THE CURVE COMPLEX AND COVERS VIA HYPERBOLIC 3-MANIFOLDS

ROBERT TANG

ABSTRACT. Rafi and Schleimer recently proved that the natural relation between curve complexes induced by a covering map between two surfaces is a quasi-isometric embedding. We offer another proof of this result using a distance estimate via hyperbolic 3-manifolds.

1. INTRODUCTION

Let S be a compact orientable surface of genus g with n boundary components. Define a simplicial complex $\mathcal{C}(S)$, called the *curve complex*, as follows. Let the vertex set be the free homotopy classes of essential non-peripheral simple closed curves. We have a simplex for every set of homotopy classes which can be realised simultaneously disjointly. In particular, two classes are adjacent in the 1-skeleton of $\mathcal{C}(S)$ if and only if they have disjoint representatives on S.

We will assume that the *complexity* $\xi(S) = 3g + n - 3$ is at least 2 in this paper. This will guarantee that $\mathcal{C}(S)$ is non-empty, connected and that all surfaces under consideration are of hyperbolic type. (There are modified definitions for the curve complex in the low-complexity cases, however, we shall not deal with them here.)

The distance $d_S(a, b)$ between vertices a and b in $\mathcal{C}(S)$ is defined to be the length of the shortest edge-path connecting them. This can be thought of as measuring how "complicated" their intersection pattern is. Endowed with this distance, the curve complex has infinite diameter and is also Gromov hyperbolic [6, 2].

The curve complex has had many applications to both the study of mapping class groups and Teichmüller theory. More recently, it was a key ingredient of the proof of the Ending Lamination Theorem [5, 1].

We will be focussed on a map between curve complexes induced by a covering map.

Suppose $P : \Sigma \to S$ is finite-degree covering map between two surfaces. The preimage $P^{-1}(a)$ of a curve $a \in \mathcal{C}(S)$ is a disjoint union of simple closed curves on

Σ. This induces a one-to-many map $\Pi : \mathcal{C}(S) \to \mathcal{C}(\Sigma)$ where $\Pi(a)$ is defined to be the set of homotopy classes of the curves in $P^{-1}(a)$.

To establish some notation, given non-negative real numbers A, B and K, write $A \prec_K B$ to mean $A \le KB + K$. We shall write $A \asymp_K B$ if both $A \prec_K B$ and $B \prec_K A$ hold.

Let (X, d) and (X', d') be pseudo-metric spaces and $f : X \to X'$ a (possibly one-to-many) function. We call f a K-*quasi-isometric embedding* if there exists a positive constant K such that for all $x, y \in X$ we have $d(x, y) \asymp_K d'(x', y')$ whenever $x' \in f(x)$ and $y' \in f(y)$. Furthermore, if the K-neighbourhood of $f(X)$ equals X' then we call f a K-*quasi-isometry* and say X and X' are K-*quasi-isometric*. Furthermore, we call K the *quasi-isometry constant*.

In this paper we give a new proof of the following theorem, originally due to Rafi and Schleimer [7].

Theorem 1.1. *Let $P : \Sigma \to S$ be a covering map of degree $\deg P < \infty$. Then the map $\Pi : \mathcal{C}(S) \to \mathcal{C}(\Sigma)$ defined above is a K-quasi-isometric embedding, where K depends only on $\xi(\Sigma)$ and $\deg P$.*

This result can be interpreted as saying that one cannot tangle or untangle two curves on S by too much when passing to finite index covers.

Theorem 1.1 was first proved in [7] using Teichmüller theory and subsurface projections. Our approach uses an estimate for distance in the curve complex via a suitable hyperbolic 3-manifold homeomorphic to $S \times \mathbb{R}$ with a modified metric. This allows us to naturally compare distances by taking a covering map between the respective 3-manifolds. The estimate (Theorem 3.1) arises from work towards the Ending Lamination Theorem [5] and is made explicit in [1]. More details will be given in Section 3.

Remark 1.2. A consequence of Theorem 1.1 and Gromov hyperbolicity of $\mathcal{C}(\Sigma)$ is that the image $\Pi(\mathcal{C}(S))$ is quasi-convex.

2. The induced map between curve complexes

Let $P : \Sigma \to S$ be a finite index covering of surfaces and assume $\xi(S) \ge 2$. The cover P induces a one-to-many map $\Pi : \mathcal{C}(S) \to \mathcal{C}(\Sigma)$ defined by declaring $\Pi(a)$ to be the set of homotopy classes in $P^{-1}(a)$. Observe that $P^{-1}(a)$ is a nonempty disjoint union of circles and that any component of $P^{-1}(a)$ must be essential and non-peripheral. Thus Π sends vertices of $\mathcal{C}(S)$ to (non-empty) simplices of $\mathcal{C}(\Sigma)$.

The preimages of disjoint curves in S must themselves be disjoint. Therefore, if we choose a preferred vertex in each $\Pi(a)$ then we can send any edge-path in $\mathcal{C}(S)$ to an edge-path in $\mathcal{C}(\Sigma)$. (We can extend this to higher dimensional simplices if so desired.) This gives us the following distance bound.

Lemma 2.1. *Let a and b be curves in $\mathcal{C}(S)$ and suppose $\alpha \in \Pi(a)$ and $\beta \in \Pi(b)$. Then*

$$d_\Sigma(\alpha, \beta) \leq d_S(a, b).$$

3. ESTIMATES FROM 3-MANIFOLDS

We now introduce the background required to state an estimate for distances in the curve complex using a suitable hyperbolic 3-manifold.

Let S be a surface with $\xi(S) \geq 2$ and fix distinct curves a and b in $\mathcal{C}(S)$. By simultaneous uniformisation, there exists a hyperbolic 3-manifold $(X, d) \cong \operatorname{int}(S) \times \mathbb{R}$ with a preferred homotopy equivalence f to S such that the unique geodesic representatives of the two curves, denoted a^* and b^*, are arbitrarily short. In fact, such a 3-manifold can be chosen so that there are no accidental cusps. For a reference, see [4].

Recall that the *injectivity radius* at a point $x \in X$ is equal to half the infimum of the lengths of all nontrivial loops in X passing through x. The ϵ-*thin part* (or just thin part) of X is the set of all points whose injectivity radius is less than ϵ. By the well known Margulis lemma, the thin part comprises of Margulis tubes and cusps whenever ϵ is less than the Margulis constant. The *thick part* of X is the complement of the thin part. We shall fix such an ϵ for the rest of this paper.

Let $\Psi_\epsilon(X)$ denote the *non-cuspidal part* of X, that is, X with its ϵ-cusps removed. We will write $\Psi(X)$ for brevity. Since X has no accidental cusps we see that if S is closed then $\Psi(X) = X$.

Define the *electrified length* of a path in $\Psi(X)$ to be its total length occurring outside the Margulis tubes of X. More formally, we take the one-dimensional Hausdorff measure of its intersection with the thick part of X. This induces a reduced pseudometric ρ_X on $\Psi(X)$ obtained by taking the infimum of the electrified lengths of all paths connecting two given points. One can show that the infimum is attained, for example, by taking a path which connects Margulis tubes by shortest geodesic segments.

The distance ρ_X shall be referred to as the *electrified distance* on $\Psi(X)$ with respect to its Margulis tubes or the reduced pseudometric obtained by electrifying the Margulis tubes.

Theorem 3.1. [1] *Let a and b be curves in $\mathcal{C}(S)$ whose geodesic representatives a^* and b^* in X have d-length at most $L \geq 0$. Then*

$$d_S(a, b) \asymp_K \rho_X(a^*, b^*)$$

where the constant K depends only on $\xi(S)$, L and ϵ.

This estimate follows from the construction of geometric models for hyperbolic 3-manifolds used in the proof of the Ending Lamination Theorem. We refer the reader to [1] and [5] for an in-depth discussion.

Remark 3.2. For Theorem 3.1 to make sense we need to ensure that a^* and b^* are indeed contained in $\Psi(X)$. This can be done using a pleated surfaces argument, such as in [3], to show that closed geodesics of bounded length in X avoid cusps provided ϵ is sufficiently small.

4. PROOF OF THEOREM 1.1

4.1. **Closed surfaces.** We first prove the main theorem for closed surfaces. The required modifications for the general case shall be dealt with in Section 4.2.

Fix a length bound L and a sufficiently small value of ϵ. Let $P : \Sigma \to S$ be a covering map. Fix curves a, b in $\mathcal{C}(S)$ and choose $\alpha \in \Pi(a)$ and $\beta \in \Pi(b)$. From Lemma 2.1, we have $d_\Sigma(\alpha, \beta) \le d_S(a, b)$ so it remains to prove the reverse inequality.

Let $(X, d) \cong \operatorname{int}(S) \times \mathbb{R}$ be a hyperbolic 3-manifold with a homotopy equivalence f to S as described in Section 3. We also assume that a^* and b^* have length at most $\frac{L}{\deg P}$ in X. There exists a covering map $Q : \Xi \to X$, where $\Xi \cong \operatorname{int}(\Sigma) \times \mathbb{R}$, and a homotopy equivalence $\tilde{f} : \Xi \to \Sigma$ such that $P \circ \tilde{f} = f \circ Q$. Note that $Q(\alpha^*) = a^*$ and $Q(\beta^*) = b^*$, hence α^* and β^* have length bounded above by L.

Let ρ_X and ρ_Ξ be the respective pseudometrics on X and Ξ obtained by electrifying their ϵ-tubes.

Lemma 4.1. *The map Q is 1-Lipschitz with respect to ρ_Ξ and ρ_X.*

Proof. Using the definition of injectivity radius and the π_1-injectivity of covering maps, we see that the ϵ-thin part of Ξ is sent into that of X. It follows that the electrified lengths of paths cannot increase under Q. $\square$

This result, together with Theorem 3.1, proves Theorem 1.1 for closed surfaces.

4.2. **Surfaces with boundary.** We now assume S has non-empty boundary. Recall that $\Psi_\epsilon(X)$ denotes X with its ϵ-cusps removed.

Lemma 4.2. *Let X be a hyperbolic 3-manifold. Choose small constants $\delta > \delta'$ and let ρ and ρ' be the pseudometrics on $\Psi_\delta(X)$ and $\Psi_{\delta'}(X)$ obtained by electrifying along their respective ϵ-tubes. Then the natural retraction*

$$r : (\Psi_{\delta'}(X), \rho') \to (\Psi_\delta(X), \rho)$$

is R-Lipschitz, where $R = \frac{\sinh \delta}{\sinh \delta'}$.

Proof. Begin with a geodesic arc γ in $\Psi_{\delta'}(X)$. We will show that the length of γ can only increase by a bounded multiplicative factor under the retraction r.

If γ is contained in $\Psi_\delta(X)$ then we are done. So suppose γ meets some δ-cusp C of X. This cusp contains a δ'-cusp C' which does not meet γ. The cusps C and C' lift to nested horoballs H and H' in the universal cover $\mathbb{H}^3$. We can arrange for the horospheres ∂H and $\partial H'$ to be horizontal planes at heights 1 and $R > 1$

respectively in the upper half-space model. Using basic hyperbolic geometry and the definition of injectivity radius, we can show that $R = \frac{\sinh \delta}{\sinh \delta'}$.

Now define π and π' to be the nearest point projections from $\mathbb{H}^3$ to H and H' respectively. Taking a lift $\tilde{\gamma}$ of γ, we see that

$$\text{length}(\tilde{\gamma} \cap H) \geq \text{length}(\pi'(\tilde{\gamma} \cap H)) = \frac{1}{R}\text{length}(\pi(\tilde{\gamma} \cap H)).$$

Thus, by projecting each arc of γ inside a δ-cusp to the boundary of that cusp, we create a new path whose length is at most $R \times \text{length}(\gamma)$. $\qquad \square$

Let $Q : \Xi \to X$ be the covering map as described in Section 4.1. Observe that

$$\Psi_\epsilon(X) \subseteq Q(\Psi_\epsilon(\Xi)) \subseteq \Psi_{\epsilon'}(X)$$

where $\epsilon' = \frac{\epsilon}{\deg P}$. As before, let ρ_Ξ and ρ_X be the pseudometrics on $\Psi_\epsilon(\Xi)$ and $\Psi_\epsilon(X)$ obtained by electrifying along their respective ϵ-tubes. Combining Lemma 4.2 with the proof of Lemma 4.1 gives us the following.

Lemma 4.3. *Let $r : \Psi_{\epsilon'}(X) \to \Psi_\epsilon(X)$ be the natural retraction. Then the composition $r \circ Q : \Psi_\epsilon(\Xi) \to \Psi_\epsilon(X)$ is R-Lipschitz with respect to ρ_Ξ and ρ_X. Moreover, the constant R depends only on $\deg P$.* $\qquad \square$

Finally, $r \circ Q(\alpha^*) = a^*$ and $r \circ Q(\beta^*) = b^*$ and so the rest of the argument follows as in Section 4.1.

Acknowledgements

References

[1] Brian H. Bowditch, *The ending lamination theorem*, in preparation, Warwick (2011).

[2] _____, *Intersection numbers and the hyperbolicity of the curve complex*, J. Reine Angew. Math. **598** (2006), 105–129. MR 2270568 (2009b:57034)

[3] _____, *Length bounds on curves arising from tight geodesics*, Geom. Funct. Anal. **17** (2007), no. 4, 1001–1042. MR 2373010 (2009h:57027)

[4] _____, *Tight geodesics in the curve complex*, Invent. Math. **171** (2008), no. 2, 281–300. MR 2367021 (2008m:57040)

[5] Jeffrey F. Brock, Richard D. Canary, and Yair N. Minsky, *The classification of kleinian surface groups, II: The ending lamination conjecture*, preprint, (2004).

[6] Howard A. Masur and Yair N. Minsky, *Geometry of the complex of curves. I. Hyperbolicity*, Invent. Math. **138** (1999), no. 1, 103–149. MR 1714338 (2000i:57027)

[7] Kasra Rafi and Saul Schleimer, *Covers and the curve complex*, Geom. Topol. **13** (2009), no. 4, 2141–2162. MR 2507116

Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK
E-mail address: robert.tang@warwick.ac.uk
URL: http://www.warwick.ac.uk/~mariam/