Symmetries of Star Products

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1 Definitions and an example

Definition 1. A *star product* [3, 4] on a manifold M is an $\mathbb{R}[\![\lambda]\!]$-bilinear associative multiplication $*$ on $C^\infty(M)[\![\lambda]\!]$ for which:

1. 1 is a unit

2. $u * v$ can be expanded in the form

$$u * v = u \cdot v + \sum_{r \geq 1} \lambda^r C_r(u, v)$$

Condition 1 requires that every C_r satisfy $C_r(u, 1) = C_r(1, u) = 0$ for $r \geq 1$. $*$ is called *differential* if the $C_r(u, v)$ are differential operators. Under this condition, Condition 1 then implies that locally constant functions are in the centre of the deformed algebra $(C^\infty(M)[\![\lambda]\!], *)$

Clearly

$$u * v - v * u = \lambda\big(C_1(u, v) - C_1(v, u)\big) + \mathcal{O}(\lambda^2),$$

and we put

$$\{u, v\}_* := C_1(u, v) - C_1(v, u).$$

This bracket $\{\ ,\ \}_*$ is a Lie bracket on $C^\infty(M)$ and a derivation in each argument. In other words, $\{\ ,\ \}_*$ is a *Poisson bracket.*

Question 1. For each Poisson bracket $\{\ ,\ \}$ on M, does there exist a star product $*$ on $C^\infty(M)[\![\lambda]\!]$ such that $\{\ ,\ \}_* = \{\ ,\ \}$?

Theorem 1 (Kontsevich, 1997, [39]). *For any Poisson bracket* $\{\ ,\ \}$ *on* M*, there exists a star product* $*$ *satisfying* $\{\ ,\ \}_* = \{\ ,\ \}$.

We call such a star product a *deformation quantisation* of the given Poisson bracket. When the Poisson bracket comes from a symplectic structure, then some quite strong results are known about the existence and uniqueness of star products. The existence problem was solved in the early 1980's (deWilde and Lecomte,

[17]) using Čech theoretic methods, and more geometrically by Omori, Maeda and Yoshioka [48] and Fedosov [20].

The difference between the symplectic case and the non-symplectic case can be seen from the centre of the deformed algebra $(C^\infty(M)[\![\lambda]\!], *)$ which we shall write as $\mathrm{Z}(M, *)$. We have seen that when $*$ is differential $\mathrm{Z}(M, *)$ contains all locally constant functions. If

$$u = u_0 + u_1\lambda + u_2\lambda^2 + \cdots \in Z(M, *)$$

then $u * v - v * u = 0$ for each $v \in C^\infty(M)$. If we write this out order by order then at first order in λ, we have

$$\{u_0, v\} = -X_v(u_0) = 0 \qquad \text{for all } v \in C^\infty(M),$$

so u_0 is central in the Poisson algebra. In the symplectic case the X_v span each tangent space which means u_0 is locally constant, that is $u_0 \in \mathrm{H}^0(M)$. In the Poisson case the central functions only need to be constant on each symplectic leaf so $Z(M, *)$ can be very much larger.

If $*$ is a differential star product, then $\mathrm{H}^0(M) \subset \mathrm{Z}(M, *)$ and hence $\frac{1}{\lambda}(u - u_0) \in \mathrm{Z}(M, *)$. We can apply this argument recursively to see that $u_1, u_2, \cdots \in \mathrm{H}^0(M)$ when $*$ is differential. Therefore, we have

$$\mathrm{Z}(M, *) = \mathrm{H}^0(M)[\![\lambda]\!]$$

for differential star products on symplectic manifolds.

From now on we restrict attention to differential star products on symplectic manifolds.

Example 1. Take Euclidean space $\mathbb{R}^{2m}$ with a symplectic form $\Omega = \frac{1}{2}\sum_{i,j} \Omega_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j$ (Ω_{ij} are constants). Let Λ^{ij} denote the inverse matrix of Ω_{ij}. Then the Poisson bracket corresponding to Ω has the form

$$\{u, v\} = \sum_{i,j} \Lambda^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j}$$

and a star product $*_M$ can be defined by

$$u *_M v := uv + \sum_{r=1}^{\infty} \frac{1}{r!} \left(\frac{\lambda}{2}\right)^r \Lambda^{i_1 j_1} \cdots \Lambda^{i_r j_r} \frac{\partial^r u}{\partial x^{i_1} \cdots \partial x^{i_r}} \frac{\partial^r u}{\partial x^{j_1} \cdots \partial x^{j_r}}.$$

This is the *Moyal star product*.

2 Symmetry

Symmetries should be defined as an automorphisms of $(C^\infty(M)[\![\lambda]\!], *)$. We call $\sigma : C^\infty(M)[\![\lambda]\!] \to C^\infty(M)[\![\lambda]\!]$ a symmetry if

1. σ is $\mathbb{R}[\![\lambda]\!]$-linear;

2. $\sigma(1) = 1$ and

$$\sigma(u * v) = \sigma(u) * \sigma(v) \quad \forall u, v \in C^\infty(M)[\![\lambda]\!]. \tag{1}$$

We expand

$$\sigma(u) = \sigma_0 + \lambda\sigma_1(u) + \cdots$$

for $u \in C^\infty(M)$ ($\sigma_i : C^\infty(M) \to C^\infty(M)$) and compare the terms of order 0 in λ in Condition 2. σ_0 must be an automorphism of $C^\infty(M)$ and therefore there exists a diffeomorphism $\tau : M \to M$ with $\sigma_0(u) = u \circ \tau^{-1}$. Looking at terms of order 1 in λ, τ preserves the Poisson bracket, and hence τ is a symplectomorphism (but this τ does not preserve C_r for $r \geq 2$ in general).

We can compare the star product $*$ and its pull-back by τ. In preparation we make the following definition:

Definition 2. Let $*_1$ and $*_2$ be two star products on M with the same Poisson bracket on $C^\infty(M)$. We say $*_1$ and $*_2$ are *equivalent* if there exists $T : C^\infty(M)[\![\lambda]\!] \to C^\infty(M)[\![\lambda]\!]$ satisfying the following condition:

1. $T(1) = 1$.

2. T can be written in the form

$$T = I + \lambda_1 T_1 + \lambda_1^2 T_2 + \cdots$$

with linear maps $T_r : C^\infty(M) \to C^\infty(M)$.

3. T satisfies

$$T(u *_1 v) = T(u) *_2 T(v), \qquad \forall u, v \in C^\infty(M)[\![\lambda]\!].$$

Condition 1 is equivalent to the condition that $T_r(1) = 0$ for all $r \geq 1$. T is called an *equivalence* between $*_1$ and $*_2$.

Remark 1. If $*_1$, $*_2$ are differential and T is an equivalence between $*_1$ and $*_2$, then each T_r is necessarily a differential operator.

If the automorphism $\sigma : C^\infty(M)[\![\lambda]\!] \to C^\infty(M)[\![\lambda]\!]$ has $\sigma_0(u) = u \circ \tau^{-1}$ then it is easy to see that $T(u) = \sigma(u \circ \tau)$ defines an equivalence between $*$ and the star product $\tau{\cdot}*$ given by $u\,(\tau{\cdot}*)\,v = ((u \circ \tau) * (v \circ \tau)) \circ \tau^{-1}$. Thus

$$\sigma(u) = T(u \circ \tau^{-1})$$

and so symmetries are deformations of symplectomorphisms.

If we start with a given symplectomorphism τ and a deformation quantisation $*$ of the Poisson bracket, we can ask if $\sigma_0(u) = u \circ \tau^{-1}$ can be deformed into a symmetry. This is clearly possible if and only if $*$ and $\tau{\cdot}*$ are equivalent.

Equivalence of star products has been studied by many people [5, 6, 37, 45, 58], and we have a definitive result given by

Theorem 2 (Deligne [15]). *If (M, ω) is a symplectic manifold there is a bijection*

$$\{\textit{star products } * \textit{ on } M \textit{ with } \{\,,\,\}_* = \{\,,\,\}\}/\textit{equivalence} \longrightarrow \frac{[\omega]}{\lambda} + \mathrm{H}^2(M)[\![\lambda]\!].$$

The series of cohomology classes $c(*)$ corresponding to the equivalence class of a given star product $*$ is called the *Deligne characteristic class* of $*$ and is represented by Fedosov's central curvature for Fedosov star products [21], see Section 5 below. See [30] for a Čech theory definition of the class. c is functorial for symplectomorphisms, so σ a symmetry of $*$ means that $c(*)$ is fixed by τ. This thus becomes a necessary condition on a symplectomorphism for it to be deformable to a symmetry.

3 Infinitesimal symmetries

Let $\mathrm{Der}(M, *)$ denote the space of all $\mathbb{R}[\![\lambda]\!]$-linear *derivations* of $(C^\infty(M)[\![\lambda]\!], *)$, i.e., $\mathbb{R}[\![\lambda]\!]$-linear maps $D : C^\infty(M)[\![\lambda]\!] \to C^\infty(M)[\![\lambda]\!]$ satisfying

$$D(u * v) = D(u) * v + u * D(v).$$

Example 2.

$$\mathrm{ad}_* u\,(v) := u * v - v * u$$

is a derivation of $(C^\infty(M)[\![\lambda]\!], *)$ which begins with λX_u. We set $a(u) := \frac{1}{\lambda}\mathrm{ad}_* u$ and call it an *almost inner derivation.* It is a deformation of the adjoint representation of the Poisson bracket.

We are assuming M is a symplectic manifold and $*$ is a differential star product. Then the map $u \mapsto a(u)$ has kernel the centre of $(C^\infty(M)[\![\lambda]\!], *)$ and so we get an exact sequence

$$0 \longrightarrow \mathrm{H}^0(M)[\![\lambda]\!] \longrightarrow C^\infty(M)[\![\lambda]\!] \xrightarrow{a} \mathrm{Der}(M, *) \longrightarrow \;\; ?$$

We denote the image of a by $\mathrm{Inn}(M, *)$, the space of almost inner derivations. Then $\mathrm{Inn}(M, *) \subset \mathrm{Der}(M, *)$ and the sequence

$$0 \longrightarrow \mathrm{H}^0(M)[\![\lambda]\!] \longrightarrow C^\infty(M)[\![\lambda]\!] \xrightarrow{a} \mathrm{Inn}(M, *) \longrightarrow 0$$

is exact.

Lemma 1. *If* $\mathrm{H}^1(M) = 0$*, then*

$$\mathrm{Inn}(M, *) = \mathrm{Der}\,(M, *).$$

Proof Let $D \in \mathrm{Der}\,(M, *)$. D can be written in the form

$$D(u) = D_0(u) + \lambda D_1(u) + \cdots .$$

Since D_0 is a derivation of $C^\infty(M)$ and of $\{\,,\,\}$,

$$\mathcal{L}_{D_0}\omega = \mathrm{d}(\mathrm{i}(D_0)\omega) = 0.$$

Thus $\mathrm{H}^1(M) = 0$ implies $\mathrm{i}(D_0)\omega = \mathrm{d}u_0$.

Consider $D - a(u_0)$, which is a derivation of order λ.

$$D - a(u_0) = \lambda a(u_1) + \mathrm{O}(\lambda^2).$$

By iterating, we get

$$D = a(u_0) + \lambda a(u_1) + \cdots = a(u_0 + \lambda u_1 + \cdots).$$

For general M, take an open set U such that $\mathrm{H}^1(U) = 0$. Then $D|_U$ is a derivation of $C^\infty(U)[\![\lambda]\!]$ and so by the Lemma there is $u \in C^\infty(U)[\![\lambda]\!]$ with $D|_U = a(u)$. u is determined up to elements of $\mathrm{H}^0(M)[\![\lambda]\!]$ and so $\mathrm{d}u$ is determined globally.

Different choices of open sets give 1-forms $\mathrm{d}u$ which agree on overlaps, so the 1-forms piece together to give a global 1-form α_D which is closed. $\mathrm{Der}\,(M,*)$ is thus linearly bijective with $Z^1(M)[\![\lambda]\!]$.

Let $c(D) := [\alpha_D]$, then we have a sequence of maps

$$0 \to \mathrm{H}^0(M)[\![\lambda]\!] \to C^\infty(M)[\![\lambda]\!] \xrightarrow{a} \mathrm{Der}\,(M,*) \xrightarrow{c} \mathrm{H}^1(M)[\![\lambda]\!] \to 0 \tag{2}$$

Claim: (2) is an exact sequence of vector spaces.

To see this, take $[\alpha] \in \mathrm{H}^1(M)[\![\lambda]\!]$, then α is locally exact, $\alpha|_U = \mathrm{d}u$ with u determined up to constants, and so $a(u)$ is independent of the choice of u, so there is a globally defined derivation D with $D|_U = a(u)$. It follows that $c(D) = [\alpha]$ showing surjectivity of c. That $\mathrm{Ker} c = \mathrm{Im} a$ follows by similar arguments.

In fact all the terms in this sequence have Lie algebra structures. On $\mathrm{Der}(M,*)$ we take the commutator

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1.$$

Then

$$\left[\frac{1}{\lambda}\mathrm{ad}_* u,\, \frac{1}{\lambda}\mathrm{ad}_* v\right] = \frac{1}{\lambda^2}\mathrm{ad}_*(u*v - v*u) = \frac{1}{\lambda}\mathrm{ad}_*\left(\frac{1}{\lambda}(u*v - v*u)\right).$$

Taking the bracket $[u,v]_* = \frac{1}{\lambda}(u*v - v*u)$ on $C^\infty(M)[\![\lambda]\!]$ makes $a : C^\infty(M)[\![\lambda]\!] \to C^\infty(M)[\![\lambda]\!]$ into a homomorphism of Lie algebras. If U is a coordinate neighbourhood with $u_i \in C^\infty(U)[\![\lambda]\!]$ such that $D_i|_U = a(u_i)$, we have

$$[D_1,\, D_2]|_U = a([u_1,\, u_2]).$$

u_i is determined modulo central elements so $[u_1, u_2]_*$ is determined globally, and hence $[u_1,\, u_2]_* = b(D_1,\, D_2)|_U$ for some $b(D_1,\, D_2) \in C^\infty(M)[\![\lambda]\!]$. Thus we get, as in [7],

$$[D_1,\, D_2] = a(b(D_1,\, D_2))$$

and therefore we can the take zero bracket on $\mathrm{H}^1(M)[\![\lambda]\!]$. This shows

Theorem 3. *The exact sequence of vector spaces (2) is an exact sequence of Lie algebras with the brackets defined above.*

If G is a Lie group of quantum symmetries,

$$\sigma : G \longrightarrow \mathrm{Aut}(C^\infty(M)[\![\lambda]\!], *)$$

then the leading term of σ gives a classical action τ of G on M and differentiation gives a homomorphism

$$\mathrm{d}\sigma : \mathfrak{g} \longrightarrow \mathrm{Aut}(\mathrm{Der}\,(M, *))$$

with $d\sigma_0(\xi) = \widetilde{\xi}$, the fundamental vector fields of the classical action on M, so $d\sigma(\xi) = \widetilde{\xi} + \mathcal{O}(\lambda)$.

4 $*$-Hamiltonian actions

Consider the diagram

$$\begin{array}{ccccccccc} 0 \to \mathrm{H}^0(M)[\![\lambda]\!] \to C^\infty(M)[\![\lambda]\!] & \stackrel{a}{\to} & \mathrm{Der}\,(M,*) & \stackrel{c}{\to} \mathrm{H}^1(M)[\![\lambda]\!] \to 0. \\ & & \uparrow d\sigma & \\ & & \mathfrak{g} & \end{array} \qquad (3)$$

Question 2. Does there exist a homomorphism μ of Lie algebras $\mu \colon \mathfrak{g} \to C^\infty(M)[\![\lambda]\!]$ such that the diagram (3) commutes? That is, $d\sigma = a \circ \mu$. If there is we call it a *quantum moment map* and say the action is *$*$-Hamiltonian*.

When $\mathrm{d}\sigma(\mathfrak{g}) \subset \mathrm{Inn}(M, *)$ then a linear map μ exists satisfying this condition and we call the action *almost $*$-Hamiltonian.* Such a linear map μ will be guaranteed to exist if $\mathrm{H}^1(M)[\![\lambda]\!] = 0$ or $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and the latter condition is equivalent to $\mathrm{H}^1(\mathfrak{g}) = 0$.

If $\mathrm{H}^2(\mathfrak{g}) = 0$ and the action is almost $*$-Hamiltonian, then we can modify μ to make it a homomorphism. This gives the Theorem

Theorem 4 (Ping Xu [59]). *If* $\mathrm{H}^1(M) = 0$ *or* $\mathrm{H}^1(\mathfrak{g}) = 0$*, then any group G of symmetries is almost $*$-Hamiltonian.*

If $\mathrm{H}^2(\mathfrak{g}) = 0$*, the ast-Hamiltonian can be chosen to be a homomorphism.*

If $\mathrm{H}^1(\mathfrak{g}) = 0$*, then the homomorphism is unique.*

When μ is a quantum moment map

$$\mu = \mu_0 + \lambda\mu_1 + \cdots$$

then μ_0 is a classical Hamiltonian for τ.

5 Connections and Fedosov star products

Let (M, ω) be a symplectic manifold. A connection ∇ on M is called *symplectic* if

$$\nabla\omega = 0, \quad T^{\nabla} = 0.$$

The space of connections is an affine space whose underlying vector space is isomorphic to $\Gamma(S^3T^*M)$.

Lichnerowicz [41] used connections to try to transfer the Moyal star product to general manifolds, only this is spoilt by curvature, so this method is effective only in the flat case. Fedosov showed how to find higher order terms to fix the associativity in the non-flat case. This was published first in Russian and then much later in [20]. The idea is to embed the tangent bundle in a larger bundle and modify the symplectic connection to be flat in this larger bundle.

$\mathcal{W}$: The formal Weyl bundle, a completion of $S^{\bullet}T^*M[\![\lambda]\!]$.

$\mathcal{W}_x$ consists of formal functions on T_xM, and the latter has a constant symplectic form ω_x, so we can put Moyal product on $\mathcal{W}$ fibrewise (which is denoted $a \circ b$, $a, b \in \Gamma\mathcal{W}$).

Denote the connection in $\mathcal{W}$ induced by ∇ by ∂.

$$\partial : \Gamma\mathcal{W} \longrightarrow \Gamma(\mathcal{W} \otimes \Lambda^1).$$

So we can extend this ∂ to $\mathcal{W} \otimes \Lambda^*$ so as to satisfy

$$\partial \circ \partial = [\bar{R}, a]_0.$$

Here, using the summation convention,

$$\bar{R} = \frac{1}{4}\omega_{rl}R^l_{ijk}y^r y^k \mathrm{d}x^i \mathrm{d}x^j$$

where $y^j \frac{\partial}{\partial x^j}$ is a point in T_xM.

Look for D with $D = \partial +$ higher order terms such that D is still a derivation of $\mathcal{W}$ and $D \circ D = 0$ holds.

Then D can be taken in the form

$$D(a) := \partial a - \delta a - [r, a]_0 \qquad \left(\delta a := \mathrm{d}x^k \wedge \frac{\partial a}{\partial y^k}\right)$$

with r a $\mathcal{W}$-valued 1-form. Then $D \circ D(a) = [\bar{R} + \delta r - \partial r + \frac{1}{2}[r, r]_0, a]_0$ so $\bar{R} + \delta r - \partial r + \frac{1}{2}[r, r]_0$ should be in the centre to give $D \circ D = 0$. This means it must be constant in y, so is a series of 2-forms on M. Therefore, taking a series of closed 2-forms $\Omega = \Omega_0 + \lambda\Omega_1 + \cdots$ we try to solve

$$\bar{R} + \delta r - \partial r + \frac{1}{2}[r, r]_0 = \Omega$$

for r.

We can define $\bar{\delta}$ so

$$(\delta\bar{\delta} + \bar{\delta}\delta)a = a - a_{00} \quad (a_{pq} \in S^p \otimes \wedge^q).$$

Then applying $\bar{\delta}$ we try to solve $r = \bar{\delta}(\Omega - \bar{R} + \partial r - \frac{1}{2}\{r, r])$ recursively. This can be done in a unique way with the above choices.

D is a derivation of $\Gamma\mathcal{W}$ so

$$\Gamma_D\mathcal{W} = \{a \in \Gamma\mathcal{W};\ Da = 0\}$$

is a subalgebra of $\Gamma\mathcal{W}$.

Theorem 5 (Fedosov [21]). *Let (M, ω) be a symplectic manifold. $a \mapsto a_0$ is a linear isomorphism $\Gamma_D\mathcal{W} \to C^\infty(M)[\![\lambda]\!]$. Let Q be the reverse mapping of this isomorphism. Then $u * v = Q^{-1}(Q(u) \circ Q(v))$ defines a star product $* = *_{\nabla,\Omega}$ on M with Deligne class $[\frac{\omega}{\lambda}] + [\Omega]$.*

This Theorem does not give all star products, but some special representatives $*_{\nabla,\Omega}$ of each equivalence class, along with a specific representative of its Deligne class. Note also that the construction is covariant for symplectomorphisms

$$\tau \cdot *_{\nabla,\Omega} = *_{\tau\cdot\nabla,\tau\cdot\Omega}.$$

It follows by differentiating this formula that a vector field X on M will be a derivation of $*_{\nabla,\Omega}$ if $\mathcal{L}_X\nabla = 0$ (so X is an infinitesimal affine transformation), $\mathcal{L}_X\omega = 0$ and $\mathcal{L}_X\Omega = 0$. Since $\omega + \lambda\Omega$ is closed then $i(X)(\omega + \lambda\Omega)$ is closed. Is it exact?

Theorem 6 (Kravchenko [40], Gutt–Rawnsley [32]). *If (M, ω) is a symplectic manifold then X is an almost inner derivation of $*_{\nabla,\Omega}$ if and only if $\mathcal{L}_X\nabla = 0$ and there exists $\mu \in C^\infty(M)[\![\lambda]\!]$ with*

$$i(X)(\omega + \lambda\Omega) = d\mu.$$

In this case $X(u) = a(u) = \frac{1}{\lambda}(\mu * u - u * \mu)$.

One may ask if there is an analogue of Theorem 6 for more general star products. For completely general star products, results are not known. There is however a class of star products which contains all the know explicit constructions and for which we have a result.

Definition 3. A star product $*$ is said to be *natural* if the coefficients $C_r(u, v)$ are differential operators of order at most r in each variable.

Remark 2. We could define a notion of *natural to order k* if the condition above holds for C_r for all $r \leq k$. Many of our results only require natural to order 2.

Theorem 7 (Gutt-Rawnsley [32]). *If $*$ is natural (to order 2) there is a unique symplectic connection ∇ such that*

$$C_1 = \frac{1}{2}\{\,,\,\} - \mathfrak{d}E,$$

$$C_2 = -\frac{1}{2}[E, \mathfrak{d}E] + \frac{1}{2}[E, \{\,,\,\}] + \frac{1}{8}\Lambda^2(\nabla^2, \nabla^2) + A_2$$

where $\mathfrak{d}$ denotes the Hochschild coboundary, $[\,,\,]$ the Gerstenhaber bracket, E is a differential operator of order at most 2, A_2 is skewsymmetric and

$$\Lambda^2(\nabla^2 u, \nabla^2 v) = \Lambda^{ij}\Lambda^{i'j'}\nabla_i\nabla_{i'}u\nabla_j\nabla_{j'}v.$$

with Λ^{ij} the Poisson tensor determined by the symplectic form ω.

This result is due to Lichnerowicz in the case where C_1 is skewsymmetric so $E = 0$.

If we have a representative Ω of the class of the natural star product $*$ then $*$ is equivalent to $*_{\nabla,\Omega}$ where ∇ is given by the Theorem. In fact one can see that the equivalence can be determined recursively in the form $\exp E$ with $E = \sum_{r\geq 1}\lambda^r E_r$ and E_r of order $\leq r+1$. This argument can be improved using a method of Bertelson–Cahen–Gutt to determine the 2-form Ω recursively at the same time and leads to a complete parametrisation of natural star products.

Theorem 8 (Gutt-Rawnsley [32]). *Given a natural star product $*$ there exist uniquely*

- *a symplectic connection ∇;*
- *a formal series of closed 2-forms $\Omega = \Omega_0 + \lambda\Omega_1 + \cdots$;*
- *a formal series of differential operators $E = \sum_{r\geq 1}\lambda^r E_r$ of the form*

$$E_r u = \sum_{k=2}^{r+1} E_r^{(k)i_1\ldots i_k}\nabla_{i_1}\cdots\nabla_{i_k}u$$

such that $u * v = \exp -E\left((\exp Eu) *_{\nabla,\Omega} (\exp Ev)\right)$.

Denote this star product by $* = *_{\nabla,\Omega,E}$ then for any symplectomorphism τ

$$\tau\cdot *_{\nabla,\Omega,E} = *_{\tau\cdot\nabla,\tau\cdot\Omega,\tau\cdot E}.$$

We then generalise Theorem 6 from Fedosov star products to natural star products:

Theorem 9 (Gutt–Rawnsley [32]). *If (M,ω) is a symplectic manifold then X is a derivation of a natural star product $* = *_{\nabla,\Omega,E}$ if and only if $\mathcal{L}_X\nabla = 0$, $\mathcal{L}_X\omega = 0$, $\mathcal{L}_X\Omega = 0$, and $\mathcal{L}_X E = 0$. X is almost inner if and only if there is a formal series of functions ρ with $i(X)(\omega + \lambda\Omega) = d\rho$ and then $X = a(\mu)$ with $\mu = \exp -E\,\rho$.*

Finally, the bibliography which follows contains a good overview of the theory surrounding my lectures. Not all entries were referred to in these notes.

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