Preferred invariant symplectic connections on compact coadjoint orbits

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Abstract. We prove the existence of at least one G-invariant preferred symplectic connection on any coadjoint orbit of a compact semisimple Lie group G. We look at the case of the orbits of $SU(3)$ and show that in this case the invariant preferred connection is unique.

Moshé Flato has been a close and wonderful friend and an inspiration for us for more than twenty years. This contribution is dedicated to him, always present in our hearts.

1. Introduction

A linear connection ∇ on a symplectic manifold (M,ω) is said to be *preferred* [1] if (i) it has zero torsion; (ii) $\nabla\omega = 0$; (iii) it is an extremal of the functional L given by

$$L(\nabla) = \frac{1}{2}\int_M r^2\omega^n \tag{1}$$

where $\dim M = 2n$, and r^2 is the square of the Ricci tensor of ∇ defined using ω. In the case of a compact surface it was shown in [1] that a preferred connection always exists. The general case is still open, but we hope to get some feeling for the existence of solutions by examining homogeneous symplectic manifolds.

A compact simply connected homogeneous symplectic manifold is known to be a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ of a compact semisimple Lie group G endowed with the classical Kirillov–Kostant–Souriau form [2, 6] and we look for G-invariant symplectic connections on these.

In this note, we prove the existence of at least one such preferred connection on $(\mathcal{O},\omega)$. For the orbits of $G = SU(3)$ we establish the uniqueness of the preferred invariant connection.

2. Positive polynomials

The existence argument is based on the following lemma on positive polynomials on a vector space.

LEMMA 1. *Let* $P\colon \mathbb{C}^N \to \mathbb{R}$ *be a real-valued polynomial of degree d. Assume (i)* $P(z) \geq 0$, $\forall z \in \mathbb{C}^N$*; (ii)* $Q_d(z) > 0$ *for* $z \neq 0$ *where* Q_d *is the homogeneous component of* P *of degree d; then* P *admits a minimum.*

Proof. Let (z_n) be a sequence in C^N such that $\lim_{n\to\infty} P(z_n) = a = \inf_{z\in\mathbb{C}^N} P(z)$. Assume that the sequence $(|z_n|)$ diverges and write $z_n = |z_n|u_n$. Passing to a subsequence, if necessary, we can assume $\lim_{n\to\infty} u_n = u$, with $Q_d(u) = b > 0$. Thus, if Q_i is the homogeneous component of P of degree i,

$$0 = \lim_{n\to\infty} |z_n|^{-d} a = \lim_{n\to\infty} Q_d(u_n) + \sum_{i=0}^{d-1} |z_n|^{i-d} Q_i(u_n) = b,$$

a contradiction. Hence the sequence (z_n) is bounded, so has a convergent subsequence. □

3. Homogeneous symplectic manifolds

We recall a well-known lemma about compact, simply connected homogeneous symplectic manifolds.

Let G be a compact semisimple Lie group and $\mathfrak{g}$ its Lie algebra. Let $\mathcal{O}$ be a coadjoint orbit in $\mathfrak{g}^*$ and $x_0 \in \mathcal{O}$ a base point. If we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using the Killing form B, then $x_0(X) = B(X_0, X)$ for an element X_0 of $\mathfrak{g}$ and the stabiliser H of x_0 is the centraliser of X_0. Let $\mathfrak{h}$ be its Lie algebra. Set

$$\omega_{x_0}(X^*_{x_0}, Y^*_{x_0}) = B(X_0, [X, Y])$$

where X^* and Y^* are fundamental vector fields on $\mathcal{O}$

$$X^*_x = \frac{d}{dt} \exp -tX.x \bigg|_{t=0}.$$

ω extends by translation by G around $\mathcal{O}$ as an invariant symplectic form on $\mathcal{O}$ called the Kirillov–Kostant-Souriau (KKS) symplectic structure [2, 6].

LEMMA 2. *Let* (M, ω) *be a connected, simply connected compact homogeneous symplectic manifold. Then* (M, ω) *is symplectomorphic to a coadjoint orbit* $\mathcal{O}$ *of a compact connected semi-simple group* G*, with its standard KKS symplectic structure.*

Proof. A connected Lie group G_0 acts transitively on M. A classical result of Montgomery [3] shows that, as M is simply connected, the maximal compact subgroup G_1 of G_0 acts transitively on M. Since M is simply connected, the action of G_1 on M is generated by Hamiltonian vector fields. Let $\mathfrak{g}_1$ be the Lie algebra of G_1; if $X \in \mathfrak{g}_1$, denote by f_X the unique smooth function on M such that

$$(1)\ \ df_X = i(X^*)\omega; \qquad (2)\ \int_M f_X \omega^n = 0.$$

Define the real-valued 2-cocycle c on $\mathfrak{g}_1$ by

$$c(X,Y) = \{f_X, f_Y\} - f_{[X,Y]}$$

where the Poisson bracket of the function $\{f_X, f_Y\} = -\omega(X^*, Y^*))$. If one computes the integral

$$\begin{aligned}\int_M c(X,Y)\omega^n &= \int_M \{f_X, f_Y\}\omega^n = \int_M X^* \cdot f_Y \omega^n \\ &= \int_M \mathcal{L}_{X^*}(f_Y\omega^n) = \int_M d\left(i(X^*) f_Y \omega^n\right) = 0.\end{aligned}$$

Since $c(X,Y)$ is a constant function, it follows that it must vanish and hence that the action is strongly Hamiltonian. The Kostant–Souriau theorem [2, 6] now ensures that the moment map $\mu\colon M \to \mathfrak{g}_1^*$

$$\mu(x)(X) = f_X(x)$$

is a G_1-equivariant, symplectic covering map of a coadjoint orbit $\mathcal{O}$ of G_1 in $\mathfrak{g}_1^*$.

The centre of G_1 acts trivially on $\mathfrak{g}_1$ and $\mathfrak{g}_1^*$ so that $\mathcal{O}$ is homogeneous under the (semisimple) derived group G of G_1. $\mathcal{O}$ is thus a coadjoint orbit of a compact semisimple Lie group G. These are known to be simply connected [4] and hence the covering map μ is a symplectic diffeomorphism.

□

4. Invariant symplectic connections

A linear connection ∇ on a symplectic manifold (M, ω) is said to be *symplectic* [7] if (i) it has zero torsion; (ii) $\nabla\omega = 0$.

It is well known that on any symplectic manifold, there exist such connections and the space of symplectic connections is an affine space modelled on the space of symmetric 3-tensors on M. Take any linear torsion-free connection ∇' on M and define a symplectic connection ∇^0

$$\omega(\nabla^0_X Y, Z) = \omega(\nabla'_X Y, Z) + \tfrac{1}{3}(N(X,Y,Z) + N(Y,X,Z))$$

where $N(X,Y,Z) = (\nabla'_X \omega)(Y,Z)$. Then any symplectic connection ∇ reads

$$\nabla_X Y = \nabla^0_X Y + A(X)Y \tag{2}$$

where the 1-form A with values in the endomorphisms of the tangent bundle is such that $\underline{A}(X,Y,Z) = \omega(A(X)Y,Z)$ is a completely symmetric 3-tensor on M.

To try to select some symplectic connections, one considers the functional $L(\nabla)$ (see (1)) on the space of symplectic connections. A symplectic connection is *preferred* if it is an extremal of this functional, hence if it satisfies the corresponding Euler-Lagrange equations

$$\mathcal{S}_{X,Y,Z}\,(\nabla_X r)(Y,Z) = 0. \tag{3}$$

for any X, Y, Z vector field on M, where $\mathcal{S}_{X,Y,Z}$ denotes the sum over cyclic permutations of X, Y and Z.

When the symplectic manifold (M,ω) is homogeneous under the action of a group G, for any $X \in \mathfrak{g}$, one defines on M the fundamental vector field X^*

$$X^*_y = \frac{d}{dt} \exp -tX.y\Big|_{t=0} \qquad y \in M.$$

For a G-invariant symplectic connection, the Ricci tensor is invariant so the condition to be preferred can be written as

$$\mathcal{S}_{X,Y,Z}\, r(\nabla_{X^*}Y^* + \nabla_{Y^*}X^*, Z^*) = 0 \tag{4}$$

Indeed

$$\begin{aligned}
(\nabla_{X^*} r)(Y^*,Z^*) &= X^*(r(Y^*,Z^*)) - r(\nabla_{X^*}Y^*,Z^*) \\
&\qquad - r(Y^*,\nabla_{X^*}Z^*) && (5)\\
&= r([X^*,Y^*],Z^*) + r(Y^*,[X^*,Z^*]) \\
&\qquad - r(\nabla_{X^*}Y^*,Z^*) - r(Y^*,\nabla_{X^*}Z^*) && (6)\\
&= -r(\nabla_{Y^*}X^*,Z^*) - r(Y^*,\nabla_{Z^*}X^*). && (7)
\end{aligned}$$

We always assume in this paper that G is compact semisimple. By Palais' principle [5], to determine G-invariant critical points of L, it is sufficient to determine the critical points of the restriction of L to the space of G-invariant symplectic connections. The equation for an invariant symplectic connection to be preferred, (4), allows to see this directly; for this we parametrise the space of invariant symplectic connections and write the Ricci tensor in terms of this parametrisation.

From now on, G is a compact connected semi-simple group, $\mathfrak{g}$ is its Lie algebra, (M,ω) is a symplectic manifold which is homogeneous under G, $x_0 \in M$ is a base point, H is its stabiliser, with Lie algebra $\mathfrak{h}$. The Killing form orthogonal complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$ is $\operatorname{Ad} H$-invariant and is identified to the tangent space to M at x_0. Let ∇^0 be an invariant symplectic connection on M (it always exists since G is compact and one can average a symplectic connection). A general invariant symplectic connection on M has the form $\nabla = \nabla^0 + A$ where A is a 1-form with values in the endomorphisms of TM such that $\underline{A}(X,Y,Z) := \omega(A(X)Y,Z)$ is completely symmetric and invariant. Hence ∇ is determined by the trilinear map

$$S\colon \mathfrak{m}\times\mathfrak{m}\times\mathfrak{m}\to\mathbb{R},\quad S(X,Y,Z)=\underline{A}_{x_0}(X^*,Y^*,Z^*)$$

which is $\operatorname{Ad} H$-invariant.

LEMMA 3. *The space $\mathcal{E}$ of invariant symplectic connections on a G- homogeneous connected symplectic manifold where G is a compact semisimple Lie group is isomorphic to the space of completely symmetric trilinear maps on $\mathfrak{m}$ which are invariant under the adjoint action of H.*

$$\mathcal{E} = (S^3\mathfrak{m}^*)^H. \tag{8}$$

This isomorphism depends on the choice of the invariant symplectic connection ∇^0 on M.

We shall now write the functional $L(\nabla)$ for an invariant connection in terms of the fundamental vector fields. More precisely, we define $D\colon \mathfrak{m}\to \operatorname{End}\mathfrak{m}$ by

$$(D(X)Y)^*|_{x_0} = \left(\nabla_{X^*}Y^*\right)(x_0). \tag{9}$$

Notice that

$$\begin{aligned}\left(\nabla_{X^*}Y^*\right)(x_0) &= 0, \quad \forall X\in\mathfrak{h};\\ \left(\nabla_{X^*}Y^*\right)(x_0) &= [X,Y]^*|_{x_0}, \quad \forall X\in\mathfrak{m}, Y\in\mathfrak{h}.\end{aligned}$$

The conditions for ∇ to be symplectic are

$$D(X)Y = D(Y)X + [X,Y]_{\mathfrak{m}}, \tag{10}$$

$$\Omega(D(Y)X,Z) + \Omega(Y, D(Z)X) = 0 \tag{11}$$

for all $X,Y,Z\in\mathfrak{m}$, where $X = X_{\mathfrak{h}} + X_{\mathfrak{m}}$ is the decomposition of an element $X\in\mathfrak{g}$ relatively to $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ and where $\Omega(X,Y)=\omega_{x_0}(X^*,Y^*)$. Equation (11) follows, as in (5,6,7), from the invariance of ω under G. Remark that it implies that $D(X)-\pi_{\mathfrak{m}}\circ \operatorname{ad} X \in sp(\mathfrak{m},\Omega)$ where $\pi_{\mathfrak{m}}$ (resp. $\pi_{\mathfrak{h}}$) is the projection of $\mathfrak{g}$ on $\mathfrak{m}$ (resp. $\mathfrak{h}$). Hence $D(X)$ is traceless for any $X\in\mathfrak{m}$.

For all $X, Y, Z \in \mathfrak{m}$, the curvature is given by

$$\begin{aligned} R(X^*, Y^*)Z^* &= \nabla_{\nabla_{Y^*} Z^*} X^* - \nabla_{\nabla_{X^*} Z^*} Y^* + \nabla_{[X^*, Y^*]} Z^* \\ &\quad - \nabla_{X^*}[Y^*, Z^*] + \nabla_{Y^*}[X^*, Z^*] \\ &= \nabla_{\nabla_{Z^*} Y^*} X^* - \nabla_{\nabla_{Z^*} X^*} Y^* + \nabla_{Z^*}[X^*, Y^*]. \end{aligned}$$

Hence

$$\begin{aligned} R(X^*, Y^*)Z^*|_{x_0} &= (D(D(Z)Y)X - D(D(Z)X)Y + D(Z)[X, Y]_{\mathfrak{m}} \\ &\quad + [Z, [X, Y]_{\mathfrak{h}}])^*|_{x_0} \\ &= (D(D(Z)Y)X - D(Y)D(Z)X - [D(Z)X, Y]_{\mathfrak{m}} \\ &\quad + D(Z)[X, Y]_{\mathfrak{m}} + [Z, [X, Y]_{\mathfrak{h}}])^*|_{x_0} \end{aligned}$$

so that the Ricci tensor is given by

$$\begin{aligned} r(Y^*, Z^*)|_{x_0} &= \operatorname{Trace}_{\mathfrak{m}}(D(D(Z)Y) - D(Y)D(Z) + \pi_{\mathfrak{m}} \circ \operatorname{ad} Y \circ D(Z) \\ &\quad - D(Z) \circ \pi_{\mathfrak{m}} \circ \operatorname{ad} Y - \operatorname{ad} Z \circ \pi_{\mathfrak{h}} \circ \operatorname{ad} Y) \\ &= -\operatorname{Trace}_{\mathfrak{m}}(D(Y)D(Z) - \operatorname{ad} Z \circ \pi_{\mathfrak{h}} \circ \operatorname{ad} Y). \end{aligned}$$

Having chosen an invariant symplectic connection ∇^0 on M with corresponding D^0 and writing a general invariant symplectic connection $\nabla = \nabla^0 + A$, one has $D = D^0 + \tilde{S}$ where $\tilde{S} : \mathfrak{m} \to \operatorname{End} \mathfrak{m}$ is defined by

$$A(X^*)Y^*|_{x_0} = (\tilde{S}(X)Y)^*|_{x_0}$$

so that, with our previous notation, $S(X, Y, Z) = \Omega(\tilde{S}(X)Y, Z)$. The Ricci tensor is

$$r(Y^*, Z^*)|_{x_0} = -\operatorname{Trace}_{\mathfrak{m}}((D^0 + \tilde{S})(Y)(D^0 + \tilde{S})(Z) + \operatorname{ad} Z \circ \pi_{\mathfrak{h}} \circ \operatorname{ad} Y).$$

PROPOSITION 4. *The preferred homogeneous symplectic connections on M are determined by the critical points of the function r^2 on the space $\mathcal{E}$ of invariant symplectic connections. This function is a real polynomial of degree 4 on $\mathcal{E}$.*

Proof. r^2 is an invariant function and so is constant on M. Thus $L = \frac{1}{2} r^2 \operatorname{vol}(M)$. If X_i is any basis for $\mathfrak{m}$, $r^2 = \sum_{ijkl} \omega^{ij} \omega^{kl} r_{ik} r_{jl}$ where ω^{ij} is the inverse matrix of $\omega_{ij} = \Omega(X_i, X_j)$ and $r_{ij} = r(X_i^*, X_j^*)|_{x_0}$. One checks that the condition for an element $\nabla^0 \in \mathcal{E}$ to be a critical point of r^2 is exactly the condition (4) for this connection to be preferred.

5. Existence

From now on, G is a compact connected semi-simple group, $\mathfrak{g}$ is its Lie algebra, $\mathcal{O}$ is a coadjoint orbit in $\mathfrak{g}^*$ with its standard KKS symplectic structure, $x_0 \in \mathcal{O}$ is a base point, H is its stabiliser, with Lie algebra $\mathfrak{h}$ and $\mathfrak{m}$ is the Killing form orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. We use the same notation as before.

As a polynomial in S, the term of highest order in the Ricci tensor is the quadratic term

$$r^{(2)}(Y,Z) = -\operatorname{Trace}_{\mathfrak{m}}(\tilde{S}(Y)\tilde{S}(Z)).$$

If X_i is any basis for $\mathfrak{m}$, we have, for any $X, Y \in \mathfrak{m}$:

$$\begin{aligned} r^{(2)}(Y,Z) &= -\sum_{ijkl}(\tilde{S}(Y)X_i)^j(\tilde{S}(Z)X_j)^i \\ &= \sum_{ijkl} S(Y,X_i,X_k)\omega^{kj}S(Y,X_j,X_l)\omega^{il}. \end{aligned}$$

where $\omega_{ij} = \omega_{x_0}(X_i^*, X_j^*)$ and ω^{ij} is the inverse matrix.

Take a maximal torus T of G in H and let $\mathfrak{t}$ be its Lie algebra. We denote by Δ the set of roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ and let

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \sum_{\alpha\in\Delta} \mathfrak{g}^{\alpha}$$

be the root space decomposition. A root space $\mathfrak{g}^{\alpha}$ is in $\mathfrak{h}^{\mathbb{C}}$ if and only if $\alpha(X_0) = 0$. We denote by Δ_c the set of complementary roots

$$\Delta_c = \{\alpha \in \Delta \mid \mathfrak{g}^{\alpha} \not\subset \mathfrak{h}^{\mathbb{C}}\} = \{\alpha \in \Delta \mid \alpha(X_0) \neq 0\}.$$

Pick a positive root system Δ^+ such that $i\alpha(X_0) > 0$ for all $\alpha \in \Delta^+ \cap \Delta_c$. The H-invariant complement $\mathfrak{m}$ of $\mathfrak{h}$ is given by

$$\mathfrak{m}^{\mathbb{C}} = \sum_{\alpha\in\Delta_c} \mathfrak{g}^{\alpha}.$$

Let $\alpha_1, \dots, \alpha_l$ be the simple roots numbered in such a way that $\alpha_1, \dots, \alpha_r$ are the simple roots in Δ_c. We use the complementary height function n_c defined by $n_c(\alpha) = \sum_{i=1}^r n_i$ if $\alpha = \sum_{i=1}^l n_i\alpha_i$. n_c is additive in the sense that if α and β are roots such that $\alpha+\beta$ is also a root then $n_c(\alpha+\beta) = n_c(\alpha)+n_c(\beta)$. All the n_i are positive for positive roots and negative for negative roots so $\alpha \in \Delta^+ \cap \Delta_c$ if and only if $n_c(\alpha) > 0$.

We extend $\mathbb{C}$-linearly $\omega_{x_0}, \underline{A}_{x_0}$ and r_{x_0} to forms on $(T_{x_0}\mathcal{O})^{\mathbb{C}}$ and S to $\mathfrak{m}^{\mathbb{C}}$. Pick a basis of root vectors $E_\alpha \in \mathfrak{g}^{\alpha}$ such that $E_{-\alpha} = \overline{E_\alpha}$. Put

$$s_{\alpha\beta\gamma} = \underline{A}_{(x_0)}(E_\alpha^*, E_\beta^*, E_\gamma^*) = S(E_\alpha, E_\beta, E_\gamma).$$

Then ∇ is determined by $s_{\alpha\beta\gamma}$ for $\alpha, \beta, \gamma \in \Delta_c$, $s_{\alpha\beta\gamma}$ is symmetric in its labels, and G-invariance of ∇ is equivalent to $\operatorname{Ad} H$-invariance of S. In particular, S is T-invariant which implies

$$s_{\alpha\beta\gamma} = 0 \quad \text{if} \quad \alpha + \beta + \gamma \neq 0.$$

Invariance of ω and ∇ implies that $\omega_{x_0}(E^*_\alpha, E^*_{\alpha'})$ and $r_{x_0}(E^*_\alpha, E^*_{\alpha'})$ are zero unless $\alpha + \alpha' = 0$. Set $\Omega_\beta = \omega_{x_0}(E^*_\beta, E^*_{-\beta}) \quad r_{\beta,-\beta} = r_{x_0}(E^*_\beta, E^*_{-\beta})$.

The function $\frac{1}{2}r^2$ defining our functional $L(\nabla)$ has the form

$$\tfrac{1}{2}r^2 = -\tfrac{1}{2}\sum_{\alpha\in\Delta_c} \Omega_\alpha^{-2} r^2_{\alpha,-\alpha}.$$

Now Ω is skewsymmetric so Ω_α is purely imaginary, and r is symmetric so $r_{\alpha,-\alpha}$ is real. Hence $\frac{1}{2}r^2$, which is a 4th order polynomial in the $s_{\alpha\beta\gamma}$ is a sum of square of real numbers hence non negative. The highest order terms of that polynomial are given by

$$-\tfrac{1}{2}\sum_{\alpha\in\Delta_c} \Omega_\alpha^{-2} R_\alpha^{(2)}$$

where $R_\alpha^{(2)} := r^{(2)}(E_\alpha, E_{-\alpha})$, so they vanish only when all the $R_\alpha^{(2)}$ vanish. But

$$\begin{aligned} R_\alpha^{(2)} &= \sum_{\beta\beta'\gamma\gamma'} s_{\alpha\beta\gamma} s_{-\alpha\beta'\gamma'} \frac{\delta_{\beta,-\beta'}}{\Omega_\beta} \frac{\delta_{\gamma,-\gamma'}}{\Omega_\gamma} \\ &= \sum_{\beta+\gamma+\alpha=0} \frac{s_{\alpha\beta\gamma} s_{-\alpha-\beta-\gamma}}{\Omega_\beta \Omega_\gamma} \\ &= \sum_{\beta+\gamma+\alpha=0} \frac{|s_{\alpha\beta\gamma}|^2}{\Omega_\beta\Omega_\gamma}. \end{aligned}$$

Now

$$\Omega_\alpha = B(X_0, [E_\alpha, E_{-\alpha}]) = \alpha(X_0) B(E_\alpha, \overline{E_\alpha})$$

and B is negative definite so $i\Omega_\alpha < 0$ if $n_c(\alpha) > 0$. Thus $\Omega_\beta\Omega_\gamma$ is positive when β and γ are roots of the opposite signs and negative when they have the same sign.

We split the sum accordingly

$$R_\alpha^{(2)} = \sum_{\substack{\beta+\gamma+\alpha=0 \\ \beta,\gamma \text{ same sign}}} \frac{|s_{\alpha\beta\gamma}|^2}{\Omega_\beta\Omega_\gamma} + \sum_{\substack{\beta+\gamma+\alpha=0 \\ \beta,\gamma \text{ opposite sign}}} \frac{|s_{\alpha\beta\gamma}|^2}{\Omega_\beta\Omega_\gamma}.$$

LEMMA 5. *If $R_\alpha^{(2)} = 0$ for all α then $s_{\alpha\beta\gamma} = 0$ for all α, β, γ.*

Proof. Suppose some $s_{\alpha\beta\gamma}$ is not zero and take such a term where the minimum of the complementary height of the labels,

$$k = \min\{|n_c(\alpha)|, |n_c(\beta)|, |n_c(\gamma)|\},$$

is as near zero as possible. By symmetry we can suppose $n_c(\alpha) = k$ and consider the expression for $R^{(2)}_\alpha$. In the terms where β, γ have the same sign, then $n_c(\beta)+n_c(\gamma) = -k$ so both $|n_c(\beta)|$ and $|n_c(\gamma)|$ are smaller than k and hence all these terms vanish in $R^{(2)}_\alpha$. The remaining terms are all of the same sign and so $R^{(2)}_\alpha = 0$ implies they all vanish too. This contradicts the initial assumption and hence the Lemma follows.

□

Thus we have proved

PROPOSITION 6. *The highest order terms in $\frac{1}{2}r^2$ are non-zero for non-zero values of the parameter S in the space of invariant symplectic connections on the compact coadjoint orbit $\mathcal{O}$.*

THEOREM 7. *Every compact coadjoint orbit $\mathcal{O}$ has an invariant preferred symplectic connection.*

Proof. r^2 is an invariant function and so is constant on $\mathcal{O}$. Thus $L = \frac{1}{2}r^2 \operatorname{vol}(M)$. By the Proposition L satisfies the conditions of Lemma 1 and hence L has at least one critical point, its absolute minimum.

□

6. Decomposition into simple factors

A coadjoint orbit of a compact semisimple Lie group G is a product of orbits of the simple ideals of $\mathfrak{g}$. In general a symplectic connection on the orbit need not be compatible with such a decomposition. However, when the connection is G-invariant, things are different as we shall now see. We may assume that the compact semisimple group G is simply connected. It is thus the direct product of simply connected compact simple groups $G = \prod_{r=1}^p G_r$ and we also have $\mathfrak{g} = \sum_{r=1}^p \mathfrak{g}_r$. If $g \in G$ and $X_0 \in \mathfrak{g}$, one has

$$\operatorname{Ad}_g X_0 = \operatorname{Ad}_{g_1} \cdots \operatorname{Ad}_{g_p}(X_1 + \cdots + X_p) = \sum_{r=1}^{p} \operatorname{Ad}_{g_r} X_r.$$

The conjugacy class $G \cdot X_0$ is thus a product of the classes $G_r \cdot X_r$.

The Killing form B of $\mathfrak{g}$ is a direct sum of the correspond Killing forms, B_r and the symplectic form ω on the orbit $\mathcal{O}$ is a direct sum of the corresponding symplectic forms ω_r, on the orbits $\mathcal{O}_r$ which correspond under the Killing form to the conjugacy classes $G_r \cdot X_r$. Thus $(\mathcal{O}, \omega)$ is a product of the orbits $(\mathcal{O}_r, \omega_r)$.

LEMMA 8. *Any G-invariant, torsion-free symplectic connection ∇ on $\mathcal{O}_r$ induces a G_r-invariant connection ∇^r on $\mathcal{O}_r$, which is torsion-free and symplectic. Furthermore ∇ is the product of the corresponding connections ∇^r.*

Proof. Pick a torsion-free invariant symplectic connection $\overline{\nabla}^r$ on $\mathcal{O}_r$ and take as a base point on $\mathcal{O}$ the product connection $\nabla^0 = \Pi_{r=1}^p \overline{\nabla}^r$. Then any invariant connection ∇ has the form $\nabla^0 + A$. Obviously, it is enough to show that A respects the product structure. Let H be the stabiliser of $X_0 = \sum_r X_r$ and H_r the stabiliser of X_r then $H = \Pi_r H_r$. If T_r is a maximal torus in H_r then $T = \Pi_r T_r$ is a maximal torus in G and the H-invariant complement $\mathfrak{m}$ is a sum $\sum_r \mathfrak{m}_r$ of complements in each simple factor. The set of roots Δ of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ is the disjoint union of sets of roots for each simple factor. We consider $\underline{A}_{x_0}(E^*_\alpha, E^*_\beta, E^*_\gamma)$ which can only be non-zero if $\alpha + \beta + \gamma = 0$ and this can only happen if all three roots α, β, γ belong to the same simple factor. Thus $\underline{A} = \sum_r \underline{A}_r$ where each $\underline{A}_r$ is extended to the other simple factors by zero. ∇ is then the product of $\overline{\nabla}^r + A_r$.

□

7. $SU(3)$

We now turn to the example of a regular orbit of $SU(3)$. Let us first fix the notation.

The Lie Algebra of $SU(3)$ is

$$\mathfrak{su}(3) = \{X \in \mathrm{End}(\mathbb{C}^3) \mid X + {}^\tau\overline{X} = 0, \mathrm{Trace}(X) = 0\}$$

and the Lie algebra $\mathfrak{t}$ of the maximal torus is the set of purely imaginary, diagonal, trace free, 3×3 matrices. As a basis of $\mathfrak{t}$ we take

$$h_1 = i(E_{11} - E_{22}), \qquad h_2 = i(E_{22} - E_{33})$$

where E_{ij} is a matrix with a 1 in the ij position and all zeros elsewhere. Let h_1^*, h_2^* be the dual basis for $\mathfrak{t}^*$. We choose as positive roots

$$\alpha = i(h_1^* - 2h_2^*), \qquad \beta = i(h_1^* + h_2^*), \qquad \alpha + \beta.$$

The element X_0 of $\mathfrak{t}$ has components

$$X_0 = p_1 h_1 + p_2 h_2$$

and is regular if

$$(p_1 + p_2)(p_1 - 2p_2)(2p_1 - p_2) \neq 0.$$

We choose positive root space vectors

$$E_\alpha = aE_{32}, \quad E_\beta = a'E_{13}, \quad E_{\alpha+\beta} = a''E_{12}$$

and the conjugate ones for negative roots.

We extend the map D to $\mathfrak{m}^{\mathbb{C}}$; the invariance condition on ∇ yields $D(E_\gamma)E_\delta \in \mathfrak{g}^{\gamma+\delta}$ and we write

$$D(E_\gamma)E_\delta = D_{\gamma,\delta}E_{\gamma+\delta} \quad (\gamma,\delta \in \Delta)$$

Using relations (10) and (11) we get

$$\begin{aligned}
D_{\beta,\alpha} &= \frac{aa'}{a''} + D_{\alpha,\beta} \\
D_{\alpha,-(\alpha+\beta)} &= \frac{a\overline{a''}}{\overline{a'}} - \frac{2p_1-p_2}{p_1+p_2}(\frac{aa'}{a''} + D_{\alpha,\beta}) \\
D_{\beta,-(\alpha+\beta)} &= -(\frac{a'\overline{a''}}{\overline{a}} + \frac{2p_1-p_2}{p_1-2p_2}D_{\alpha,\beta}) \\
D_{-(\alpha+\beta),\alpha} &= -\frac{2p_1-p_2}{p_1+p_2}(\frac{aa'}{a''} + D_{\alpha,\beta}) \\
D_{-(\alpha+\beta),\beta} &= -\frac{2p_1-p_2}{p_1-2p_2}D_{\alpha,\beta}
\end{aligned}$$

and the conjugate ones. The space of homogeneous symplectic connections is of dimension 2; we shall identify it with $\mathbb{C}$ and take as coordinate z where

$$z = 2\frac{a''}{aa'}D_{\alpha,\beta}.$$

The Lagrangean has the form $\frac{1}{2}r^2 = \frac{L'+L''}{16.9.(p_1+p_2)^2(p_1-2p_2)^2}$ where

$$\begin{aligned}
L' = {} & 3z^2\overline{z}^2(2p_1-p_2)^2 + 12z\overline{z}(z+\overline{z})(2p_1-p_2))(p_1-p_2) \\
& +2(z+\overline{z})^2(7p_1^2+7p_2^2-13p_1p_2) + 8z\overline{z}[(2p_1-p_2)^2 \\
& +(p_1-2p_2)^2] + 48(z+\overline{z})(p_1-2p_2)(p_1-p_2)
\end{aligned}$$

and L'' does not depend on z.

The critical points correspond to the value of z such that $\frac{\partial L'}{\partial \overline{z}}$ vanishes. The imaginary part of this condition factorises in the form

$$(z-\overline{z})P(z,\overline{z}) = 0$$

where $P(z,\overline{z})$ is strictly positive. Thus, if we write $z = x+iy$, we have $y = 0$ and critical points become zeros of the real polynomial $Q(x) := 3x^3(2p_1-p_2)^2 + 18x^2(2p_1-p_2)(p_1-p_2) + 4x((2p_1-p_2)^2 + 4x(7p_1^2+7p_2^2-13p_1p_2) + (p_1-2p_2)^2) + 24(p_1-2p_2)(p_1-p_2)$. Since Q has odd degree, it has at least one real root. Now the derivative Q' is a second order polynomial whose discriminant is strictly negative on the interior of the Weyl chambers; hence $Q' > 0$ in this domain and thus Q has only one zero.

We summarise this analysis by

THEOREM 9. *On any regular adjoint orbit $\mathcal{O}$ of $SU(3)$ there exists a unique $SU(3)$-invariant preferred symplectic connection.*

The theorem is also true for non-regular orbits of $SU(3)$ since these are Hermitean symmetric spaces ($\mathbb{C}P^2$) and in this case there is only one invariant symplectic connection.

We know by Theorem 7 that L achieves its minimum on the invariant symplectic connections. Hence this minimum is the unique critical point we just found.

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