Symmetric symplectic spaces
with Ricci-type curvature

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Abstract
We determine the isomorphism classes of symmetric symplectic manifolds of dimension at least 4 which are connected, simply-connected and have a curvature tensor which has only one non-vanishing irreducible component - the Ricci tensor.

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Moshé Flato has been a close and wonderful friend and an inspiration for us for more than twenty years. This contribution is dedicated to him, always present in our hearts.

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1 Introduction

On any symplectic manifold \((M, \omega)\) the space of symplectic connections (linear connections \(\nabla\) with vanishing torsion and such that \(\nabla\omega = 0\)) is infinite dimensional. In order to select a smaller family of symplectic connections, a variational principle was introduced in [2]. This principle has Euler-Lagrange equations

\[
(\nabla_X r)(Y, Z) + (\nabla_Y r)(Z, X) + (\nabla_Z r)(X, Y) = 0
\]

for all vector fields \(X, Y, Z\); \(r\) denotes the Ricci tensor of \(\nabla\)

\[
r(X, Y) = \text{Tr}(Z \mapsto R(X, Z)Y).
\]

In [2] the case where \(\dim M = 2\) was examined in complete detail so we shall assume throughout the \(\dim M \geq 2\).

It was observed in [3] that the field equations (1) are identically satisfied if one assumes that the irreducible component of the curvature, denoted there by \(W\) (see also [5]), vanishes

\[
W = 0. \tag{2}
\]

The tensor \(W\) is the symplectic analogue of the Weyl or conformal curvature of a Riemannian connection. The vanishing of \(W\) (equation (2)) is equivalent to the requirement that the curvature tensor \(R\) of \(\nabla\) is expressed in terms of its Ricci tensor by

\[
R(X, Y)Z = \frac{1}{2(n + 1)} \left[ 2\omega(X, Y)AZ + \omega(X, Z)AY - \omega(Y, AZ)X - \omega(Y, Z)AX + \omega(X, AZ)Y \right] \tag{3}
\]

where \(\dim M = 2n, n \geq 2\), where \(X, Y, Z\) are vector fields and where \(A\) is the Ricci tensor viewed as an endomorphism of the tangent bundle using \(\omega\):

\[
r(X, Y) = \omega(X, AY). \tag{4}
\]

The Ricci tensor is symmetric so \(A\) is an infinitesimal symplectic endomorphism of each tangent space.

Equations (2) (or (3)) imply the existence of a 1-form \(u\) on \((M, \omega)\) such that

\[
(\nabla_X r)(Y, Z) = \omega(X, Y)u(Z) + \omega(X, Z)u(Y). \tag{5}
\]

If \(u = 0\), then \(\nabla r = 0\) and since \(R\) is expressed in terms of \(r\) (3), \(\nabla\) is locally symmetric.

The condition \(W = 0\) also appears as the integrability condition for the almost complex structure naturally defined from a symplectic connection on \((M, \omega)\) on the manifold \(J(M)\) of almost complex structures on \(M\) which are compatible with \(\omega\).

In this note we prove, amongst other things, the following two results.
Theorem 1 Let \((M, \omega) = (M_1, \omega_1) \times (M_2, \omega_2)\) be symplectic manifolds of dimension greater than zero and \(\nabla = \nabla_1 + \nabla_2\) be a symplectic connection. If \(W^\nabla = 0\) then \(\nabla, \nabla_1, \nabla_2\) are flat.

Theorem 2 Let \((M, \omega, s)\) be a connected, simply-connected, symmetric symplectic space of dimension \(2n(\geq 4)\); let \(\nabla\) be its canonical invariant symplectic connection and let \(r\) be its Ricci tensor; let \(A\) be the corresponding endomorphism
\[
\omega(X, AY) = r(X, Y).
\]
Assume \(W^\nabla = 0\). Then
\[
A^2 = \lambda I_d
\]
for some real number \(\lambda\).

If \(\lambda \neq 0\), the transvection group \(G\) of \((M, \omega, s)\) is semisimple and, up to coverings, \(M = G/K\) with either \(G = SL(n+1, \mathbb{R})\) and \(K = GL(n, \mathbb{R})\) or \(G = SU(p + 1, q)\) and \(K = U(p, q)\) where \(\dim M = 2n\), \(p + q = n\).

If \(\lambda = 0\) and \(\text{Rank}(A) > 1\), the transvection group \(G\) of \((M, \omega, s)\) is neither solvable nor semisimple. The radical of \(G\) is a 2-step unipotent if \(\text{Rank}(A) < n\) and abelian in \(\text{Rank}(A) = n\). If \(\lambda = 0\) and \(\text{Rank}(A) = 1\), the transvection group \(G\) of \((M, \omega, s)\) is solvable.

2 Proof of Theorem 1

Let \((M, \omega) = (M_1, \omega_1) \times (M_2, \omega_2)\) be symplectic manifolds and \(\nabla = \nabla_1 + \nabla_2\) be a symplectic connection. Then \(R(X, Y)Z = R_1(X_1, Y_1)Z_1 + R_2(X_2, Y_2)Z_2\) where \(X = X_1 + X_2\), \(Y = Y_1 + Y_2\), \(Z = Z_1 + Z_2\) and suffices indicate components tangent to \(M_1\) and \(M_2\), respectively. Then also \(r(X, Y) = r_1(X_1, Y_1) + r_2(X_2, Y_2)\). On the other hand, the relation between \(W, W_1\) and \(W_2\) involves cross terms \(C(X, Y)Z\):
\[
W(X, Y)Z = W_1(X_1, Y_1)Z_1 + W_2(X_2, Y_2)Z_2 + C(X, Y)Z.
\]
These can be read off equation (3). Then \(W = 0\) implies \(W_1 = 0\), \(W_2 = 0\) and \(C = 0\). We have
\[
C(X_1, Y_1)Z_2 = \frac{1}{2(n+1)} \left[-2\omega(X_1, Y_1)A_2Z_2\right]
\]
so \(A_2 = 0\) and interchanging 1 and 2 we see also \(A_1 = 0\). Thus \(r_1 = 0\) and \(r_2 = 0\), and hence \(R_1 = 0\) and \(R_2 = 0\).

3 Some facts about symmetric symplectic spaces

Affine symmetric spaces are studied in Loos [4], symplectic symmetric spaces are studied in Bielavsky [1].
Definition 3 A symmetric symplectic manifold is a triple \((M, \omega, s)\) where \(M\) is a smooth connected manifold, where \(\omega\) is a smooth symplectic form on \(M\) and where \(s\) is a smooth map \(M \times M \to M, (x, y) \mapsto s_x(y)\), such that:

(i) for each \(x \in M\), \(s_x\) is an involutive symplectic diffeomorphism of \((M, \omega)\) (called the symmetry at \(x\)) and \(x\) is an isolated fixed point of \(s_x\),

(ii) \(s_x s_y s_x = s_x s_{s_y(x)}\) for all \(x, y \in M\).

The transvection group \(G\) of \((M, \omega, s)\) is the group generated by products of an even number of symmetries.

We recall below some general facts about symmetric spaces ([4], [1]).

1. \((M, \omega, s)\) has a unique connection \(\nabla\) such that \(\nabla \omega = 0\) and such that each symmetry \(s_x\) is an affine transformation of \((M, \nabla)\). Observe that \(s_{x \cdot x} = -\text{Id}_{T_x M}\) because \((s_{x \cdot x})^2 = \text{Id}_{T_x M}\) and \(x\) is an isolated fixed point of \(s_x\). Since \(\omega_x(\nabla_X Y, Z) = \frac{1}{2}(\omega_x(\nabla_X Y, Z) + (s_x \cdot \omega)_x(\nabla_X Y, Z))\), the connection is given by

\[
\omega_x(\nabla_X Y, Z) = \frac{1}{2} X_x(\omega(Y + s_x \cdot Y, Z))
\]

for \(x \in M\), where \(X, Y, Z\) are vector fields on \(M\) and \((s_x \cdot Y)_y = s_x \cdot Y_{s_x(y)}\). This connection \(\nabla\) has no torsion and is thus a symmetric connection. The symmetry \(s_x\) coincides with the geodesic symmetry around \(x\), since an affinity is determined by its 1-jet at a point.

2. The automorphism group \(\text{Aut} = \text{Aut}(M, \omega, s)\) of \((M, \omega, s)\) is the set of symplectic automorphisms \(\varphi\) of \((M, \omega)\) such \(\varphi \circ s_x = s_{\varphi(x)} \circ \varphi, \forall x \in M\). It is the intersection of the affine group of \((M, \nabla)\) and the symplectic diffeomorphism group of \((M, \omega)\). It is thus a Lie group containing the transvection group so acts transitively on \(M\) (since any two points in \(M\) can be joined by a broken geodesic).

Choose a base point \(o\) in \(M\). Denote by \(\bar{\sigma}\) the conjugation by the symmetry \(s_x\), it is an involutive automorphism of \(\text{Aut}\).

Let \(K'\) denote the stabilizer of \(o\) in \(\text{Aut}\) and let \(A^\sigma\) (respectively \(A^\sigma_o\)) denote the group of fixed points of \(\bar{\sigma}\) in \(\text{Aut}\) (respectively its connected component). Then \(A^\sigma_o \supseteq K' \supseteq A^\sigma\).

Hence, if \(a\) (respectively \(\mathfrak{e}'\)) is the Lie algebra of \(\text{Aut}\) (respectively \(K'\)) and if \(\sigma = \bar{\sigma} \cdot 1_d\), then \(\mathfrak{e}'\) is the subalgebra of \(a\) of fixed points of \(\sigma\).

3. Let \(p = \{X \in a \mid \sigma(X) = -X\}\). Then \(a = \mathfrak{e}' \oplus p\).

Denote by \(\pi'\) the projection \(\text{Aut} \to M\) given by \(\pi'(g) = g \cdot o\). Then \(\pi'_{o, p} : p \to T_o M\) is a linear isomorphism which identifies the tangent space \(T_o M\) with \(p\).
Denote by \( \operatorname{Exp}: T_o M \to M \) the exponential map given by the connection \( \nabla \) at the point \( o \) and by \( \operatorname{exp} \) the exponential map from the Lie algebra \( \mathfrak{a} \) to the Lie group \( \text{Aut} \).

Observe that \( s_{\operatorname{Exp} \circ X} s_o, X \in T_o M \), is an affinity in \( G \) which realises the parallel transport along \( \operatorname{Exp} t X \), since \( s_{\operatorname{Exp} \circ X(a)} \) for any \( a \in \mathbb{R} \) maps a vector field which is parallel along the geodesic \( \operatorname{Exp} t X \) to another such parallel vector field. Hence \( s_{\operatorname{Exp} \circ X(a)} s_o = \operatorname{exp} t X, \forall X \in \mathfrak{p} \).

It follows that the transvection group \( G \), which is stable by \( \tilde{\sigma} \), is the connected Lie subgroup of \( \text{Aut}(M, \omega, s) \) whose Lie algebra is

\[
\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \quad \text{where} \quad \mathfrak{t} = [\mathfrak{p}, \mathfrak{p}].
\] (7)

Indeed, if \( G_1 \) denotes that subgroup, clearly by the above \( G_1 \subset G \) and the parallel transport along a geodesic \( \operatorname{Exp} t X \) is in \( G_1 \), but then any \( x \in M \) can be written as \( x = g \cdot o \) for \( g \in G_1 \) hence \( s_x s_o = g s_o g^{-1} s_o = g \tilde{\sigma}(g^{-1}) \in G_1 \) and \( G \subset G_1 \).

Let \( K \) denote the stabilizer of \( o \) in \( G \). Its Lie algebra is \( \mathfrak{t} \) and \( \mathfrak{t} = \{ X \in \mathfrak{g} \mid \sigma(X) = X \} \). Since the Lie group \( G \) acts effectively on \( M \), the representation of \( K \) on \( T_o M, k \mapsto k_s o \), is faithful so \( \mathfrak{t} \) acts faithfully on \( \mathfrak{p} \).

(4) Denote by \( \pi \) the projection \( \pi: G \to M \) where \( \pi(g) = g \cdot o \). Denote by \( X^* \) the vector field on \( M \) which is the image under \( \pi_* \) of the right invariant vector field on \( G \), i.e. \( X^*_{g \cdot o} = \frac{d}{dt} \exp t X \cdot g \cdot o |_{t = 0} \). Observe that \([X^*, Y^*] = -[X, Y]^* \). Since \( \omega \) is invariant under \( G \), formula (6) yields \( \omega_x (\nabla_{X} Y^*, Z^*) = \frac{1}{2} \omega_x ([Y^*, X^* + s_x \cdot X^*], Z^*) \) so \( (\nabla_{X^*} Y^*)_x = [X^*, Y^*] + \frac{1}{2} [Y^*, X^* + s_x \cdot X^*] \). But \( s_{g \cdot o} X^* = g \cdot s_o g^{-1} \cdot X^* = (\text{Ad} g \sigma(\text{Ad} g^{-1} X))^* \) so the connection has the form

\[
(\nabla_{X^*} Y^*)_x = ([Y, \text{Ad} g(\text{Ad} g^{-1} X)]_\mathfrak{p})^{g \cdot o} \quad \text{where} \quad Z_\mathfrak{p} \text{ denotes the component in } \mathfrak{p} \text{ of } Z \in \mathfrak{g} \text{ relatively to the decomposition } \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{t} \text{ and where } [\cdot, \cdot] \text{ is the bracket in } \mathfrak{g}.
\] (8)

Since any \( G \)-invariant tensor on \( M \) is parallel, the curvature tensor of \((M, \nabla)\) is parallel \((\nabla R = 0)\) and if \( X, Y, Z \) belong to \( \mathfrak{p} \), one has,

\[
R_o(X_o^*, Y_o^*) Z_o^* = -([X, Y], Z)_o^*.
\] (9)

**Definition 4** A **symmetric symplectic triple** is a triple \((\mathfrak{g}, \sigma, \Omega)\) where \( \mathfrak{g} \) is a finite dimensional real Lie algebra, \( \sigma \) is an involutive automorphism of \( \mathfrak{g} \) such that if we write \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \) with \( \sigma = \text{Id}_\mathfrak{t} \oplus -\text{Id}_\mathfrak{p} \), then

- \([\mathfrak{p}, \mathfrak{p}] = \mathfrak{t} \);
- the action of \( \mathfrak{t} \) on \( \mathfrak{p} \) is faithful.
and where $\Omega$ is a non-degenerate skewsymmetric 2-form on $\mathfrak{p}$, invariant by $\mathfrak{t}$ under the adjoint action.

We have seen above that to any connected symmetric symplectic manifold $(M, \omega, s)$, when one chooses a base point $o \in M$, one associates a symmetric symplectic triple $(\mathfrak{g}, \sigma, \Omega)$ with $\mathfrak{g}$ the Lie algebra of its transvection group $\mathfrak{t}$, with $\sigma$ the differential at the identity of the conjugation by the symmetry $s_o$ and with $\Omega = \omega_o$ with the identification between $T_oM$ and $\mathfrak{p}$.

Reciprocally, given a symmetric symplectic triple $(\mathfrak{g}, \sigma, \Omega)$, one builds a simply-connected symmetric symplectic space $(M, \omega, s)$ with $M = G/K$ where $G$ is the simply-connected Lie group with Lie algebra $\mathfrak{g}$ and $K$ is its connected subgroup with Lie algebra $\mathfrak{k}$, with $\omega$ the $G$-invariant 2-form on $M$ whose value at $\epsilon K$ is given by $\Omega$ (identifying $T_oK M$ and $\mathfrak{p}$ via the differential of the canonical projection $\pi: G \to G/K$) and with symmetries defined by $s_{\pi(\epsilon)} s(g) = \pi(g \tilde{\sigma}(g^{-1} g'))$ where $\tilde{\sigma}$ is the automorphism of $G$ whose differential at $\epsilon$ is $\sigma$.

## 4 Proof of Theorem 2

Consider a symmetric symplectic space $(M, \omega, s)$ and assume that its canonical invariant symplectic connection $\nabla$ has a curvature with $W = 0$.

Since $\nabla R = 0$, the Ricci tensor $r$ and its associated endomorphism $A$ (where $r(X, Y) = \omega(X, AY)$) are covariantly constant and hence $A$ commutes with the curvature endomorphisms

$$AR(X, Y) = R(X, Y)A.$$  

This implies, when we substitute $R$ by its expression in terms of $A$ into equation (3)

$$\omega(X, Z) A^2 Y + \omega(Y, Z) A^2 X = \omega(Y, A^2 Z)X - \omega(X, A^2 Z) Y.$$

If $Y \neq 0$ is arbitrary, $Z = Y$, and we pick $X$ so that $\omega(X, Y) = 1$, then $\omega(Y, A^2 Y) = \omega(A Y, A Y) = 0$, so $A^2 Y = \lambda_Y Y$ for some function $\lambda_Y$. Substituting back into the equation shows that $\lambda_Y = \lambda$ is independent of $Y$, and since $A$ is covariant constant, $\lambda$ must be constant.

Remark that if $\lambda \neq 0$ then $r$ is a non-degenerate parallel symmetric bilinear form so $\nabla$ is its Levi-Civita connection and $(M, r, s)$ is a pseudo-Riemannian symmetric space.

Let $G$ be the transvection group of our symmetric symplectic space. Choose a base point $o \in M$ and let $(\mathfrak{g}, \sigma, \Omega)$ be the symmetric triple associated to $(M, \omega, s)$. Let $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$ be the decomposition of the Lie algebra of $G$ into the $+1$ and $-1$ eigenspaces of $\sigma$. Then $\Omega(X, Y) = \omega_o(\mathfrak{X}^*_o, \mathfrak{Y}^*_o)$ and with a slight abuse of notations we denote by $R$ the map $R: \mathfrak{p} \times \mathfrak{p} \to \text{End}(\mathfrak{p})$ so
that \((R(X,Y)Z)_e = R_e(X_e,Y_e)Z_e\) and by \(A\) the map \(A : p \to p\) so that \((A(X))_e = A_e(X_e)\).

Since \(\mathfrak{e}\) acts faithfully on \(p\), we view \(\mathfrak{e}\) as a subset of \(\text{End}(p)\); by formula (9),

\[
\mathfrak{e} = \{R(X,Y) \in \text{End}(p) \mid X,Y \in p\}
\]  

(10)

and the brackets on \(\mathfrak{g} \subset p \oplus \text{End}(p)\) are

\[
[(C,X), (D,Y)] = [(C,D) - R(X,Y), CY - DX]
\]  

(11)

where \(C,D \in \mathfrak{e} \subset \text{End}(p)\), and \(X,Y \in p\).

Define the 1-form on \(p\) corresponding to a vector \(X \in p\) by \(\underline{X} = i(X)\Omega\). Formula (3) giving the curvature when \(W = 0\) is equivalent to

\[
R(X,Y) = k \left( 2\Omega(X,Y)A + AY \otimes \underline{X} - AX \otimes \underline{Y} + X \otimes \underline{AY} - Y \otimes \underline{AX} \right)
\]

where \(k = 1/(m+2)\) if \(m = \dim M = 2n\). Note that, for a symplectic symmetric space built from a Lie algebra \(\mathfrak{g} = \mathfrak{e} + p\) whose bracket of \(p\) into \(\mathfrak{e}\) is given by this formula, then the canonical connection will have curvature given by this formula and hence \(W\) will vanish.

Define \(B = Y \otimes \underline{X} - X \otimes \underline{Y}\). Clearly \(B\) satisfies \(\Omega(U, BV) = \Omega(BU, V)\) and any antisymplectic endomorphism of \(p\) can be written as a sum of such operators. Then

\[
R(X,Y) = k(\text{Tr}(B)A + AB + BA)
\]

and, if we put \(B' = k(B + \frac{1}{2} \text{Tr}(B)I)\), the RHS becomes \(C = AB' + B'A\).

**Lemma 5** For any \(\lambda\),

\[
\mathfrak{e} = \{C = AB + BA \mid B \in \text{End}(p) \text{ and } \Omega(X, BY) = \Omega(BX, Y)\}.
\]

If \(\lambda \neq 0\) then \(\mathfrak{e}\) is the set of endomorphisms \(C \in \text{End}(p)\) which are infinitesimally symplectic and commute with \(A\).

**Proof** The first part follows from the considerations above and the fact that the map \(B \mapsto B + \frac{1}{2} \text{Tr}(B)\) is a bijection on the space of antisymplectic endomorphisms of \(p\). \(C\) commutes with \(A\) since \(AC = \lambda B' + AB'A = CA\). Also \(\Omega(X,CY) = -\Omega(AX, B'Y) + \Omega(B'X, AY) = -\Omega(B'AX, Y) - \Omega(AB'X, Y) = -\Omega(CX, Y)\).

Conversely, if \(\lambda \neq 0\), given \(C\) commuting with \(A\) and such that \(\Omega(X, CY) = -\Omega(CX, Y)\), let \(B = \frac{1}{2} \lambda^{-1} AC\); then

\[
BA + AB = \frac{1}{2} \lambda^{-1} 2\lambda C = C.
\]

\(\Box\)
4.1 Case $\lambda > 0$

Write $\lambda = a^2$, $a > 0$. Then $p = V^+ \oplus V^-$ where $V^\pm = \{X \in p \mid AX = \pm aX\}$. Let $P^\pm$ be the projection onto $V^\pm$. Then $A = a(P^+ - P^-)$. Clearly

$$
\omega(V^+, V^+) = \omega(V^-, V^-) = 0
$$

$$
R(V^+, V^+) = R(V^-, V^-) = 0
$$

$$
R(X, Y) = 2ka(\Omega(X, Y)(P^+ - P^-) - Y \otimes X - X \otimes Y),
$$

for $X \in V^+$, $Y \in V^-$. It follows that $V^\pm$ are Lagrangian subspaces of $p$. Identifying $V^-$ with $(V^+)^*$ via $Y \mapsto \sum_{V^+}$ and renaming $V^+$ as $V$, we have identified $p$ with $V \oplus V^*$ with its standard symplectic structure $\Omega(X + \xi, X' + \xi') = -\langle X, \xi \rangle + \langle X', \xi \rangle$, and $A$ acts as $+a$ on $V$, $-a$ on $V^*$. With this notation the curvature has the form

$$
R(X, \xi) = 2ak(-\langle X, \xi \rangle(Id_V - Id_{V^*}) + \xi \otimes X - X \otimes \xi).
$$

The symplectic centraliser of $A$ can then be identified with $\text{End}(V) = \mathfrak{gl}(V)$, identifying the element in $\text{End}(p) = \text{End}(V \oplus V^*)$ given by

$$
\begin{pmatrix}
C & 0 \\
0 & -C
\end{pmatrix}
$$

with the element $C \in \mathfrak{gl}(V)$.

So $\mathfrak{f} = \mathfrak{gl}(V)$ and as a vector space $\mathfrak{g} = \mathfrak{gl}(V) \oplus V \oplus V^*$ with the brackets

$$
[(C, X, \xi), (C', X', \xi')] = ([C, C'] + 2ka(\langle X, \xi \rangle - \langle X', \xi \rangle)I
$$

$$
+ 2kaX \otimes \xi' - 2kaX' \otimes \xi,
$$

$$
CX' - C'X, -tC\xi' + tC'\xi).
$$

The map $j: \mathfrak{g} \to \mathfrak{gl}(V \oplus \mathbb{R})$ given by

$$
j(C, X, \xi) = \begin{pmatrix}
C - 2k \text{Tr}(C)I & sX \\
-s\xi & -2k \text{Tr}(C)
\end{pmatrix}
$$

has the brackets above provided $s^2 = 2ka$.

Thus when $\lambda > 0$, $M = G/K$ where $G = SL(n+1, \mathbb{R})$, $K = GL(n, \mathbb{R})$. The involution $\sigma$ is given by

$$
\sigma \begin{pmatrix}
C & v \\
\xi & -\text{Tr}(C)
\end{pmatrix} = \begin{pmatrix}
C & -v \\
-\xi & -\text{Tr}(C)
\end{pmatrix}
$$

and, writing $(X, \xi)$ for $\begin{pmatrix}
0 & X \\
\xi & 0
\end{pmatrix}$, the symplectic form is given by

$$
\Omega((X, \xi), (X', \xi')) = -\langle X, \xi \rangle + \langle X', \xi \rangle.
$$
The curvature of the canonical connection on this symplectic symmetric space at the base point $eK$ is

$$R((X, \xi), (X', \xi'))((X'', \xi'')) = (X''((X', \xi) - \langle X, \xi' \rangle) - X \langle X'', \xi' \rangle + X'((X'', \xi) - \langle X', \xi'' \rangle) - \xi((X', \xi) - \langle X', \xi' \rangle))$$

$$r((X, \xi), (X', \xi')) = (n + 1)((X, \xi') + \langle X', \xi \rangle)$$

$$A(x, \xi) = (n + 1)(x, -\xi)$$

and formula (3) holds so $R$ is of Ricci-type.

4.2 Case $\lambda < 0$

We write $\lambda = -b^2$ where $b < 0$. If we put $J = b^{-1}A$ then $J$ defines a complex structure on the vector space $p$. We write $V$ for $p$ viewed as an $n$-dimensional complex vector space. $V$ has a (pseudo-)Hermitean structure given by

$$\langle X, Y \rangle = \Omega(X, JY) + i\Omega(X, Y)$$

which is $C$-linear in the second variable. The infinitesimally symplectic transformations which commute with $A$, or equivalently $J$, are the complex linear transformations of $V$ which are skew-Hermitean with respect to this Hermitean structure. Thus $f$ is the (pseudo-) unitary Lie algebra $u(V, \langle \cdot, \cdot \rangle)$.

The curvature has the form

$$R(X, Y) = kb (2\Omega(X, Y)J + Y \otimes (X, \cdot) - X \otimes (Y, \cdot))$$

Then $g = u(V, \langle \cdot, \cdot \rangle) \oplus V$ with bracket

$$[(C, X), (C', X')] = [[C, C'] + kb(X \otimes (X', \cdot) - X' \otimes (X, \cdot)) - 2\Omega(X, X')J, CX' - C'X].$$

and $g$ can be identified with $su(V \oplus C, \langle \cdot, \cdot \rangle)$ via

$$j(C, X) = \begin{pmatrix}
C - 2k \text{Tr}(C)I & sX \\
-\pi(X, \cdot) & -2k \text{Tr}(C)
\end{pmatrix}$$

with

$$\langle ((v, r), (w, t)) \rangle = \langle v, w \rangle + \pi t$$

provided

$$s\pi = -kb.$$
Hence when \( \lambda < 0 \) then \( M = G/K \) with \( \mathfrak{g} = \mathfrak{su}(p + 1, q) \), \( p + q = n \), \( \mathfrak{k} = \mathfrak{u}(p, q) \),

\[
\sigma \left( \begin{array}{cc}
C & v \\
-\langle v, . \rangle & - \text{Tr}(C)
\end{array} \right) = \left( \begin{array}{cc}
C & -v \\
\langle v, . \rangle & - \text{Tr}(C)
\end{array} \right)
\]

and

\[
\Omega(v, w) = \text{Im} \langle \langle v, w \rangle \rangle.
\]

The curvature of the canonical connection on this symmetric symplectic space at \( \epsilon K \) is

\[
R(v, w)z = v\langle w, z \rangle - w\langle v, z \rangle + z(\langle v, w \rangle + \langle w, v \rangle)
\]

\[
r(v, z) = -2(n + 1)\langle v, z \rangle
\]

\[
A(v) = -2(n + 1)iv
\]

and formula (3) holds so \( R \) is of Ricci-type.

### 4.3 Case \( \lambda = 0 \)

In this case \( A \) is nilpotent since \( A^2 = 0 \). Let \( Z = \text{Image} \ A \) and \( \tilde{Z} = \text{Ker} \ A \). Then \( Z \subset \tilde{Z} \), and \( Z \) and \( \tilde{Z} \) are symplectic orthogonals of each other. If \( V \) denotes a complement for \( Z \) in \( \tilde{Z} \), then the restriction of \( \Omega \) to \( V \) is non-degenerate. \( Z \) is contained in the \( \Omega \)-orthogonal of \( V \); let \( Z' \) be a complement so that \( V^\perp = Z \oplus Z' \). \( V^\perp \) is a symplectic subspace and \( Z \) is maximal isotropic so we can also suppose that \( Z' \) is maximal isotropic. \( \Omega \) gives a duality of \( Z \) with \( Z' \).

In other words, we have written \( p \) as \( Z \oplus Z^* \oplus V \) where \( Z \oplus Z^* \) has its standard symplectic structure and \( V \) is a symplectic vector space. \( A \) is non-zero only on \( Z^* \) and maps it isomorphically onto \( Z \), and as such it is symmetric. In block form, the symplectic structure \( \Omega \) is given by

\[
\begin{pmatrix}
0 & -I & 0 \\
I & 0 & 0 \\
0 & 0 & J'
\end{pmatrix}
\]

and \( A \) by

\[
\begin{pmatrix}
0 & A' & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

where \( A' \), by a suitable choice of basis is diagonal with \( \pm 1 \) on the diagonal. An easy calculation shows that matrices of the form \( AB + BA \) with \( \Omega(X, BY) = \Omega(BX, Y) \) have the
form

\[
\begin{pmatrix}
K & L & -iMJ^t \\
0 & -iK & 0 \\
0 & M & 0
\end{pmatrix}
\]

where \(iKA' + A'K = 0, iL = L\). The matrices with \(K = 0\) form an ideal which is 2-step nilpotent (abelian when \(\text{Rank } A = n = \frac{1}{2} \dim M\)) and the matrices with \(L = M = 0\) a subalgebra isomorphic to \(so(p, q)\), where \(p + q = r = \text{Rank } A\), \(p\) the number of +’s and \(q\) the number of −’s in \(A'\) (hence \((p, q)\) is the signature of the non-degenerate symmetric bilinear form naturally induced on \(p/\text{Ker } A\) by the Ricci tensor \(\Omega(X, Ay)\)).

The bracket of \(p\) into \(t\) is given, using formulas (11) and (3) by

\[
\begin{pmatrix}
u \\
v' \\
w
\end{pmatrix}
, \begin{pmatrix}
u' \\
v'' \\
w'
\end{pmatrix}
= -k \begin{pmatrix}
\tilde{K} = A'(v''v - v'lv') & 0 & -i\tilde{M}J^t \\
0 & -i\tilde{K} & 0 \\
0 & -\tilde{M} & 0
\end{pmatrix}
\]

where \(\tilde{L} = A'B + iBA' + 2(\text{Tr } B + iJ'w'A')A'\) with \(B = v'^tv' - v'lv\) and \(\tilde{M} = -i(A'(v''w - v'w'))\).

Then \(\mathfrak{g} = \mathfrak{t} + \mathfrak{p} = \{(K, L, M, u, v, w) | K \in so(p, q), L \in \text{Mat}(r \times r, \mathbb{R}), iL = L, M \in \text{Mat}(2n - 2r \times r, \mathbb{R}), u \in Z = \mathbb{R}^r, v \in Z^*, w \in W = \mathbb{R}^{2n-2r}\}. The brackets are given, with obvious notations, by

\[
[(K, L, M), (K', L', M')] = ([K, K'], L'', -M'K' + M''K)
\]

where \(L'' = KL' - L'K - K'L + L''K - iMJ'M + iM'J'M,\)

\[
[(K, L, M), (u, v, w)] = (Ku + Lv - iMJ'w, -iKv, Mv),
\]

\[
[(u, v, w), (u', v', w')] = (-kA'(v''v - v'lv'), -kL, kL(A'(v''w - v'w')))
\]

where \(\tilde{L}\) is defined as above.

We can combine \(so(p, q)\) with \(Z^*\) to give \(so(p, q + 1)\) via

\[
(K, v) \mapsto \begin{pmatrix}
K & -k^{1/2}A'v \\
-k^{1/2}i_v & 0
\end{pmatrix}.
\]

The subset \(\mathfrak{r} = \{(0, L, M, u, 0, w) \in \mathfrak{g}\}\) is a 2-step nilpotent ideal of \(\mathfrak{g}\) (abelian when \(r = n\) i.e. when the rank of the Ricci tensor is half the dimension of the manifold). Hence, when \(p + q = r > 1\), \(\mathfrak{r}\) is the radical of \(\mathfrak{g}\) and the semisimple Levi factor of \(\mathfrak{g}\) is isomorphic to \(so(p, q + 1)\).
5 Some corollaries

Corollary 6 Let \((M_i, \omega_i, s_i), i = 1, 2\) be symmetric symplectic spaces of the same dimension 2n with \(W_i = 0\) with semisimple transvection groups \(G_i\). Then \(G_1^C = G_2^C\).

Proof \(SL(n + 1, \mathbb{R})\) and \(SU(p + 1, q)\) both have \(SL(n + 1, \mathbb{C})\) as complexification. \(\square\)

Corollary 7 Let \((M, \omega, s)\) be a compact, simply-connected symmetric symplectic space of dimension 2n such that \(W = 0\) then \((M, \omega, s)\) is \(\mathbb{P}_n(\mathbb{C})\).

Proof This follows immediately from the list in Theorem 2. The only case where \(G/K\) is compact is when \(G = SU(n + 1)\) and \(K = U(n)\). \(\square\)

In dimension 4 we have the following list of possibilities (up to coverings) for \(M\):

- \(SL(3, \mathbb{R})/GL(2, \mathbb{R})\);
- \(SU(1, 2)/U(2)\);
- \(SU(2, 1)/U(1, 1)\);
- \(SU(3)/U(2)\);
- \(\lambda = 0\) cases corresponding to:
  - Rank \(A = 1, p = 0\) or \(p = 1\);
  - Rank \(A = 2, p = 0, p = 1\) or \(p = 2\).

References


