On dunce hats and the Kervaire conjecture

Colin Rourke

The dunce hat Z is a CW complex with one 0–cell, one 1–cell and one 2–cell attached to the 1–cell by the word $t^2\overline{t}$ (throughout the paper $\overline{x}$ will mean x^{-1}), see for example Zeeman [8]. By **a dunce hat** I shall mean a CW complex with one each of 0, 1 and 2–cells, as before, but with the 2–cell attached to the 1-cell by an arbitrary (unreduced) word $w(t,\overline{t})$, the simplest interesting dunce hat being Z. In this paper I shall point out a connection between dunce hats and a conjecture in combinatorial group theory, known as the Kervaire conjecture, and then use the dunce hat connection to suggest possible counterexamples to the Kervaire conjecture.

Transversality for 2–dimensional CW complexes

The main tool will be the theory of transversality for CW complexes of Buonchristiano, Rourke and Sanderson [1; chapter 7]. This theory gives automatic "pictures" to illustrate any situation in homotopy theory. I shall need here only the simplest non-trivial case of the general theory and for completeness I shall describe this case from scratch. All my CW complexes will be connected with a single 0–cell $*$ (the base-point). A **transverse 1–complex** K just means a 1–dimensional CW complex. A map $f\colon S^1 \to K$ is **transverse** if f^{-1}(1–cells) consists of intervals with disjoint closures on each of which f is a characteristic map for a 1–cell.

So the essential part of the definition of a transverse map $f\colon S^1 \to K$ is that the intervals in S^1 where f crosses the 1–cells of K are spaced out — separated by intervals which map to the base-point of K (see figure 1).

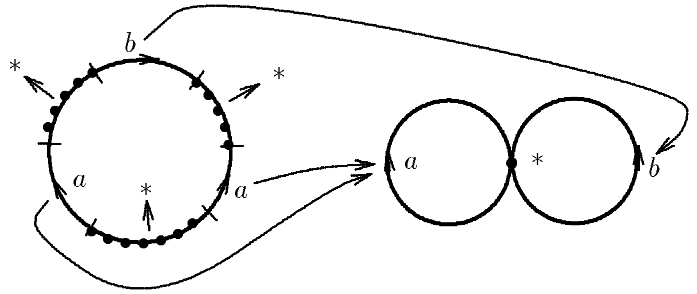


Figure 1

A **transverse 2–complex** is a 2–dimensional CW complex K such that the 2–cells are attached by transverse maps to the 1-skeleton.

Let M be a surface. A map $f\colon M \to K$ is **transverse** if :

(1) $f|\partial M$ is a transverse map to the 1–skeleton of K.

(2) f^{-1}(2–cells) consists of a number of discs with disjoint closures in the interior of M, on each of which f is a characteristic map for a 2–cell of K.

(3) Let $M' = \mathrm{cl}(M - f^{-1}(\text{2–cells}))$ and let $g = f|M'$ then g^{-1}(1–cells) consists of a number of framed 1–manifolds with disjoint closures such that, on each framing line, g is a characteristic map for a 1–cell of K.

A transverse map $f\colon M \to K$ gives rise to a picture in M of the type illustrated in figure 2.

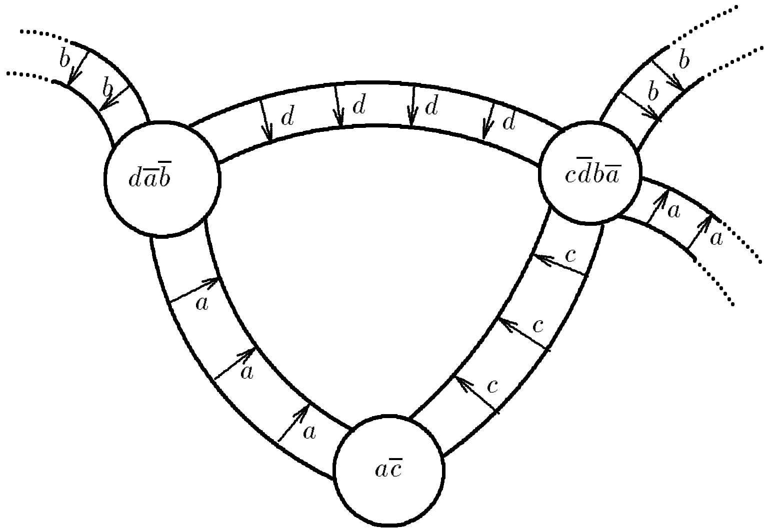


Figure 2

In figure 2, the discs are mapped to 2–cells of K, which are attached by the word written in the disc, and the framing lines (arrowed) are mapped to the 1–cell labelled by the same letter. The unmarked regions in figure 2 are all mapped to the base-point.

I shall call the picture in M coming from a transverse map $M \to K$ a K–**diagram in** M. I usually draw K–diagrams in equivalent simplified form as in figure3.

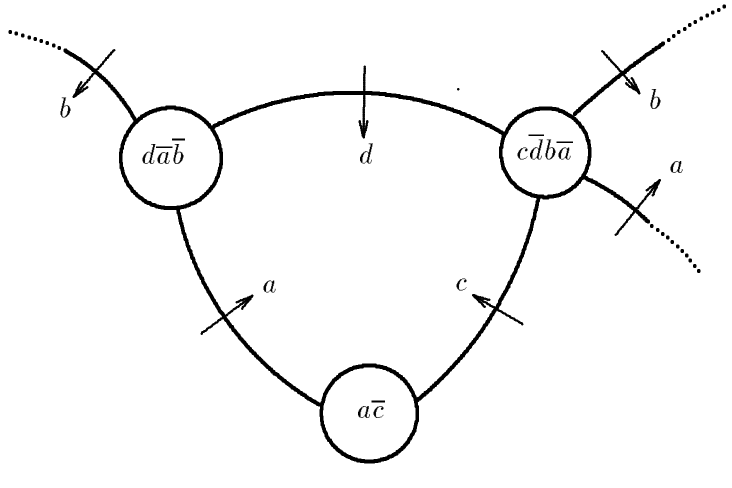


Figure 3

It is easy to see that any map $f\colon M \to K$ can be approximated by a transverse map and it is also easy to recover the map from the diagram. Thus changes to the diagram correspond precisely to changes to the (transverse) map.

The Kervaire conjecture

Let G be a non-trivial group. Adjoin to G a new generator t and add a new relator r.

Conjecture $\dfrac{G * \langle t \rangle}{[r]}$ *is non-trivial.*

This conjecture is implied by the following conjecture which I shall call the Kervaire conjecture :

Kervaire conjecture *Suppose the total exponent of t in r is non-zero then the natural homomorphism $G \to \dfrac{G * \langle t \rangle}{[r]}$ is injective.*

To see the connection between the Kervaire conjecture and diagrams choose a 2–complex K such that $\pi_1(K) = G$ then $\dfrac{G * \langle t \rangle}{[r]}$ is $\pi_1(L)$ where $L = K \cup 1\text{–cell} \cup 2\text{–cell}$.

The Kervaire conjecture is equivalent to the statement that if $f\colon D^2 \to L$ is a map such that $f(\partial D^2) \subset K$ then $f|\partial D^2$ is inessential in K. Making everything transverse we obtain an L–diagram in D^2 such that t does not appear in ∂D^2. I call such a diagram a **counterexample**. A counterexample is **trivial** if the word on ∂D^2 is trivial in G. So the Kervaire conjecture is equivalent to the statement that all counterexamples are trivial!

Now let Q be the dunce hat corresponding to the occurrences of t in r. (Ie read r as a word in $t, \bar{t}$ omitting all elements of G.) There is a map $p\colon L \to Q$ (obtained by squeezing K to a point), $p \circ f\colon D^2 \to Q$ is still transverse, so a counterexample gives rise to a Q–diagram in D^2 which does not meet ∂D^2. At first sight this Q–diagram contains far less information than the original counterexample since all information connected with G has been suppressed but the surprising fact is that it contains all the information necessary to reconstruct a "universal" counterexample which will be non-trivial if the original one is non-trivial. Thus the Kervaire conjecture is equivalent to a conjecture about dunce hats.

The dunce hat conjecture

Let Q be a dunce hat and let T be a connected Q–diagram in S^2. Let $w = w(t, \bar{t})$ be the word defining Q and let F be the free group on the (cyclic) "spaces" in w. Ie enlarge $w = t^{\epsilon_1} t^{\epsilon_2} \dots t^{\epsilon_n}$, $\epsilon_i = \pm 1$, to read $t^{\epsilon_1} x_1 t^{\epsilon_2} x_2 \dots t^{\epsilon_n} x_n$ and let $F = F(x_1, x_2, \dots, x_n)$. Each region of $S^2 - T$ is a disc and, reading round the disc, we obtain a word in

$x_1, x_2, \ldots, x_n$; for example see figure 4.

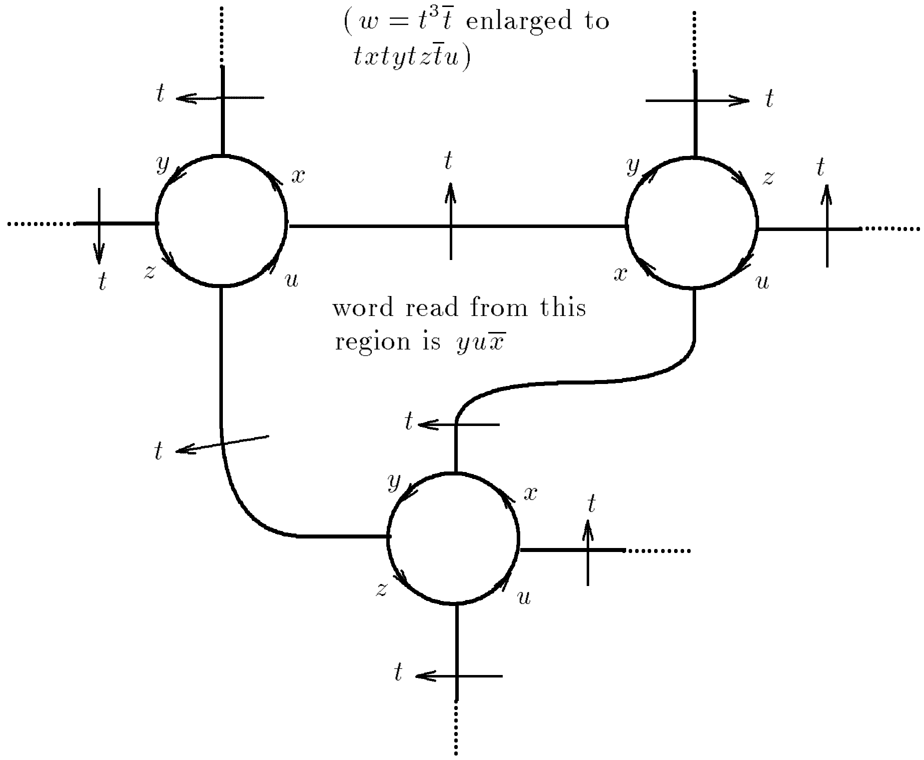


Figure 4

Let the words obtained in this way be $w_1, w_2, \ldots, w_t$.

Dunce Hat Conjecture *For each i, w_i is a consequence of $w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n$ (ie w_i is in the normal closure in F of $w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n$).*

Theorem *The Kervaire conjecture is equivalent to the dunce hat conjecture.*

Proof Let T be a diagram in D^2 which is a non-trivial counterexample to the Kervaire conjecture where $f\colon D^2 \to L$ is the corresponding map (as above). Let T' be the associated Q–diagram (where Q is the appropriate dunce hat) then T' can be regarded as a diagram in S^2 with one region selected for special attention (the "outside"). If T' is disconnected then, looking at each connected component in turn (innermost first), we can see that there must be a non-trivial counterexample with corresponding Q–diagram connected. Thus, without loss, we may assume that T' is connected.

Let $x_1, x_2, \ldots, x_n$ be the new letters associated to Q and $w_1, w_2, \ldots, w_t$ the words associated to T', as above, where w_t corresponds to the outside region. Define G' to be the group $\langle x_1, x_2, \ldots, x_n | w_1, w_2, \ldots, w_{t-1}\rangle$ (with generators $x_1, x_2, \ldots, x_n$ and relators $w_1, w_2, \ldots, w_{t-1}$). Consider the homomorphism $F \to G$ given by $x_i \mapsto g_i$ where $r = t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 \ldots t^{\epsilon_n} g_n$. Since the images of $w_1, w_2, \ldots, w_{t-1}$ map to the boundaries of discs mapped into K, they go to zero in G and hence the homomorphism factors via a homomorphism $G' \to G$ such that $w_t \mapsto$ the homotopy class of $f(\partial D^2)$. Hence $w_t \neq 0$ in G' and we have a counterexample to the dunce hat conjecture.

Conversely, suppose T is a Q–diagram representing a counterexample to the dunce hat conjecture and suppose for definiteness that $w_t \notin [w_1, \ldots, w_{t-1}]$. Identify S^2 with D^2 (after cutting a hole in the t-th region) and let $G' = \dfrac{F}{[w_1, \ldots, w_{t-1}]}$ as before. Choose K with $\pi_1(K) = G'$ and $L = K \cup 1\text{–cell} \cup 2\text{–cell}$ where the 1–cell is labelled t and the 2–cell is attached by $t^{\epsilon_1}x_1t^{\epsilon_2}x_2\ldots t^{\epsilon_n}x_n$ Then the diagram T (with extra labels $x_1, \ldots, x_n, w_1, \ldots, w_{t-1}$) defines a map $f\colon D^2 \to L$ such that $\partial D^2 \mapsto w_t \neq 0$ in G' (ie essential in K). Therefore we have a non-trivial counterexample to the Kervaire conjecture. $\square$

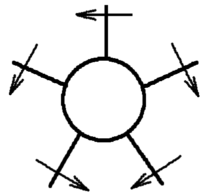
The last half of the proof just given is the construction of the "universal" counterexample from T, which I mentioned earlier.

Searching for counterexamples

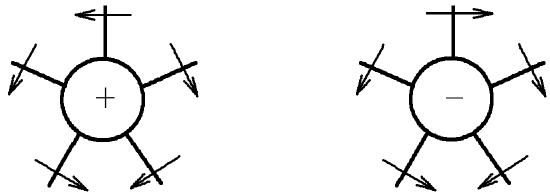
The theorem replaces the Kervaire conjecture by a conjecture about all Q–diagrams. I shall call a Q–diagram trivial or non-trivial according as the universal counterexample, constructed in the second half of the proof, is trivial or non-trivial. The Kervaire conjecture becomes the statement that all Q–diagrams are trivial (where Q is a dunce hat with non-zero total exponent of t in $w(t, \bar{t})$).

Thus a way to tackle the conjecture is to search through Q–diagrams for various Q in the hope of either finding a non-trivial diagram or seeing a general reason why they must all be trivial.

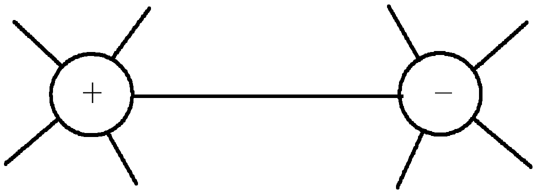
Lyndon [5] implies that all Q–diagrams where Q is given by word t^n, are trivial and Howie [4] has proved that all Z–diagrams ($w = t^2\bar{t}$) are trivial; Edjvet and Howie [2] have announced a proof for the only remaining case of length four ($w = t^3\bar{t}$)), so the first interesting case to look at is length 5 and for the rest of this paper I shall concentrate on the particular case $w = t^3\bar{t}^2$. In this case a disc representing w has the following form



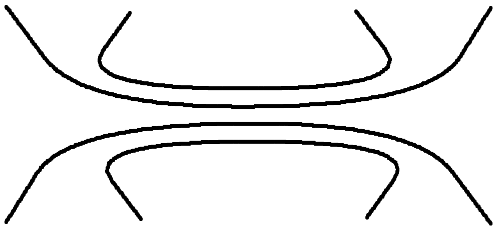
(where the label t on all arrows has been suppressed). In a Q–diagram, the disc can appear in either of two forms, according to orientation, and we shall label them + or − thus :



To be **interesting** a Q–diagram must be (a) connected and (b) **reduced** ie not containing any pairs of this type



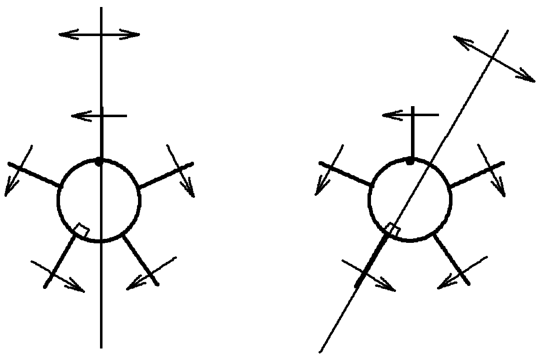
where the joining 'leg' represents the same occurence of t or $\bar{t}$ in w. If a Q–diagram is not reduced then it can be simplified as suggested below :



and if the original diagram is non-trivial, the simplified one must also be.

Diagrams made of "units"

One way of making interesting diagrams is to use the two 'axes of symmetry' of the disc :



(note the key and to locate these axes).

Using either axis of symmetry, discs of the same size can be joined in circuits (which I call **units**). Figure 5 shows a + unit of size 4,

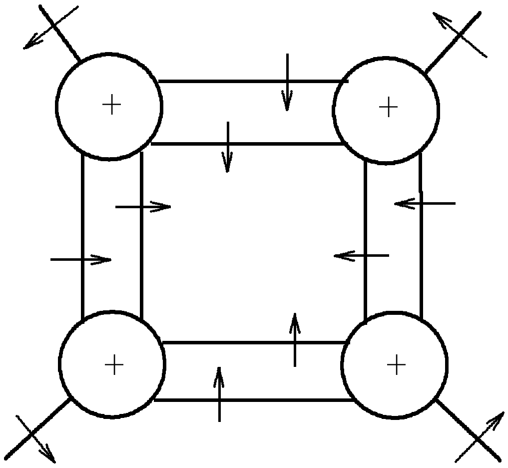


Figure 5

for which I shall use the abbreviated notation $\boxed{+^4}$.

If different axes of symmetry are used for the two signs (for + units and for − units) then any diagram made of units will automatically be interesting. For an example see figure 6 :

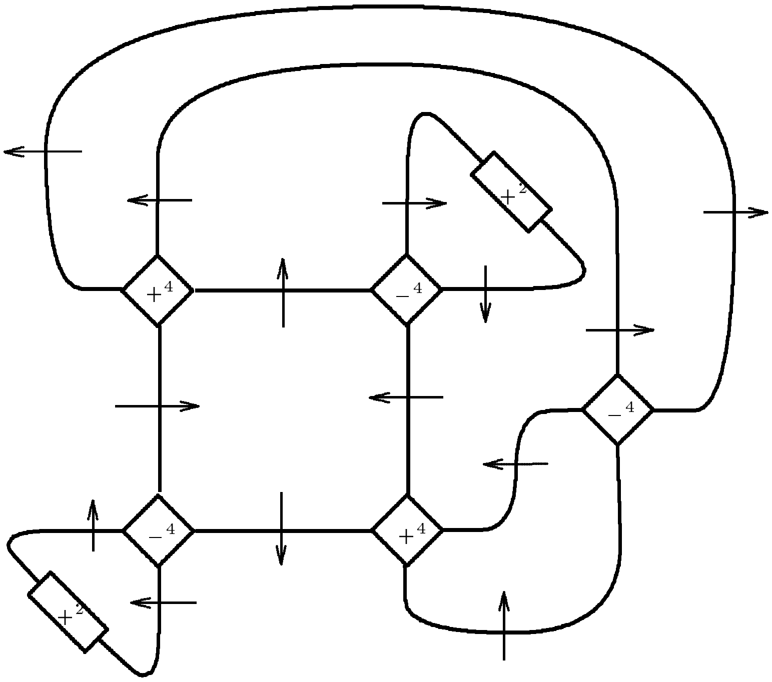
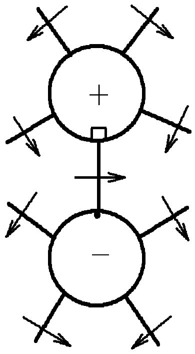


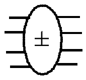
Figure 6

Unfortunately Short [6] has proved that any digram made of units is trivial (with no restriction on the length of w)!

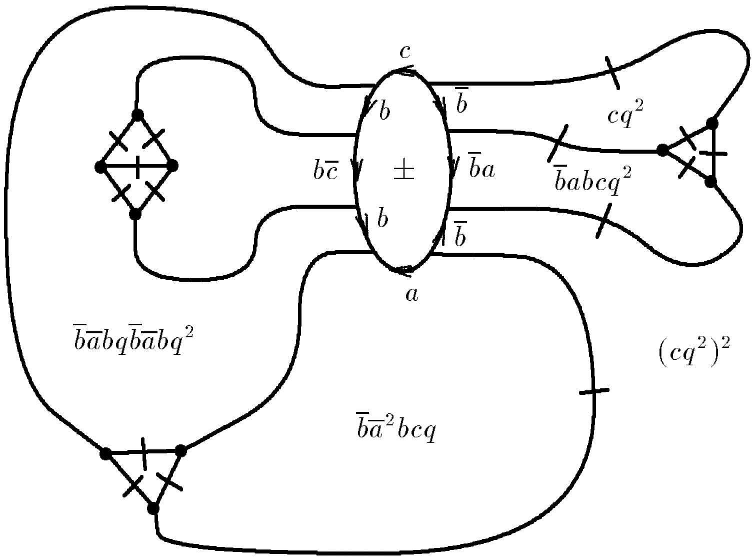
Modified unit diagrams

The next attempt to construct non-trivial examples is to modify unit diagrams by introducing other elements. For example if the subdiagram



which I shall abbreviate to  is introduced then the diagram will still be interesting.

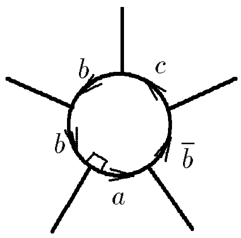
A specific example of a modified unit diagram with all the F labels (and words w_i) is drawn in figure 7.

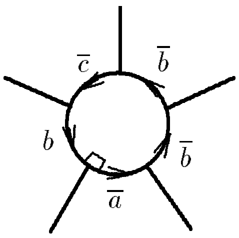


where $q = \overline{b}\overline{a}bc$ and all unlabelled regions carry one of q^3, a^2, c^3

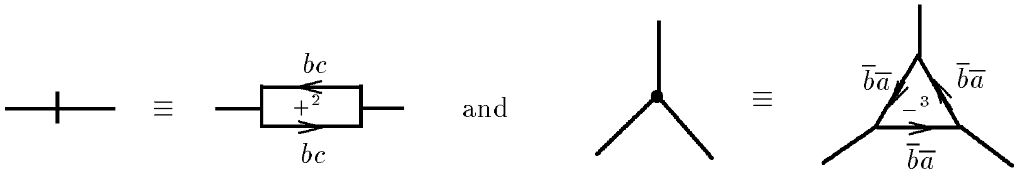
Figure 7

In figure 7, elementary discs are labelled :

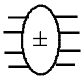


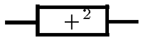


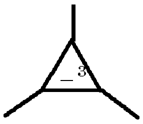
the repeating b's being forced by the units and I have used the following abbreviations :



It can be checked that this diagram is trivial, but the triviality is by no means obvious at first sight and leaves the hope that a modified example might turn out to be non-trivial.

Question Must every diagram made of one 

element and 

and 

units be trivial?

Acknowledgements I proved the theorem (but did not publish it) about 10 years ago. In some sense it is well-known, similar results being used by other authors [3,7]. An equivalent theorem appears in Short's thesis [6], where the initial work on unit diagrams can be found and the proof of their triviality.

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Mathematics Unstitute, University of Warwick, COVENTRY CV4 7AL, UK