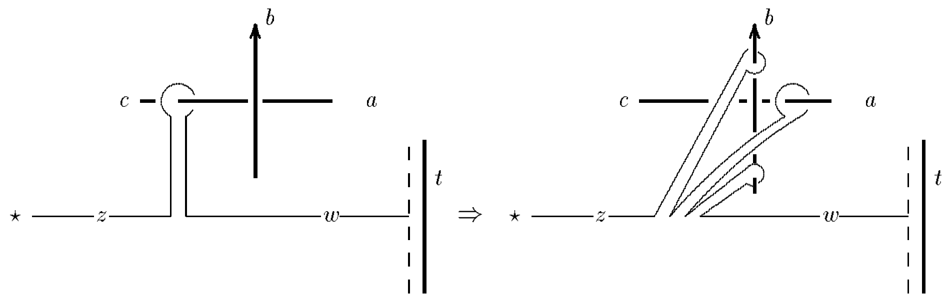
But examining moves (1) to (5) of lemma 4.1 we see that the only non-trivial part to be checked is that if a^w is altered by either a primary or a secondary substitution using relators of X then $\mu(a^w)$ is altered by a homotopy.

Primary substitution. Replace c^w by a^{bw} where $c = a^b$ is a relator.



The picture shows a typical situation, and the required homotopy can be seen.

Secondary substitution. Replace t^{wcz} by $t^{w\bar{b}abz}$ where $c = a^b$.

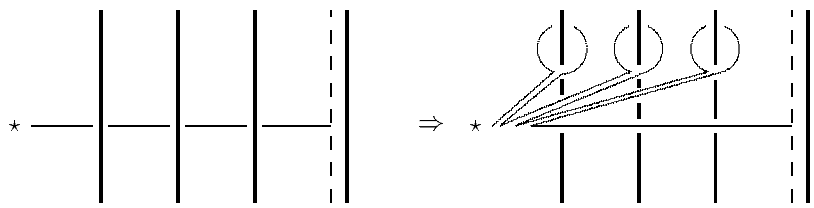


The picture again shows a typical situation, and the required homotopy can be seen.

We have defined

$$\lambda : \Gamma(L) \to X \qquad \mu : X \to \Gamma(L).$$

It is clear from the definitions that $\lambda \circ \mu = \mathrm{id}_X$ whilst to see that $\mu \circ \lambda = \mathrm{id}_\Gamma$ we take an arbitrary path γ and deform it into $\mu \circ \lambda(\gamma)$ by pulling "feelers" back to the base-point, as illustrated in the following figure.



□

Remarks

(1) If we regard the relators R in the diagram presentation as group relators, then we obtain the Wirtinger presentation of the fundamental group. This verifies the fact we already know: the associated group is the fundamental group of the link, (Corollary 3.3).

(2) A similar analysis can be carried out for an embedding of M^n in S^{n+2}: we obtain a "diagram" by projecting onto $\mathbb{R}^{n+1}$ in general position and regarding top dimensional strata (n–dimensional sheets) as "arcs" to be labelled by generators and $(n-1)$–dimensional strata (simple double manifolds) as "crossings" to be labelled by relators. In general position a homotopy between paths only crosses the $(n-1)$ strata and a proof along the lines of the theorem can be given that this determines a finite presentation of the fundamental rack.

(3) There is a general process for obtaining a (not necessarily finite) presentation for *any* codimension 2 embedding by using an analogue of the edge-path presentation for the fundamental group. We leave the details to the interested reader.

We now turn to presentations of the fundamental augmented rack for links in general 3–manifolds. We shall need to enlarge the concept of presentation and this is the content of the remainder of the section.

We shall consider two stages of generalisation. The first (allowing operator relations) does not essentially change the class of racks being considered.

Operator relations

The concept of rack presentation can be generalised by allowing relations which apply only at operator level. For example here is an alternative presentation for the cyclic rack C_n using an operator relation:

$$[a \ : \ a^n \equiv 1].$$

A presentation for a rack with operator relations comprises three sets: a set S of generators, a set R_P of primary relators (as in the first definition of presentation, above) and a set R_O of **operator relators** which are words $w \in F(S)$, to be understood as relations at the operator level $w \equiv 1$.

This concept appears to be more general than the earlier one but in fact it is not:

4.8 Lemma *An operator relator is equivalent to* n *primary relators where* $n = |S|$.

Proof We shall show that the operator relator $w \equiv 1$ is equivalent to the n primary relators

$$a_1^w = a_1, \; a_2^w = a_2, \; \ldots, a_n^w = a_n \quad \text{where} \quad S = \{a_1, a_2, \ldots, a_n\}. \tag{$*$}$$

Since $w \equiv 1$ implies each of the primary relators $a_i^w = a_i$ it suffices to prove the converse, i.e. that $x^w = x$ is a consequence of $(*)$ for each $x \in \mathrm{FR}(S)$. Write $x = a_j^t$, $t \in F(S)$ say and use induction on the length of t.

Suppose that $t = t_1 a_k^\varepsilon$, $\varepsilon = \pm 1$ and for definiteness suppose that $\varepsilon = +1$. By induction

$$a_j^{t_1} = a_j^{t_1 w}$$

is a consequence of $(*)$. Then using the relator moves we have the following consequences:

$$\begin{aligned} x &= a_j^t = a_j^{t_1 a_k} \\ &= a_j^{t_1 w a_k} \quad \text{(move (d))} \\ &= a_j^{t_1 w a_k^w} \quad \text{(move (e) using } a_k^w = a_k) \\ &= a_j^{t_1 w \overline{w} a_k w} \quad \text{(definition)} \\ &= a_j^{t_1 a_k w} = a_j^{tw} = x^w. \end{aligned}$$

The case $\varepsilon = -1$ is similar. □

General presentations

The final generalisation of presentations is to allow operator **generators** as well:

Definitions Given sets S, T the **extended free rack** $\mathrm{FR}(S,T)$ is defined by

$$\mathrm{FR}(S,T) := S \times F(S \cup T) = \{a^w \mid a \in S, w \in F(S \cup T)\}$$

with rack operation given by

$$(a^w)^{(b^z)} = a^{w z^{-1} b z}.$$

The proof that this is a rack is formally identical to the case of the usual free rack (1.3 example 9).

A **general presentation** of a rack comprises four sets: S_P, S_O the **primary** and **operator** generators and R_P, R_O the **primary** and **operator** relators, where elements of R_P are statements of the form $a^w = b^z$ where $a^w, b^z \in \mathrm{FR}(S_P, S_O)$, and elements of R_O are words $w \in \mathrm{FR}(S_P, S_O)$

Now given a general presentation define the congruence $\sim$ on $\mathrm{FR}(S,T)$ to be the smallest congruence containing:

(1) $x \sim y$ if $(x = y) \in R_P$

(2) $z^x \sim z^y$ if $(x = y) \in R_P$

(3) $z^w \sim z$ if $w \in R_O$.

Then the rack generated by the presentation is defined to be:

$$[\, S_P, S_O : R_P, R_O \,] := \frac{\mathrm{FR}(S,T)}{\sim}.$$

It is worth examining a simple example in some detail because the operator generators in general introduce a non-finiteness in any possible primary generating set.

Example $\quad S_P = \{a\} \quad S_O = \{u\} \quad R_P = R_O = \emptyset$.

Here

$$X = [\, S_P, S_O : R_P, R_O \,] = \mathrm{FR}(\{a\}, \{a,u\}) = \{a^w \mid w \in F(a,u)\}$$

the rack structure is given by

$$(a^w)^{(a^z)} = a^{wz\overline{a}z}.$$

So as a *set* X can be identified with the free group $F(a,u)$ but the rack structure is not conjugacy.

Notice that a^w is in the same orbit as a^z if and only if w and z have the same total degree in u and the set of orbits is in bijective corespondence with the set of cosets of $\mathrm{Ker}(F(a,u) \to F(u))$ in $F(a,u)$. Therefore X has infinitely many orbits and hence cannot have a finite primary presentation.

Presentations of augmented racks

The example makes it clear that a general presentation has operator structure not implied by the rack structure, thus a general presentation fits naturally with the idea of augmented racks.

Let $X = [\, S_P, S_O : R_P, R_O \,]$ be a general presentation and let G be the *group* presented by $< S_P \cup S_O, R_P \cup R_O >$. Then there is a natural map $\partial : X \to G$ and G acts on X by the formula for the rack operation. Therefore X is an augmented rack, which we denote

$$\widehat{X} = [\, S_P, S_O : R_P, R_O \,]_G$$

the **augmented rack presented by** $[\, S_P, S_O : R_P, R_O \,]$.

Note: Do not confuse G with the associated group $As(X)$. In the simpler case without operator generators, we can see from lemma 4.8 that G and $As(X)$ are in general different (in this simpler case G is a quotient of $As(X)$: the operator relations do not become trivial in $As(X)$, but *central*, see the first line of the proof of the lemma).

In the general case, even if the presentation of X is finite, $As(X)$ does not necessarily have a finite presentation: it is generated by S_P and all conjugates $w^{-1}aw$ where $a \in S_P$ and $w \in F(S_O)$ with relators R_P and commutators of generators by elements of R_O. However there is an important special case in which the rack (and hence the associated group) **does have** a finite primary presenation, given in the lemma below.

Note that since $\partial : X \to G$ is a rack homomorphism from X to G_{conj}, lemma 2.1 gives a homomorphism $\partial_\sharp : As(X) \to G$.

4.9 Lemma *Suppose that the presentation of X is finite and that $\partial_\sharp$ is onto, then X has a finite primary presentation.*

Proof We shall show how to replace one operator generator by a finite number of primary generators; the result then follows from lemma 4.8.

Let $t \in S_O$. Since $\partial_\sharp$ is onto, we can write t as an element of $As(X)$ as a product of elements of X, i.e.

$$t = a_1^{w_1} a_2^{w_2} \dots a_n^{w_n} \quad \text{in} \quad As(X).$$

Since the operator group is a quotient of the associated group, this implies that

$$t \equiv a_1^{w_1} a_2^{w_2} \dots a_n^{w_n}.$$

Introduce n new primary generators, $b_1, b_2, \dots, b_n$ together with n primary relators $b_i = a_i^{w_i}$ for $i = 1, 2, \dots, n$. Then we can substitute $b_1 b_2 \dots b_n$ for t at operator level and the operator generator t is now redundant and can be deleted. □

Augmented presentations There is also the useful concept of an **augmented presentation** of an augmented rack. This comprises an explicit group G, a set S of generators, a function $\partial : S \to G$ and a set R of relators which are statements of the form $x = y$ where $x, y \in F(S)$, which respect ∂, i.e. such that $\partial x = \partial y$ in G.

The presentation defines an augmented rack $[S : R]_G$ by defining $\sim$ on $\text{FR}(S, G)$ to be the smallest congruence containing:

(1) $x \sim y$ if $(x = y) \in R$

(2) $z^x \sim z^y$ if $(x = y) \in R$

(3) $z^{gh} \sim z^k$ if $gh = k$ in G

and setting

$$[S : R]_G = \frac{\text{FR}(S, G)}{\sim}.$$

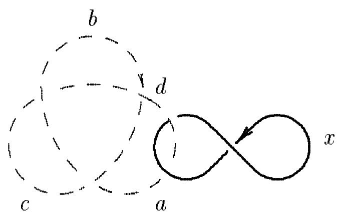
If G is finitely presented, then we can convert this to a general presentation, by adding the presentation of G as operator generators and relators.

Remark There are Tietze type theorems for all the more general classes of presentations, which we shall leave the reader to formulate and prove.

The fundamental augmented rack of a link in a 3–manifold

We finish the section by explaining how to read a presentation of the fundamental augmented rack of a link L in a closed orientable 3–manifold from a diagram. Recall that such a manifold can be obtained by surgery on a framed link in S^3. Thus we can represent L by a diagram in which some of the curves (which we think of as 'red' curves) are the surgery curves, and others ('black' curves) are the actual link components.

We label the black arcs by primary generators and the red arcs by operator generators, then at each crossing where the underarc is black we read a primary relator by the usual rule and at each crossing where the underarc is red we read an operator relator. Finally, for each red curve we read a further operator relator by reading round the curve and noting undercrossings. The whole process is illustrated by the example in the following diagram, where the 'red' surgery curve has been drawn with broken lines.



The presentation is

$$[\{x\},\{a,b,c,d\} : \{x = x^{\bar{x}d}\}, \{b \equiv \bar{d}cd, c \equiv \bar{b}db, a \equiv \bar{c}bc, a \equiv \bar{x}dx, \bar{c}\overline{db}x \equiv 1\}]$$

where the first four operator relations come from the crossings with 'red' underarcs and the final operator relation is obtained by reading undercrossings round the red curve.

We leave the details of the proof of the following theorem to the reader.

4.10 Theorem *The fundamental augmented rack $\widehat{\Gamma}(L)$ is the augmented rack presented by any diagram for L.*

Sketch of proof The group G corresponding to the presentation is the group $\pi_1(Q_0)$ of the link. That the rack is the fundamental rack is then proved in a similar way to theorem 4.7. □

4.11 Corollary *The fundamental rack of a link in a homotopy 3–sphere has a finite primary presentation.*

Proof This follows from lemma 4.9, and the remark below corollary 3.3.

5. The Main Classification Theorem.

The main result of this section is the classification theorem (theorem 5.1 below) which states that the fundamental augmented rack of an irreducible link in a closed connected 3–manifold is a complete invariant for both the link *and* the ambient 3–manifold.

Since for simply connected 3–manifolds the augmented rack and the plain rack coincide, we deduce that the plain fundamental rack is a complete invariant in this case.

There is also a classification theorem for more general links in 3–manifolds, including any link in S^3, which involves the concept of an **oriented** rack. This will be considered at the end of the section.

Definition A link $L : M \subset Q^3$ is **irreducible** if Q is a closed connected 3–manifold and $Q_0 = \text{closure}(Q - N(M))$ is P^2 irreducible.

Remarks

(1) For a link in S^3 irreducibility is the same as being **non-split**, i.e. L is not the disjoint separated union of two non-trivial sublinks.

(2) For a general 3–manifold irreducibility is equivalent to Q_0 being sufficiently large (because a 3–manifold with boundary tori is sufficiently large if and only if it is P^2 irreducible). Thus other reasonable names for an irreducible link would be **non-split** or **sufficiently large**.

(3) Note however that Q^3 *need not be irreducible*: it is well known that *any* closed connected 3–manifold contains an irreducible link, in fact an irreducible knot. For example, as Dale Rolfson remarked to us, the spine of any open book decomposition of a 3–manifold is an irreducible link.

(4) Irreducibility can be detected algebraically from the fundamental rack. This follows from the following lemma.

5.1 Lemma *A semi-framed link $L : M \subset Q^3$ in a closed connected 3–manifold which contains no homotopy 3–sphere summands is reducible if and only if the fundamental augmented rack $\widehat{\Gamma}(L)$ is a non-trivial free product.*

If Q^3 is a homotopy sphere, then the result is also true with $\widehat{\Gamma}(L)$ replaced by the plain rack $\Gamma(L)$.

Proof If the link is reducible, then $\widehat{\Gamma}(L)$ is $X * Y$ where X and Y are non-trivial and are the augmented racks of the connected summands; this can easily be checked from definitions.

Conversely, if $\widehat{\Gamma}(L)$ is a non-trivial free product, then $G = \pi_1(Q_0)$ is a non-trivial free product. It then follows from a standard result in 3–manifolds [12; theorem 7.1] that Q_0 is a connected sum, i.e. L is reducible.

If Q is a homotopy sphere then $\pi_1(Q_0)$ is the associated group of the plain rack $\Gamma(L)$ (corollary 3.3), and the proof is similar to the proof for the augmented rack. □

We now give the main result of this section:

5.2 Classification theorem

The fundamental augmented rack is a complete invariant for irreducible semi-framed links in closed, connected 3–manifolds.

More precisely suppose that $L : M \subset Q$ and $L' : M' \subset Q'$ are two irreducible semi-framed links in closed, connected 3–manifolds and suppose that the fundamental augmented racks are isomorphic:

$$\widehat{\Gamma}(L) \cong \widehat{\Gamma}(L').$$

Then there is a homeomorphism $Q \to Q'$ carrying M to M' as semi-framed submanifolds.

Proof We shall use Waldhausen's classification theorem for P^2 irreducible, sufficiently large 3–manifolds, see Hempel [12; theorem 13.6].

We need a couple of observations.

Observation 1 The orbits of $\widehat{\Gamma}(L)$ are in one–one correspondence with the components of M.

Observation 2 A choice of **base path system** — that is a base point for Q_0 and one on each component of ∂Q_0, in the parallel manifold if appropriate, together with base paths from the base points on ∂Q_0 to the one for Q_0 — is the same as a choice of representatives $\alpha_1, \alpha_2, \ldots, \alpha_t$ of elements of $\widehat{\Gamma}(L)$ one from each orbit.

Thus a choice $x_1, x_2, \ldots, x_t$ of elements of $\widehat{\Gamma}(L)$, one from each orbit, is equivalent to a choice of base path system, up to equivalence generated by homotopy through base path systems.

Now assume that L is framed and choose a base path system for Q_0 and let $x_1, x_2, \ldots, x_t$ be the corresponding choice of elements of $\widehat{\Gamma}(L)$. Then we can read off the following items of the corresponding π_1 system:

$\pi_1(Q_0)$ (the group for $\widehat{\Gamma}(L)$).

The x_i–meridinal subgroup (generated by ∂x_i) for $i = 1, 2, \ldots, t$.

The x_i–longitudinal subgroup, namely its stabiliser in $\pi_1(Q_0)$, see 3.5.

Now the isomorphism of $\widehat{\Gamma}(L)$ with $\widehat{\Gamma}(L')$ carries these items to corresponding items in the π_1 system for Q'_0 given by the base path system determined by the images of $x_1, x_2, \ldots, x_t$. It follows from Waldhausen's theorem that there is a homeomorphism $Q_0 \to Q'_0$ realising this isomorphism of π_1 systems and the correspondence of listed items. Since meridinal subgroups go to meridinal subgroups, this homeomorphism extends to a homeomorphism $Q \to Q'$ carrying M to M', and since longitudinal subgroups go to longitudinal subgroups, this carries M to M' as framed submanifolds.

The proof in the general (semi-framed) case is the same with the longitudinal subgroups replaced by peripheral subgroups on the unframed components. □

Remark The proof of theorem 5.2 makes it clear that the fundamental augmented rack of a link is an algebraic gadget which encapsulates all the information in the fundamental group and peripheral group structure of the complement, without the need for any unnatural choice of base path system. Thus it is the precise algebraic input for Waldhausen's theorem when applied to link complements in closed 3–manifolds.

5.3 Corollary *For a homotopy sphere the* **plain** *fundamental rack is a complete link invariant. More precisely suppose that $L : M \subset Q$ and $L' : M' \subset Q'$ are two irreducible semi-framed links where Q, Q' are homotopy 3–spheres and suppose that the plain fundamental racks are isomorphic:*

$$\Gamma(L) \cong \Gamma(L').$$

Then there is a homeomorphism $Q \to Q'$ carrying M to M' as semi-framed submanifolds, in particular Q and Q' are homeomorphic.

Proof This follows at once from the theorem and the remark below corollary 3.3. □

Remarks

(1) The corollary is deceptively strong: the plain fundamental rack is a complete invariant for **the homotopy spheres themselves** as well as for the links in them. Thus if there is a non-trivial homotopy sphere H^3, then any irreducible link or knot in H^3 will have a fundamental rack **different from the fundamental rack of any link in** S^3. It follows that any rack invariant which vanishes for classical links (links in S^3) could theoretically be used to detect a counterexample to the Poincaré conjecture. We shall return to these ideas in section 7.

(2) There is no chance whatsoever that the corollary could be extended to general 3–manifolds. Indeed we have the following observation about the plain fundamental rack:

5.4 Lemma

Suppose $L : M \subset Q$ is a semi-framed link and that $p : Q' \to Q$ is a covering. Let $M' = p^{-1}(M)$ and $L' : M' \subset Q'$. Then $\Gamma(L) \cong \Gamma(L')$.

Proof This follows at once from the path lifting property of covering spaces. □

Orientations for racks

The proof of theorem 5.2 fails for reducible links because the fundamental rack does not determine the orientation of the components of M. Although the longitudinal *subgroups* are invariant under isomorphism , specific *longitudes* are not. For example consider the disjoint union of two trefoil knots K and K' in S^3. If K say is reflected (changing right-hand to left-hand trefoil or vice versa) then the link changes but the fundamental rack remains the free product of two trefoil racks.

Moreover the conclusion of theorem 5.2 only gives a homeomorphism between the links which may not respect orientations of the components of the link or the ambient space. Consider an oriented link L. Then L has an inverse L^* where the orientations of each component are reversed. In section 2 we considered inverted racks where the new binary operation is $a^{\overline{b}}$. The fundamental rack of the inverse link L^* is the inverted rack $\Gamma(L)^*$ of the fundamental rack $\Gamma(L)$. If the orientation of space is reversed the mirror link $\overline{L}$ is obtained. The fundamental rack of the link $\overline{L^*}$ is isomorphic to the fundamental rack of the link L under an isomorphism induced by the space reversing homeomorphism. It follows that the fundamental rack is not a complete invariant for oriented links L which are not equivalent to their inverted mirror image.

We can avoid these difficulties and extend the theorem to general framed links in S^3 by introducing orientations for racks:

5.5 Definition An **orientation** for the fundamental rack Γ of a framed link in S^3, is a choice, for each component (orbit of Γ), of generator of the (cyclic) stabiliser.

An oriented rack carries the extra information which enables the orientation of the components to be recreated from the algebra. Using oriented racks, the main theorem extends to arbitrary framed links in 3–manifolds.

5.6 Theorem *The oriented fundamental augmented rack is a complete invariant for oriented and semi-framed links in closed connected 3–manifolds which contain no homotopy 3–sphere summands.*

Proof Decompose the fundamental rack $\Gamma(L)$ into a free product of indecomposable racks. By lemma 5.1 this corresponds to the decomposition of Q_0 into its connected summands. A similar decomposition applies to $\Gamma(L')$. Now apply theorem 5.2 to each piece, and then the resulting homeomorphism carries each piece of L to the corresponding piece of L', with determinate orientations. Thus the homeomorphism can be pieced together along the separating spheres to yield the required homeomorphism. □

Remark The result can be extended to 3–manifolds which are not closed under the extra condition that each connected summand of Q_0 meets $\partial N(M)$.

6. Invariants of Links

We have shown that the fundamental rack is a complete invariant for irreducible links in S^3. It follows that, theoretically at least, all invariants of such links can be derived from the fundamental rack. In this section we shall briefly indicate how some old and new invariants are defined in terms of the fundamental rack.

The subject of rack invariants is enormous. Because a rack is such a simple algebraic object (as simple as a group), there are an enormous number of "naturally occurring" racks, some of which we have mentioned in previous sections. Each such rack gives rise to link invariants, and it follows that it is absurdly easy to define new (or apparently new) invariants in this way. In the brief time that we have been studying racks, we have found far more examples of "new" invariants than we have had time to investigate, or even to decide whether they are really new. So this section is just a bare introduction to the subject and we intend to return to study it in the depth that it deserves in future papers.

We shall here consider two ways to define invariants:

(1) **Representation invariants**

(2) **Functorial invariants**

Representation invariants are defined by considering rack homomorphisms (representations) to 'known' racks, and functorial invariants are defined by transforming the fundamental rack (by a functor) into a one of a class of racks with more easily computable invariants.

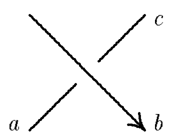
A third method of defining invariants is given by the rack space of the fundamental rack. Any topological invariant of this space is *a fortiori* a link invariant. These invariants are strongly connected with the concepts of **cobordism of links** and the **rack space**, see [10,11], investigated in our future paper on the rack space.

Throughout the section, L will denote a semi-framed link and Γ its fundamental rack.

Representation invariants

Let X be any fixed rack. Then the set $\Omega = \mathrm{Hom}(\Gamma, X)$ of representations (i.e. rack homomorphisms) of Γ in X is a link invariant. If X has any extra structure then this set inherits similar extra structure. For example if X is a **topological rack** (i.e. X is a topological space, the rack operation is continuous in both variables, and $a \mapsto a^b$ is a homeomorphism for each $b \in X$) then Ω is also a topological space the **representation space** of Γ in X.

Now suppose that L is a classical link (i.e. a link in S^3) and suppose that D is a diagram for L. Then a representation $\rho \in \Omega$ of Γ in X can interpreted in terms of the diagram D as a **labelling** of D. In other words each arc of D is labelled by an element of X so that at each crossing the labels satisfy the rule $c = a^b$ where a, b, c are indicated in the following diagram:



(6.1)

Any extra structure on Ω has the obvious interpretation in terms of labels.

If X is a finite rack, then the set Ω of representations can be enumerated in a systematic manner by enumerating all labellings satisfying the above rule. We conjecture that there exists a countable sequence $\{X_i\}$ of finite racks such that the sequence $\{\mathrm{Hom}(\Gamma, X_i)\}$ distinguishes all irreducible classical links.

Now if X is a quandle then there is always the **trivial** representation which is obtained by labelling each arc by the same element. Therefore the crudest invariant with a quandle X is the existence or otherwise of a non-trivial representation. If X is not a quandle there may be no representation at all and the crudest invariant is the existence or otherwise of *any* representation.

6.2 Examples

Example 1 *Conjugation racks*

Let X be a conjugation rack (i.e. a union of conjugacy classes in a group G) then, by corollary 2.2, representations of Γ in X are in bijective correspondence with representations (homomorphisms) of the associated group in G. In the case when L is a classical link, the associated group is the fundamental group and this case has been extensively studied in the literature, see for example the book of Burde and Zieschang [4]. The next example gives a specific case.

Example 2 *The Dihedral Rack.*

Let L be a classical link and let X be the dihedral rack R_n. Representations in R_n may be described as follows: let the arcs of any diagram of L be coloured with the n colours $0, 1, \ldots, n-1$ such that at each crossing if x_a, x_b, x_c are the three colours assigned to the arcs labelled a, b, c in figure 6.1 then the following equation holds;

$$x_c \equiv 2x_b - x_a \mod n$$

If n is prime it is well known that these equations have a non-constant solution if and only if n divides $\Delta(-1)$, the **determinant** of L.

If $n = 3$ this is the well known property of being 3–colourable. For instance the determinant of the trefoil is 3 and of course the trefoil is 3–colourable.

In general a representation into any finite rack could be interpreted as a suitable "colouring scheme" for the diagram.

Example 3 *The Alexander Rack*

Let Λ be the ring of Laurent polynomials $\mathbb{Z}[t, t^{-1}]$ in the variable t with integer coefficients. Any $\Lambda-$module has the structure of a quandle with the rule $a^b := ta + (1-t)b$. The equations needed for a representation to this quandle are

$$x_c = tx_a + (1-t)x_b$$

where the unknowns x correspond as before to figure 6.1. Let $f(t)$ be an irreducible polynomial over the integers. Then the equations above have a non-trivial solution with t a root of f if and only if f divides the Alexander polynomial $\Delta(t)$. We shall consider a more substantial Alexander invariant later in the section.

Example 4 *The Dodecahedral Rack*

If X is the reflection rack whose elements are the edges of a dodecahedron, then the existence of a representation can used to distinguish knots which have determinant ± 1 and so have no non trivial representation to R_n. An example is 10_{124}, see Joyce [16] for details and Azcan [1] for generalisations using Coxeter groups.

All the above examples have used quandles. The remaining three examples of representation invariants use non-quandle racks.

Example 5 *The Cyclic Rack*

Consider the cyclic rack C_n of order n. Define the **total writhe** of a component of a link to be the framing number of the component plus the sum of its linking numbers with the other components. It is easy to see that the link has a representation to C_n if and only if the total writhe of each component is divisible by n.

Example 6 *The (t,s)–Rack*

The (t,s)–rack is a generalisation of the Alexander rack defined in 1.3 example 6. The two dimensional real plane has the structure of a Λ_s module if t, s act linearly as the matrices

$$\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \quad \begin{pmatrix} -u & -u \\ 1 & 1 \end{pmatrix}$$

where $u \neq 0$.

Let X denote this rack. If we seek a non-trivial representation of Γ in X, then we find a number of linear equations in u have to be satisfied. For instance if we take the standard diagram representing the trefoil and with writhe 3 then the existence of a representation depends on the solution of the equations

$$\begin{array}{ccccccccc} ux_1 & - & y_1 & - & uz_1 & - & uz_2 & = & 0 \\ ux_2 & - & y_2 & + & z_1 & + & z_2 & = & 0 \\ uy_1 & - & z_1 & - & ux_1 & - & ux_2 & = & 0 \\ uy_2 & - & z_2 & + & x_1 & + & x_2 & = & 0 \\ uz_1 & - & x_1 & - & uz_1 & - & uz_2 & = & 0 \\ uz_2 & - & x_2 & + & y_1 & + & y_2 & = & 0. \end{array}$$

The polynomial which is the determinant of the matrix of these equations is an invariant of the framed knot.

A more general Λ_s–module structure on a 2–dimensional vector space is given by the matrices

$$\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \quad \begin{pmatrix} -u-x & -y(u+x) \\ y^{-1}(1+x) & 1+x \end{pmatrix}$$

where u and y are non-zero. This leads to a 3–variable polynomial invariant. We have not yet decided whether these "new" polynomials contain any really new information.

These polynomials are invariants of the *framed* knot or link. However there are corresponding invariants of the *unframed* knot or link, see remark 6.3 below

Example 7 *Matrix racks*

There is a way of associating a rack to any matrix or affine group and this leads to whole families of new and computable polynomial invariants for knots and links.

The general construction is this: Given a set X with an action by the group G, we can define a rack structure on $G \times X$ by the formula

$$(g, x)^{(h,y)} := (h^{-1}gh,\ x \cdot h). \qquad (*)$$

If we apply this in the case when G is a matrix or affine group and X the corresponding vector space, then the formula gives a rack structure on (some subset of) a linear space.

A simple example is given by the group of dilations acting on the plane:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b \\ c \end{pmatrix} \qquad a, b, c \in \mathbb{R} \quad a \neq 0$$

then the rack structure is given by

$$(a, b, c, x, y)^{(d,e,f,z,t)} = (a, db - ae + e, dc - af + f, dx + f, dy + e)$$

where a, b, c represent the element of the group as in the equation above, and x, y are the coordinates of the point of the plane.

The formula can be used to define several polynomial invariants including the multivariable Alexander polynomial (in fact this polynomial comes from an even simpler example: the group of affine transformations of a 1–dimensional space, see example 3 below 7.6 and Devine [8] for details).

6.3 Remarks

(1) When using a non-quandle rack, the invariants found for a classical link will depend in general on the *framing* of the link (i.e. the writhe of the diagram). However there is a way to define an infinite family of corresponding invariants of the *unframed* link: we choose arbitrarily integers corresponding to the components of the link and then we choose to frame the link with the unique framing given by setting the writhes equal to these integers. For each choice of integers, we have in this way an invariant of the unframed link.

(2) The general construction for a rack given in $(*)$ above can be further generalised. Let X be a set on which a *rack* Y acts (i.e, for each $y \in Y$ we have a bijection $x \mapsto x \cdot y$ of X such that y^z acts like $\overline{z}yz$, where $x \cdot \overline{z}$ means the pre-image of x under the action of z), then almost the same formula gives a rack structure on $Y \times X$

$$(y, x)^{(t,u)} := (y^t, x \cdot t).$$

Functorial Invariants

Invariants of links can be obtained by applying a functor to the fundamental rack. One advantage of this method of defining invariants is that they automatically apply to arbitrary racks and not just to classical racks.

6.4 Examples

Example 1 *The Associated Group*

The associated group functor is an example. However, as we have seen in proposition 3.2, this leads to an existing topological invariant $\pi_2(Q, Q_0)$, the associated group of the link.

Example 2 *The Alexander module*

The following is, we believe, a new invariant which generalises the definition of the Alexander polynomial of a knot or link.

Let Λ be the ring of Laurent polynomials $\mathbb{Z}[t, t^{-1}]$ in the variable t with integer coefficients. As we have seen above any Λ–module has the structure of a quandle with the rule $a^b := ta + (1-t)b$. This means that $\mathbb{MOD}$, the category of Λ–modules is a subcategory of $\mathbb{RACKS}$, the category of racks. There is a functor

$$A : \mathbb{RACKS} \longrightarrow \mathbb{MOD}$$

the **Alexander functor**, which is a left adjoint to the inclusion defined as follows; If X is a rack let $A(X)$ be the Λ–module with generators $\{u_a | a \in X\}$ and relations

$$u_{a^b} = tu_a + (1-t)u_b \qquad a, b \in X.$$

In the case of classical links the Alexander functor takes the fundamental rack into the usual Alexander module. However our invariant is defined for an arbitrary codimension 2 link.

For the generalisation to Alexander modules with many variables see Devine [8].

Example 3 *The (t,s)–module*

The Alexander module can be generalised to give a non-quandle module by replacing Λ by Λ_s in the last example and modifying the relations to

$$u_{a^b} = tu_a + su_b \qquad a, b \in X.$$

Example 4 *Verbal Groups*

The associated group functor can be generalised.

Let X be a rack. We consider the group with generators $\{g_x | x \in X\}$ and relations

$$g_{a^b} = w(g_a, g_b)$$

where $w(x, y)$ is a fixed word in two variables. In order for the group to be well defined, w has to satisfy two conditions which are the analogues of the rack laws. Examples are $w = y^{-1}xy$ which yields the associated group and $w = yx^{-1}y$ which yields the **associated core group**. Kelly [18] has proved that these are the *only* such examples. However, as Kelly shows, there is a further generalisation: replace g_x by a fixed set of say n generators and w by n words in $2n$ variables; there are then many new examples. For instance in the case $n = 2$, Kelly found, in a computer search, more than 500 new examples including several infinite families.

There is much work to be done to categorise and to calculate these new invariants.

7. Racks and Braids

In this section we shall explore the relationship between racks and braids. We shall show that there is a faithful representation of the braid group B_n on n strings in the automorphism group $Aut(\mathrm{FR}_n)$ of the free rack on n elements. This representation can be used to define and to calculate link invariants. Moreover a classical result of Artin can be adapted to characterise the image of the represention, and we shall apply this result to give a characterisation of the fundamental rack of a link in a 3–manifold.

We call a rack **classical** if it is the fundamental rack of a link in S^3. We shall give separate characterisations for classical racks and for the fundamental augmented racks of links in general oriented 3–manifolds. These results complement the main classification theorem of section 5.

Unfortunately the characterisations are far from practical. Practical versions would be very important. In particular an algorithmic version would partially solve the homeomorphism problem for S^3. We shall discuss this and the connection with the Poincaré conjecture at the end of the section.

The braid groups

Let $L_n : \{P_1, \ldots, P_n\} \subset D^2$ be a fixed link comprising n points in the interior of the 2–disc. Let L_n^+ be the framed version of L_n, which we can think of as comprising n standard little discs which are reduced copies of D^2.

A **braid** on n strings is an equivalence class of links $\beta : A \subset D^2 \times I$, where A comprises n arcs each of which meets every level $D^2 \times \{t\}$ in a single point, and such that $\beta \cap (D^2 \times \{i\}) = L_n$ for $i = 0, 1$. The equivalence is isotopy through similar links. Similarly a **framed braid** is an equivalence class of framed links of the same type, such that $\beta \cap (D^2 \times \{i\}) = L_n^+$ for $i = 0, 1$.

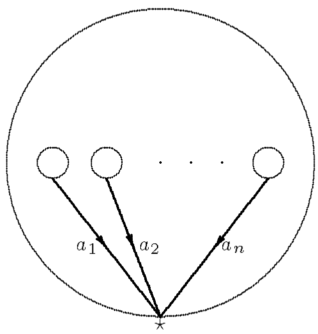
It is well known that braids on n strings form a group the **braid group** B_n, with composition defined by stacking two braids one above the other. Similarly, framed braids form the **framed braid group** FB_n. A braid or framed braid determines a permutation $\pi \in S_n$ of $\{P_1, \ldots, P_n\}$ by following the strings from top to bottom, and this defines surjective homomorphisms $B_n \to S_n$ and $FB_n \to S_n$.

Now an unframed braid has a standard framing defined by transporting the little discs down the strings, keeping them parallel throughout. In general a framed braid can be regarded as a braid with an integer attached to each string, which counts the total rotation of the little disc as the string is traversed from top to bottom. The standard framing corresponding to zeros. In composition the integers on the two pieces of the string are added.

Thus FB_n is the **wreath product** $\mathbb{Z} \wr B_n$ that is the semi-direct product of $\mathbb{Z}^n$ with B_n where the action of B_n on $\mathbb{Z}^n$ is given by permuting the factors using the homomorphism of B_n to S_n.

The representations of B_n and FB_n

Let β be a framed braid. We shall consider the fundamental racks $\Gamma(L_n^+)$ and $\Gamma(\beta)$. Now a braid can be 'unbraided' and any framing can be 'untwisted' hence, as a link, $\beta \cong L_n^+ \times I$. Therefore the fundamental racks are isomorphic. Moreover the fundamental rack $\Gamma(L_n^+)$ can be identified with the free rack $\mathrm{FR}_n = \mathrm{FR}\{a_1, a_2, \ldots, a_n\}$ on n generators as pictured:



Thus we have isomorphisms

$$\mathrm{FR}_n = \Gamma(L_n^+) \xrightarrow{i_0} \Gamma(\beta) \xleftarrow{i_1} \Gamma(L_n^+) = \mathrm{FR}_n$$

where i_0, i_1 are induced by inclusions of L_n^+ in β at $D^2 \times 0$, $D^2 \times 1$ respectively. The composition $i_1^{-1} i_0$ is an isomorphism

$$\beta_* : \mathrm{FR}_n \to \mathrm{FR}_n$$

of the free rack.

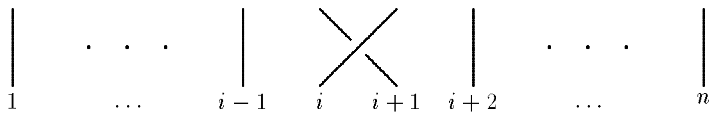
It is easy to check that $\beta \mapsto \beta_*$ is a homomorphism

$$\lambda : FB_n \to Aut(\mathrm{FR}_n).$$

In theorem 7.3 below we shall show that λ is a faithful representation of the braid groups in the automorphism group of the free rack.

Example

Let σ_i be the braid which is the simple interchange of the ith and $(i+1)$st strings and which keeps the other strings fixed. Let σ_i have the standard framing.



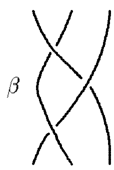
Then σ_i induces the automorphism σ_{i*}

$$\begin{cases} a_i & \mapsto & a_{i+1}^{a_i} \\ a_{i+1} & \mapsto & a_i \\ a_j & \mapsto & a_j \quad j \neq i, i+1. \end{cases}$$

The automorphism β_* for any braid β with standard framing can now be calculated since β can be written as a word in the σ_i.

The general form of an automorphism of FR_n is $a_i \mapsto a_{\pi(i)}^{w_i}$ where π is a permutation of the set $\{1, 2, \ldots, n\}$ and w_i for $i = 1, 2, \ldots, n$ are words in the free group $F(a_1, \ldots, a_n)$, which satisfy certain conditions obtained by considering the analogue of Nielsen theory for racks. This theory is given in an appendix to the paper. We shall not need to consider these conditions in detail in this section. In general the above formula defines a monomorphism of FR_n to itself, (indeed the same formula defines a monomorphism of the free group) and this is also proved in the appendix, see corollary 8.6.

7.1 Example Let β be the following illustrated braid, again with standard framing:



then β_* is given by

$$\pi = (13) \quad w_1 = a_2^{-1} a_1^{-1} a_2 \quad w_2 = a_2^{-1} a_1 a_2 a_3 a_2^{-1} a_1^{-1} a_2 \quad w_3 = a_2.$$

For a braid with non-standard framing, we decompose the braid into interchanges and twists (the identity braid but with one string, say the i–th, framed ± 1). The automorphism for a twist on the i–th string is

$$\begin{cases} a_i & \mapsto & a_i^{\pm a_i} \\ a_j & \mapsto & a_j \quad j \neq i. \end{cases}$$

Later in the section we shall show how to read the automorphism β_* from the diagram, without decomposing β into elementary interchanges and twists.

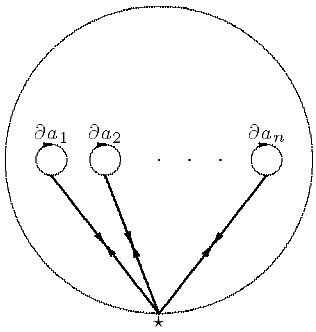
Similar considerations apply to unframed braids, and we have the representation $\overline{\lambda} : B_n \to Aut(\mathrm{FQ}_n)$ where FQ_n is the free quandle on n generators. An element of $Aut(\mathrm{FQ}_n)$ is again defined by a permutation π and n words w_i but now the words w_i are only determined up to premultiplication by powers of $a_{\pi(i)}$ (because

$a_{\pi(i)}^{a_{\pi(i)}} = a_{\pi(i)}$ in a quandle). Note that FQ_n is the associated quandle to FR_n and that the natural homomorphism $Aut(\mathrm{FR}_n) \to Aut(\mathrm{FQ}_n)$ is given by using the same permuation π and words w_i. Further we can regard $Aut(\mathrm{FQ}_n)$ as a subgroup of $Aut(\mathrm{FR}_n)$ by choosing to premultiply w_i by the unique power of $a_{\pi(i)}$ which makes the total power of $a_{\pi(i)}$ in w_i zero. A general element of $Aut(\mathrm{FR}_n)$ is then an element of $Aut(\mathrm{FQ}_n)$ together with an integer for each w_i giving the total degree of $a_{\pi(i)}$. Thus we have another wreath product

$$Aut(\mathrm{FR}_n) = \mathbb{Z} \wr Aut(\mathrm{FQ}_n)$$

and the homomorphism $\lambda : FB_n \to Aut(\mathrm{FR}_n)$ carries one wreath product structure into the other.

Finally we have connections with the free group F_n on n generators. The associated groups of both $\Gamma(L_n)$ and $\Gamma(L_n^+)$ can be identified with the fundamental group $\pi_1(D_0^2)$ of the disc minus the n little discs, which is the free group $F_n = F(\partial a_1, \partial a_2, \ldots, \partial a_n)$ on the n loops illustrated:



We shall usually use the symbols $a_1, a_2, \ldots, a_n$ for the generators of F_n, rather than the more accurate $\partial a_1, \partial a_2, \ldots, \partial a_n$, whenever no confusion is likely to arise. Given a possibly framed braid β, then β_* induces an automorphism $\beta_{*\sharp}$ of the free group F_n which can be described in the same way as β_* using the fundamental groups of β and L_n in place of the fundamental racks. Thus we have further representations of B_n and FB_n in the automorphism group $Aut(F_n)$ of the free group.

In summary we have the following commuting diagram.

$$\begin{array}{ccccc} B_n & \xrightarrow{\bar{\lambda}} & Aut(\mathrm{FQ}_n) & \longrightarrow & Aut(F_n) \\ \downarrow & & \downarrow & & \\ \mathbb{Z} \wr B_n & \xrightarrow{\lambda} & \mathbb{Z} \wr Aut(\mathrm{FQ}_n) & & \\ \| & & \| & & \\ FB_n & & Aut(\mathrm{FR}_n) & & \end{array}$$

A classical result of Artin shows that $B_n \to Aut(F_n)$ is injective. In fact all the maps in the diagram are injective.

Remark It can be readily seen that the homomorphism $Aut(\mathrm{FQ}_n) \to Aut(F_n)$ is injective. This is because the image of the automorphism $a_i \mapsto a_{\pi(i)}^{w_i}$ in $Aut(F_n)$ is given by $a_i \mapsto w_i^{-1}a_{\pi(i)}w_i$ and then $a_{\pi(i)}$ and w_i are determined by this word in the free group up to premultiplication of w_i by a power of $a_{\pi(i)}$.

The Artin condition

Definition The permutation $\pi \in S_n$ and words $w_i \in F_n$, $i = 1, 2, \ldots n$ are said to satisfy the **Artin condition** if the identity

$$\prod_{i=1}^{n} a_i = \prod_{i=1}^{n} w_i^{-1} a_{\pi(i)} w_i$$

holds in the free group. In this case, we say that the homomorphism $g : \mathrm{FR}_n \to \mathrm{FR}_n$ defined by $a_i \mapsto a_{\pi(i)}^{w_i}$ is **Artin**. In theorem 7.3 below, we shall show that Artin homomorphisms are in fact automorphisms.
For example in 7.1 above

$$\begin{aligned}\prod_{i=1}^{n} & w_i^{-1} a_{\pi(i)} w_i \\ &= (a_2^{-1}a_1a_2\ a_3\ a_2^{-1}a_1^{-1}a_2)(a_2^{-1}a_1a_2a_3^{-1}a_2^{-1}a_1^{-1}a_2\ a_2\ a_2^{-1}a_1a_2a_3a_2^{-1}a_1^{-1}a_2)(a_2^{-1}\ a_1\ a_2) \\ &= a_1a_2a_3.\end{aligned}$$

A similar definition works for FQ_n since the words $w_i^{-1}a_{\pi(i)}w_i$ are well defined by the automorphism.

7.2 Lemma *An automorphism $g \in Aut(FR_n)$ or $Aut(FQ_n)$ determined by a braid or framed braid is Artin.*

Proof The element $\partial a_1 \partial a_2 \cdots \partial a_n$ of the free group regarded as an element of $\pi_1(D_0^2)$ is represented by the boundary ∂D^2 of the disc. But, from the definition of the induced automorphism of the free group, this element maps to itself, since ∂D^2 can be homotoped down $\partial D^2 \times I$ outside the braid, from top to bottom.

But the right hand side of the Artin condition is precisely the image of this element under the induced automorphism. □

7.3 Theorem *An Artin homomorphism is an automorphism. Moreover the homomorphism $\lambda : FB_n \to Aut(FR_n)$ defined above is injective and the image of λ consists precisely of Artin automorphisms.*

Remark By the commuting diagram given earlier, the theorem is equivalent to the classical result of Artin, see Birman [2; theorem 1.9], where a combinatorial proof can be found. However, the rephrasing of the result in terms of racks has an advantage because it leads to a simple geometric proof along the lines of the main result of section 5.

Proof To prove the theorem we need to show that if g is an Artin homomorphism of FR_n to itself then there is a unique braid β such that $\beta_* = g$. We shall start by proving the there is a homeomorphism h of D_0^2 (i.e. D^2 with n little discs around $P_1, P_2, \ldots, P_n$ removed) such that $h|\partial D^2 = \text{id}$ and $h_* = g$. Moreover h is unique up to isotopy.

To prove this we shall use the 2–dimensional version of Waldhausen's theorem [12; theorem 13.1]. A homeomorphism h of D_0^2 is determined up to isotopy by its effect on $\pi_1(D_0^2) = F_n$ and the peripheral structure. But the elements $a_1, a_2, \ldots, a_n \in \Gamma$ define a base path system for D_0^2 (see the proof of theorem 5.2) and their images $g(a_1), g(a_2), \ldots, g(a_n)$ another base path system. Moreover by corollary 8.6 g induces a monomorphism $g_\sharp$ of $\pi_1(D_0^2)$ to itself and we have the peripheral structure given by the inner loops $\partial a_1, \partial a_2, \ldots, \partial a_n$ and the outside loop $\partial a_1 \partial a_2 \cdots \partial a_n$ which is the image in $As(\Gamma)$ of $a_1 a_2 \cdots a_n$. Thus the n inner loops are mapped by $g_\sharp$ to corresponding inner loops in the other system and the Artin condition says precisely that the outer loop maps to itself. Thus by the 'Waldhausen' theorem quoted there is a homeomorphism h, unique up to isotopy, such that $h|\partial D^2 = \text{id}$ and $h_* = g$.

The connection with braids is well known. We extend h to a homeomorphism $D^2 \to D^2$ by inserting the little discs (which are permuted by parallel translations). Then there is an isotopy relative to the boundary of the identity to h and this isotopy restricted to the little discs gives a braid β such that $\beta_* = g$. This sets up an isomorphism between the braid group and the group of isotopy classes of homeomorphisms of D^2 which satisfy the same conditions as h, therefore β is unique. □

Reading the automorphism from the braid

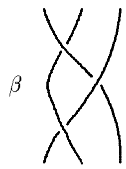
We now give the promised recipe for reading the automorphism β_* from the braid β. Assume that we have a diagram for the braid and assume first that the framing is standard. Orient the strings of the braid downward and label the fixed points at the top of the braid $P_1^+, P_2^+, \ldots, P_n^+$. In a similar fashion label the fixed points at the bottom of the braid $P_1^-, P_2^-, \ldots, P_n^-$.

Starting at the bottom of the braid label the arc starting at P_j^- by $a_j, j = 1, 2, \ldots, n$. Now continue up the braid and label arcs which start at a crossing points using the rules for labelling arcs given in section 4. The labels are all elements of the free rack $\text{FR}(a_1, a_2, \ldots, a_n)$.

Suppose now that the top arc of the string which started at P_j^- finishes at P_i^+ with label a_j^w. Then put $w_i = w$ and $\pi(i) = j$. In other words the label on P_i^+ is $a_{\pi(i)}^{w_i}$, and the automorphism can be read from the top labels, as we see from the following example.

Example We shall check the rule for the braid of example 7.1:

$$\beta_* \left\{ \begin{array}{ccc} a_1 & a_2 & a_3 \\ \downarrow & \downarrow & \downarrow \\ a & b & c \end{array} \right.$$



$$a = a_3^{\overline{a_1^{a_2}}} = a_3^{\overline{a_2 a_1} a_2}$$

$$b = a_2^a = a_2^{\overline{a_2} a_1 a_2 a_3 \overline{a_2 a_1} a_2}$$

$$c = a_1^{a_2}$$

For general framings, we correct the framing of the diagram by inserting little twists (see lemma 4.5), and use the same method. By the results of section 4 (see in particular figures 3 to 6) it makes no difference where the little twists are inserted, or how the braid is respresented as a diagram.

To prove that the method gives the correct result, we make the following observations:

(1) It gives the correct result for a braid which is a simple interchange (σ_i) or a single twist. This is readily checked by hand.

(2) The method gives a homomorphism $\mu : B_n \to Aut(\mathrm{FR}_n)$.

To see this consider the effect of stacking the braid β' on top of the braid β. If the i–th point at the bottom of β is labelled a_i and at the top is labelled $a_{\pi(i)}^{w_i}$, then the labels at the top of the combined braid are obtained from those for β' by substituting $a_{\pi(i)}^{w_i}$ for a_i. But this is precisely how the composition $\beta'_* \circ \beta_*$ of the two automorphism of FR_n is formed.

Since any framed braid can be decomposed into simple interchanges and twists, it now follows that $\lambda = \mu$, i.e. the method gives the correct result.

We can now give our characterisation of classical racks.

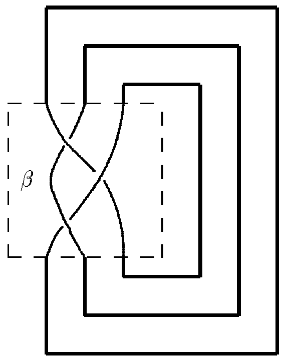
7.4 Theorem: Characterisation of classical racks

A rack is the fundamental rack of a framed link in S^3 if and only if it has a primary presentation of the form

$$[\, a_1, a_2, \ldots, a_n \ : \ a_1 = a_{\pi(1)}^{w_1}, a_2 = a_{\pi(2)}^{w_2}, \ldots, a_n = a_{\pi(n)}^{w_n} \,]$$

where $\pi, w_1, w_2, \ldots, w_n$ satisfy the Artin condition.

Proof Let β be a framed braid. The **closure** $C(\beta)$ of β is the link in S^3 obtained by joining the top of the braid to the bottom by n arcs 'round the back' with standard framing.



By a theorem of Alexander (see e.g. [2]) any framed link $L : M \subset S^3$ can be represented as the closure $C(\beta)$ of a braid (in fact this can be done so that the framing of L is given by the standard framing of β though we shall not need to use this fact).

Now we can read a presentation for the fundamental rack $\Gamma(L)$ from the diagram for $C(\beta)$ by the methods of section 4. Moreover, by using the same lebels as in the above discussion, we see that $\Gamma(L)$ has the presentation $[\ldots, a_i, \ldots | \ldots, a_i = a_{\pi(i)}^{w_i}, \ldots]$ where $a_i \mapsto a_{\pi(i)}^{w_i}$ is the atomorphism of FR_n induced by β. Thus $\Gamma(L)$ has a presentation of the required form by theorem 7.3.

Conversely suppose we are given a rack Γ with a presentation of this form then, again by theorem 7.3, there is a braid β which induces the automorphism $a_i \mapsto a_{\pi(i)}^{w_i}$. Then the closure of β has fundamental rack isomorphic to Γ. □

Remarks

(1) The theorem has content. For example any rack whose associated group has torsion is not classical.

(2) There is a similar characterisation of the fundamental racks of semi-framed or unframed links in S^3. Here the presentation has the same form but with extra relations $a_j^{a_j} = a_j$ corresponding to the unframed components. The proof is essentially the same.

(3) Markov's theorem (see e.g. [2]) can be combined with the theorem to give an algebraic classification of classical links. By Markov's theorem, any two braids which have isotopic closures are related by a series of moves. The resulting presentations of the fundamental rack are therefore also related by a series of moves through presentations of the same form. We leave the details to an interested reader.

Links in general 3–manifolds

We now adapt the last result to give an algebraic characterisation for the fundamental augmented rack of a link in any closed orientable 3–manifold. As at the end of section 4, we shall regard the 3–manifold as given by surgery on a link in S^3 and by Alexander's theorem we can represent a diagram for the surgery curves together with the actual link as the closure of a braid β, with t 'red' strings and n 'black' strings, where the closures of the 'red' strings represent the surgery curves and the 'black' the link curves. By a suitable conjugacy of the braid, we may suppose that the t 'red' strings start (and finish) at the t right-most positions.

Let $\pi \in S_n$ be the permutation given by the black strings and $\sigma \in S_t$ the permutation given by the red strings. Denote by $\pi|\sigma$ the permutation in S_{n+t} of all $n+t$ strings.

We need to decompose σ into its cycle decomposition. By a further conjugacy of the braid we can suppose that this decomposition is in fact of the form

$$\sigma = (1, 2, \ldots, l_1)(l_1+1, l_1+2, \ldots, l_2)\cdots(l_{p-1}+1, \ldots, l_p).$$

Suppose that β induces the automorphism

$$\begin{aligned} a_i &\mapsto a_{\pi(i)}^{w_i} \quad i = 1, 2, \ldots, n \\ b_i &\mapsto b_{\sigma(i)}^{z_i} \quad i = 1, 2, \ldots, t \end{aligned}$$

of the free rack FR_{n+t}, where we have used $b_1, b_2, \ldots, b_n$ for the last t generators in order to distinguish the surgery strings from the genuine link strings. Note that w_i and z_i are words in $F_{n+t} = F(a_1, \ldots, a_n, b_1, \ldots, b_t)$.

We can now read off an augmented presentation of the fundamental augmented rack of L using the recipe given at the end of section 4.

S_P The primary generators $\{a_1, a_2, \ldots, a_n\}$.

S_O The operator generators $\{b_1, b_2, \ldots, b_t\}$.

R_P The primary relators $\{a_i = a_{\pi(i)}, i = 1, 2, \ldots, n\}$.

R_O The operator relators $\{b_i \equiv b_{\sigma(i)}, i = 1, 2, \ldots, t\}$, together with the p further operator relators

$$\{z_{l_1} z_{l_1-1} \cdots z_1, \quad z_{l_2} z_{l_2+1} \cdots z_{l_1+1}, \quad \ldots, \quad z_{l_p} z_{l_p+1} \cdots z_{l_{p-1}+1}\}.$$

The last set of operator relators are the ones which come from reading around the surgery curves noting undercrossings.

The following theorem is proved in a similar way to the last theorem:

7.5 Theorem: Characterisation of fundamental augmented racks in 3–manifolds

An augmented rack is the fundamental rack of a framed link in a closed oriented 3–manifold if and only is it has a presentation of the form listed above such that the permutation $\pi|\sigma$ and the words $w_1, w_2, \ldots, w_n, z_1, z_2, \ldots, z_t$ satisfy the Artin condition.

Remarks

(1) Again there is a semi-framed version of the theorem, which we leave the reader to formulate.

(2) Again the theorem can be combined with a 'Markov' theorem to give an algebraic classification of links in terms of moves through presentations of the same type. The appropriate theorem here is the extension of Markov's theorem to general 3–manifolds contained in Lambropoulou's thesis [21].

Invariants

The connections between braids and racks can be used to define and calculate invariants of classical links. As with all discussions of invariants in this paper, we shall content ourselves here with a brief outline of the methods and return to discuss the subject in greater depth in future papers.

The key idea is an extension of an idea of Brieskorn [3; proposition 3.1]. Given any rack X, there is an action of the automorphism group $Aut(\mathrm{FR}_n)$ on X^n as follows. Let $f \in Aut(\mathrm{FR}_n)$ and let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in X^n$. Define $j_{\mathbf{x}}$ to be the unique rack homomorphism $\mathrm{FR}_n \to X$ such that $a_i \mapsto x_i$ for each i. Then the action f_* of f on X^n is given by

$$f_*(\mathbf{x}) := (j_{\mathbf{x}}(f(a_1)), j_{\mathbf{x}}(f(a_2)), \ldots, j_{\mathbf{x}}(f(a_n))).$$

For an automorphism $f \in Aut(\mathrm{FR}_n)$ which comes from a braid, f_* has a simple interpretation: label the strings at the bottom by the elements $x_1, x_2, \ldots, x_n \in X$, use the usual rules to carry the labels up the braid. Then the labels at the top are $j_{\mathbf{x}}(f(a_1)), j_{\mathbf{x}}(f(a_2)), \ldots, j_{\mathbf{x}}(f(a_n))$. That the two descriptions of the action coincide follows from the method of reading the automorphism from the braid given earlier in the section.

This action can be used to define R–matrices, see remark 7.7 near the end of the paper.

Now let β be a fixed framed braid and let β_* be the corresponding automorphism of FR_n. Consider the closure $C(\beta)$ and let Γ be the fundamental rack of $C(\beta)$. Then a representation of Γ in X is a labelling of $C(\beta)$ by elements of X i.e. an n–tuple $(x_1, x_2, \ldots, x_n) \in X^n$ such that $x_i = j_{\mathbf{x}}(f(a_i))$ for each i, in other words *a fixed point of* f_*. The following result now follows quickly.

7.6 Proposition *The set of representations $\Omega = Hom(\Gamma, X)$ is in natural bijection with the fixed point set of f_*.*

Remark This result can be extended to a larger class of racks. Let $f \in Aut(\mathrm{FR}_n)$ and let $\sim$ be the smallest congruence on FR_n such that $x \sim f(x)$ for all $x \in \mathrm{FR}_n$. Define the **almost classical rack** FR_n/f to be FR_n quotiented out by the congruence $\sim$.

The proposition applies to almost classical racks, because a representation ρ of FR_n/f in X determines the n–tuple $\mathbf{x} = (\rho a_1, \rho a_2, \ldots, \rho a_n)$ such that $f_*(\mathbf{x}) = \mathbf{x}$, and conversely.

The usefulness of the result is best demonstrated by examples:

Example 1 *The extended Burau representation*

Let Λ be the ring of Laurent polynomials $\mathbb{Z}[t,t^{-1}]$ in the variable t with integer coefficients and with rack operation $a^b = ta+(1-t)b$. Then the action of $Aut(\mathrm{FR}_n)$ on $X^n = \Lambda^n$ is an extension of the Burau representation and gives a representation $B : Aut(\mathrm{FR}_n) \to GL_n(\Lambda)$.

Now let $f = \lambda(\beta)$ as before, then the fixed point set of f_* is defined by the equation $B(f)\mathbf{x} = \mathbf{x}$ (where $B(f)$ is an $n \times n$ matrix with entries in Λ). It follows that the eigenspace of $B(f)$ with eigenvalue 1 is an invariant of the link. The condition for this eigenspace to be non-empty, namely $|B(f) - I| = 0$, is the Alexander polynomial of the link.

In fact substantially more invariants of $B(f)$ than just this eigenspace are invariants of the link, since the matrix determines the Alexander module, see example 4 below.

Example 2 *The (t, s)–rack*

We shall describe the extension of the above Burau representation to the ring Λ_s explicitly. The automorphism group of the free rack $Aut(\mathrm{FR}_n)$ is generated by permutation of the generators and **elementary** isomorphisms:

$$\begin{cases} a_i & \mapsto & a_i^{a_j} \\ a_k & \mapsto & a_k \quad k \neq i. \end{cases}$$

The corresponding matrices are permutation matrices and "elementary" matrices obtained from the unit matrix by replacing the i–th diagonal entry by t and the (i, j)–th entry by s. If $i = j$ then replace the i–th diagonal entry by $s + t$. In explicit examples, for instance the (2×2) matrix examples of 6.2 example 6, the entries are regarded as blocks and the blocks s and t replaced by the appropriate smaller matrices.

Again the eigenspace contains polynomial information and again we can deduce further invariants, see example 5 below.

Example 3 *A sample matrix rack*

Let X be the matrix rack (6.2 example 7) obtained by considering the action of the 1–dimensional affine group $x \mapsto ax + b$ on $\mathbb{R}^1$. Then the rack structure on X is given by

$$(a, b, x)^{(c,d,y)} = (a, cb - ad + d, cx + d).$$

The corresponding representation $Aut(\mathrm{FR}_n) \to X^n = \mathbb{R}^{3n}$ can be quickly written down explicitly as in the last example, and is multilinear.

Several multivariable polynomials can be read from this representation including the multivariable Alexander polynomial (Devine [8]).

Remark In all these examples there is link information contained in the eigenspaces other than that with eigenvalue 1, because they correspond to representations in the appropriate projectivised linear rack.

Functorial invariants

So far we have used the connection with braids to indicate how to read off *representation* invariants of almost classical racks. But *functorial* invariants can also be read. The observation that we need is the following: Suppose we are given a functor

$$\mathcal{F} : \mathbb{RACKS} \longrightarrow \mathbb{RACKS}$$

and an automorphism $f \in Aut(\mathrm{FR}_n)$. Let $\mathcal{F}(n)$ denote $\mathcal{F}(\mathrm{FR}_n)$, then by functoriality we have the induced automorphism $f_\flat : \mathcal{F}(n) \to \mathcal{F}(n)$. Therefore if $\Gamma = \mathrm{FR}_n/f$ then $\mathcal{F}(\Gamma) = \mathcal{F}(n)/f_\flat$.

Example 4 *The Alexander module*

The relevant functor here was defined in 6.3 example 2. Here $\mathcal{F}(n)$ is the free n–dimensional Λ module and $f_\flat$ is the Burau matrix $B(f)$. It follows that all the invariants of $B(f)$ which are invariants of the module $\mathcal{F}(n)/B(f)$ are invariants of the link. For example we could take all the polynomials in the Smith normal form.

Example 5 *The (t,s)–module*

Once again we can generalise to the (t,s)–rack. Here $B(f)$ is replaced by the (t,s) matrix described explicitly in example 2 above (or in particular matrix representations, by the block matrix obtained by substitution) and again many polynomials invariants of the link can be read. For example we can take one of the variables to be the "variable" and the rest to be "fixed" and then consider the Smith normal form, which gives us in general n multivariable polynomials.

7.7 Remark *Racks and R–matrices*

The representation of $Aut(\mathrm{FR}_n)$ as permutations of X^n defined earlier restricts to a representation of B_n. In detail this is defined as follows. There is a bijection $T : X^2 \to X^2$ defined by $T(a,b) := (b, a^b)$ and further bijections $T_i : X^n \to X^n$ for all $n \geq 2$ and $1 \leq i < n$ defined by $T_i := I_{i-1} \times T \times I_{n-i-1}$ where I_i is the identity map on X^i. The representation $B_n \to \mathrm{Perm}(X^n)$ of the braid group is then given by $\sigma_i \to T_i$.

Now suppose that X is in fact a based module over a ring Λ then this representation is given by an "R–matrix" with entries in Λ (cf. Kauffman [17] for details in the case of the Alexander rack).

The other examples of racks which are also modules given earlier in the paper also define R–matrices in this way. This suggests the possibility of using the theory of racks to define 3–manifold invariants in the spirit of Reshetikhin and Turaev [26].

Final remarks

(1) Algorithms

We finish this paper by making some remarks on the homeomorphism problem for S^3 and the Poincaré conjecture. The classification of classical racks given in theorem

7.3 is far from algorithmic. Now an algorithmic version would be very important. Suppose we are given a possible counterexample H^3 to the Poincaré conjecture. Then we can always find an irreducible link $L : M \subset H^3$ (see remark (3) at the beginning of section 5). We can then read off an explicit primary presentation for the (plain) fundamental rack $\Gamma(L)$ by the results of section 4 (especially lemmas 4.8 and 4.9). Now apply the algorithm. By deciding whether $\Gamma(L)$ is a classical rack we are *a fortiori* deciding whether H^3 is S^3 by corollary 5.3. Thus we would have solved the problem of deciding whether a homotopy sphere is S^3.

The following considerations suggest that an algorithmic version may not be too hard to find. Consider a primary presentation $[S : R]$ of a rack Γ. Let R consist of the n equations $w_i = z_i$ where $i = 1, 2, \ldots, n$. Call a word u in conjugates of the w_i and the z_i and their inverses a **generalised Artin condition** (GAC) if

(1) each appearance of a w_i^x is **balanced** by an appearance of z_i^{-x} or vice versa,

(2) $u = 1$ in $F(S)$.

Note that a particular relator can appear (as balanced pairs) many times.

Now the set of GAC's is 'invariant' under the Tietze moves (moves (1) to (5) of lemma 4.1) in the sense that any particular GAC is tranformed in an obvious way by one of these moves. The problems is that there are now many "trivial" Artin conditions. However, it can be shown that a GAC corresponds to an element of $\pi_2(B\Gamma)$ (where is $B\Gamma$ is the rack space) and for an irreducible classical rack this group is infinite cyclic generated by the diagram for the link, see [10,11]. Hence for an irreducible classical rack there is essentially only one non-trivial GAC. In particular examples it is easy to decide whether a GAC is non-trivial and it seems reasonable that an algorithm to do this can be constructed. Thus the problem reduces to finding a computable bound on the length (and the lengths of the conjugating elements) of the possible non-trivial GAC in terms of the original presentation.

(2) Obstructions

The other side of the coin is the construction of *obstructions* to a rack being classical. What we seek is a class of rack invariants which vanish for classical racks. Such invariants could then be applied to detect a possible counterexample H^3 to the Poincaré conjecture, by applying them to the fundamental rack Γ of an irreducible link in H^3.

There are several suitable theoretical invariants, for example $\pi_2(B\Gamma)$ where is $B\Gamma$ is the rack space (which must be $\mathbb{Z}$ for an irreducible classical rack). We do not at the time of writing know of an effectively computable invariant. However there are, as we have seen, many infinite classes of computable rack invariants. Therefore it seems extremely hopeful that a suitable class can be found.

Note that there is an effective algorithm to list all homotopy spheres provided by Rêgo and Rourke [25]. If there were an effective algorithm to decide whether a homotopy sphere is S^3, or else some invariants which might detect a non-trivial homotopy sphere, then these could be used in a computer search for a possible counterexample to the Poincaré conjecture.

8. Appendix : Nielsen theory for racks

In this appendix we characterise automorphisms of the free rack FR_n as products of elementary automorphisms. This result is an analogue for racks of Nielsen theory for groups. We shall in fact use Nielsen theory for groups, following the treatment given in Lyndon and Schupp [23] pages 4 to 17.

Let F_n denote the free group on the basis $\{a_1, \ldots, a_n\}$. Denote the reduced length of the word $w \in F_n$ by $|w|$. If $\mathbf{u} = \{u_1, \ldots, u_k\}$ is a set of elements of F_n we let $\mathbf{u}^{-1} = \{u_1^{-1}, \ldots, u_k^{-1}\}$ be the set of inverses and let $\mathbf{u}^{\pm 1} = \mathbf{u} \cup \mathbf{u}^{-1}$.

A set $\mathbf{u}$ is called **Nielsen reduced** if the following conditions hold.

N0 If $u \in \mathbf{u}$ then $u \neq 1$.

N1 If $u,v \in \mathbf{u}^{\pm 1}$ and $uv \neq 1$ then $|uv| \geq \max\{|u|,|v|\}$.

N2 If $u,v,w \in \mathbf{u}^{\pm 1}$ and $uv \neq 1$ and $vw \neq 1$, then $|uvw| > |u| - |v| + |w|$.

8.1 Lemma *Let $\mathbf{u} = \{u_1, \ldots, u_k\}$ be Nielsen reduced. If $w = v_1 \cdots v_r$ where $v_i \in \mathbf{u}^{\pm 1}$ and all $v_i v_{i+1} \neq 1$, then $|w| \geq r$.*

Proof For each $v \in \mathbf{u}^{\pm 1}$ let v_0 be the longest initial segment of v that cancels in any product uv where $u \in \mathbf{u}^{\pm 1}$ and let v_1 be the longest final segment of v that cancels in any product vw where $w \in \mathbf{u}^{\pm 1}$. Note that $v_1 = (v^{-1})_0^{-1}$.

Then we can write $v = v_0 m v_1$ where by N2 $|m| \geq 1$.

So in the product $w = v_1 \cdots v_r$ there is always at least an irreducible subword $m_1 \cdots m_r$ and the result follows. □

8.2 Corollary *Let $\mathbf{u} = \{u_1, \ldots, u_k\}$ be Nielsen reduced and suppose in addition that $\mathbf{u}$ generates F_n. Then $\mathbf{u}^{\pm 1} = \mathbf{a}^{\pm 1}$ where $\mathbf{a}$ is the basis $\{a_1, \ldots, a_n\}$ of F_n. (Note in particular that $k = n$.)*

Proof Let the basis element a_i be written as a product $a_i = v_1 \cdots v_r$ where $v_i \in \mathbf{u}^{\pm 1}$ and all $v_i v_{i+1} \neq 1$. Then $1 = |a_i| \geq r$. So r is forced to be unity and $a_i = v_j$ for some j. □

8.3 Definition *PC-type*

Consider a permutation $\pi \in S_n$ of $\{1, 2, \ldots, n\}$ and n words w_i for $i = 1, 2, \ldots, n$ in the free group F_n. Corresponding to this data is the set of n words $a_{\pi(i)}^{w_i}$ in F_n where $i = 1, 2, \ldots, n$, obtained by permuting and conjugating the generators, which we shall call a set of words of **permutation–conjugacy type**, or **PC–type** for short. We shall also use this name for the set obtained by inverting some of the elements of this set.

Let $\mathbf{u} = \{u_1, u_2, \ldots, u_n\}$ be a set of words of PC–type; we shall consider the following two **double Nielsen transfomations** which preserve PC–type:

T1 replace u_i by $u_j^{-1} u_i u_j$ where $j \neq i$;

T2 replace u_i by $u_j u_i u_j^{-1}$ where $j \neq i$.

8.4 Lemma *Let $\mathbf{u} = \{u_1, u_2, \ldots, u_n\}$ be a set of words of PC–type. Suppose that $|u_i u_j| < |u_i|$ then either $|u_j^{-1} u_i u_j| < |u_i|$ or $|u_i u_j u_i^{-1}| < |u_j|$.*

Proof We first observe that we cannot have $|u_i| = |u_j|$ because then $|u_i u_j| < |u_i|$ implies that at least half of u_i, u_j cancel in the product and the middle letter of u_i cancels with that of u_j. But this is impossible since the words are of PC–type and their middle letters are different generators.

Now assume that $|u_j| < |u_i|$. We shall show $|u_j^{-1} u_i u_j| < |u_i|$.

Write $u_i = w^{-1} a w$ where a is one of the generators $a_1, a_2, \ldots a_n$, or an inverse. Then since w has length greater than half of u_j, more than half of u_j cancels with w, i.e.

$$w = bx \quad u_j = x^{-1} c \quad \text{where} \quad |c| < |x|$$

Therefore

$$|u_j^{-1} u_i u_j| = |c^{-1} x x^{-1} b^{-1} a b x x^{-1} c| = |c^{-1} b^{-1} a b c| < |x^{-1} b^{-1} a b x| = |u_i|.$$

In the case when $|u_j| > |u_i|$ then $|u_i u_j| < |u_j|$ and we can show in a similar way that $|u_i u_j u_i^{-1}| < |u_j|$. □

8.5 Lemma *Let $\mathbf{u} = \{u_1, u_2, \ldots, u_n\}$ be a set of words of PC–type. Then $\mathbf{u}$ can be carried by a sequence of moves T1 and T2 above to a set $\mathbf{v}$ of words of PC–type which is Nielsen reduced.*

Proof The condition N0 is automatically satisfied so assume that $\mathbf{u}$ does not satisfy N1. Then there is a pair $u, v \in \mathbf{u}^{\pm 1}$ such that $|uv| < |u|$ and $uv \neq 1$. Then by the last lemma there is a transformation T1 or T2 which reduces $\sum |u_i|$. Therefore if we apply T1 and T2 until $\sum |u_i|$ is minimum the condition N1 will hold.

Now consider a triple $u, v, w \in \mathbf{u}^{\pm 1}$ such that $uv \neq 1, vw \neq 1$. Then by N1 we have $|uv| \geq |v|$ and $|vw| \geq |v|$. It follows that that part of v which cancels in the product uv is no more than half of v. Likewise that part of v which cancels in the product vw is also no more than half of v. So we can write in reduced form $u = ap^{-1}, v = pbq, w = q^{-1}$. Notice that $b \neq 1$ because v is one of a set of PC–words and hence has odd reduced length. So $uvw = abc$ is reduced and $|uvw| = |u| - |v| + |w| + |b| > |u| - |v| + |w|$. □

8.6 Corollary *A set of words $\mathbf{u} = \{u_1, u_2, \ldots, u_n\}$ of PC–type forms the basis of a free subgroup of F_n of rank n and hence the endomorphism of F_n defined by $a_i \mapsto u_i$ is injective.*

Proof By the last lemma we can assume without loss that our set of words is Nielsen reduced and the result now follows from lemma 8.1. □

Elementary automorphisms of the free rack

Let FR_n denote the free rack $\mathrm{FR}(a_1, a_2, \ldots, a_n)$.

We shall consider the following elementary automorphisms:

$$p_{i,k} : \begin{cases} a_i & \mapsto & a_i^{a_k} \\ a_j & \mapsto & a_j \quad j \neq i \end{cases}$$

$$s_\pi : a_i \mapsto a_{\pi(i)} \quad \pi \in S_n .$$

8.7 Theorem : characterisation of automorphisms of the free rack *Any automorphism of FR_n is a product of elementary automorphisms.*

Proof Let f be an automorphism of FR_n and let $f(a_i) = a_{\pi(i)}^{w_i}$. We need to consider $a_{\pi(i)}^{w_i}$ as a word in F_n as well as an element of FR_n and we shall use u_i for this word in order to avoid confusion.

If $i \neq k$, the effect of an elementary automorphism of the first type is to realise a double Nielsen transformation on set of words $\{u_i\}$. (If $i = k$ the automorphism has no effect on the words.)

Therefore by lemma 8.5 we may assume that this set is Nielsen reduced. But by corollary 8.2 this implies that these words are a permutation of the words a_i, $i = 1, \ldots, n$ and therefore by an elementary automorphism of the second type, we can assume that $u_i = a_i$ for each i. But then the automorphism is given by $a_i \mapsto a_i^{a_i^{n_i}}$ for some integers n_i and is therefore a product of elementary automorphisms of first type for $i = k$. □

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