Natural splittings of independence complexes

Michał Adamaszek

Warwick Mathematics Institute and DIMAP

WATACBA
November 2011

Introduction

Definition

- G — undirected, simple graph
- $\mathrm{Ind}(G)$ — *independence complex* of G
  - vertices: $V(G)$
  - faces $\equiv$ independent (stable) sets in G

Introduction

Definition

- G — undirected, simple graph
- $\mathrm{Ind}(G)$ — *independence complex* of G
  - vertices: $V(G)$
  - faces $\equiv$ independent (stable) sets in G

$$\mathrm{Ind}(\bullet) = \bullet$$

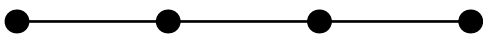
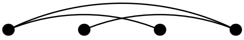
Definition

- G — undirected, simple graph
- $\mathrm{Ind}(G)$ — *independence complex* of G
    - vertices: $V(G)$
    - faces $\equiv$ independent (stable) sets in G

$$\mathrm{Ind}(\bullet\!\!\longrightarrow\!\!\!\bullet) \quad = \quad \bullet \qquad \bullet \quad = \quad S^0$$

Definition

- G — undirected, simple graph
- $\mathrm{Ind}(G)$ — *independence complex* of G
    - vertices: $V(G)$
    - faces $\equiv$ independent (stable) sets in G

$\mathrm{Ind}(P_4) = \mathrm{Ind}($  $) =$  $\simeq *$

Introduction

Definition

- G — undirected, simple graph
- $\mathrm{Ind}(G)$ — *independence complex* of G
  - vertices: $V(G)$
  - faces $\equiv$ independent (stable) sets in G

independence complexes

$\equiv$

flag complexes

$\equiv$

clique complexes

$$\mathrm{Cl}(G) = \mathrm{Ind}(\overline{G})$$

Introduction

Definition

- G — undirected, simple graph
- $\mathrm{Ind}(G)$ — *independence complex* of G
  - vertices: $V(G)$
  - faces $\equiv$ independent (stable) sets in G

$$\mathrm{Ind}(H_1 \sqcup H_2) = \mathrm{Ind}(H_1) * \mathrm{Ind}(H_2)$$

In particular, if G has an isolated vertex then

$$\mathrm{Ind}(G) \simeq *$$

Also:

$$\mathrm{Ind}(\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet \ \sqcup \ G) = S^0 * \mathrm{Ind}(G) = \Sigma\,\mathrm{Ind}(G).$$

Introduction

Definition

- G — undirected, simple graph
- $\mathrm{Ind}(G)$ — *independence complex* of G
    - vertices: $V(G)$
    - faces $\equiv$ independent (stable) sets in G

Functoriality:

- $G - e \hookrightarrow G$ induces an inclusion

$$\mathrm{Ind}(G) \hookrightarrow \mathrm{Ind}(G - e)$$

- $H \subset G$ — *induced* subgraph inclusion induces

$$\mathrm{Ind}(H) \hookrightarrow \mathrm{Ind}(G)$$

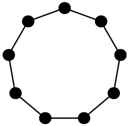
Motivating question

Motivating question

$\operatorname{Ind}(P_n) \simeq \Sigma \operatorname{Ind}(P_{n-3})$

Motivating question

$$\mathrm{Ind}(P_n) \simeq \Sigma\,\mathrm{Ind}(P_{n-3})$$

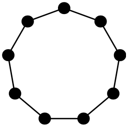


$$\mathrm{Ind}(C_n) \simeq \Sigma\,\mathrm{Ind}(C_{n-3})$$

[Kozlov'99]

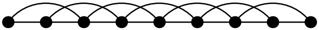
Motivating question

$$\mathrm{Ind}(P_n) \simeq \Sigma\,\mathrm{Ind}(P_{n-3})$$



$$\mathrm{Ind}(C_n) \simeq \Sigma\,\mathrm{Ind}(C_{n-3})$$

[Kozlov'99]

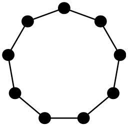


$$\mathrm{Ind}(P_n^r) \simeq \Sigma\,\mathrm{Ind}(P_{n-(r+2)}^r) \vee \dots \Sigma\,\mathrm{Ind}(P_{n-(2r+1)}^r)$$

[Engström'09]

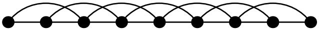
Motivating question

$$\mathrm{Ind}(P_n) \simeq \Sigma\,\mathrm{Ind}(P_{n-3})$$



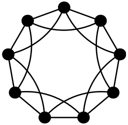
$$\mathrm{Ind}(C_n) \simeq \Sigma\,\mathrm{Ind}(C_{n-3})$$

[Kozlov'99]



$$\mathrm{Ind}(P_n^r) \simeq \Sigma\,\mathrm{Ind}(P^r_{n-(r+2)}) \vee \cdots \Sigma\,\mathrm{Ind}(P^r_{n-(2r+1)})$$

[Engström'09]



$$\mathrm{Ind}(C_n^r) \simeq ???$$

Two cofibre sequences

G — graph, v — vertex, e — edge

$$\mathrm{Ind}(G \setminus N[v]) \hookrightarrow \mathrm{Ind}(G \setminus v) \hookrightarrow \mathrm{Ind}(G) \to \Sigma\,\mathrm{Ind}(G \setminus N[v]) \to \cdots$$

$$\Sigma\,\mathrm{Ind}(G \setminus N[e]) \hookrightarrow \mathrm{Ind}(G) \hookrightarrow \mathrm{Ind}(G - e) \to \Sigma^2\,\mathrm{Ind}(G \setminus N[e]) \to \cdots$$

Two cofibre sequences

G — graph, v — vertex, e — edge

$$\mathrm{Ind}(G \setminus N[v]) \hookrightarrow \mathrm{Ind}(G \setminus v) \hookrightarrow \mathrm{Ind}(G) \to \Sigma\,\mathrm{Ind}(G \setminus N[v]) \to \cdots$$

$$\Sigma\,\mathrm{Ind}(G\setminus N[e]) \hookrightarrow \mathrm{Ind}(G) \hookrightarrow \mathrm{Ind}(G-e) \to \Sigma^2\,\mathrm{Ind}(G\setminus N[e]) \to \cdots$$

Proof of (2).

$$\mathrm{Ind}(G-e) = \mathrm{Ind}(G) \bigcup_{\mathrm{Ind}(e \sqcup (G\setminus N[e]))} e * \mathrm{Ind}(G \setminus N[e])$$

Consequences: $\mathrm{Ind}(G)$ vs. $\mathrm{Ind}(G \setminus v)$

$$\mathrm{Ind}(G \setminus N[v]) \hookrightarrow \mathrm{Ind}(G \setminus v) \hookrightarrow \mathrm{Ind}(G) \to \Sigma\,\mathrm{Ind}(G \setminus N[v]) \to \cdots$$

Consequences: $\mathrm{Ind}(G)$ vs. $\mathrm{Ind}(G \setminus v)$

$$\mathrm{Ind}(G \setminus N[v]) \hookrightarrow \mathrm{Ind}(G \setminus v) \hookrightarrow \mathrm{Ind}(G) \to \Sigma\,\mathrm{Ind}(G \setminus N[v]) \to \cdots$$

- If $\mathrm{Ind}(G \setminus N[v]) \simeq *$ then

$$\mathrm{Ind}(G) \simeq \mathrm{Ind}(G \setminus v)$$

Consequences: $\mathrm{Ind}(G)$ vs. $\mathrm{Ind}(G \setminus v)$

$$\mathrm{Ind}(G \setminus N[v]) \hookrightarrow \mathrm{Ind}(G \setminus v) \hookrightarrow \mathrm{Ind}(G) \to \Sigma\,\mathrm{Ind}(G \setminus N[v]) \to \cdots$$

- If $\mathrm{Ind}(G \setminus N[v]) \simeq *$ then

$$\mathrm{Ind}(G) \simeq \mathrm{Ind}(G \setminus v)$$

e.g. when $N(w) \subset N(v)$ *for some* w (*fold lemma*)

Consequences: $\mathrm{Ind}(G)$ vs. $\mathrm{Ind}(G \setminus v)$

$$\mathrm{Ind}(G \setminus N[v]) \hookrightarrow \mathrm{Ind}(G \setminus v) \hookrightarrow \mathrm{Ind}(G) \to \Sigma\,\mathrm{Ind}(G \setminus N[v]) \to \cdots$$

- If $\mathrm{Ind}(G \setminus N[v]) \simeq *$ then

$$\mathrm{Ind}(G) \simeq \mathrm{Ind}(G \setminus v)$$

  e.g. when $N(w) \subset N(v)$ for some w (fold lemma)

- If $\mathrm{Ind}(G \setminus N[v]) \hookrightarrow \mathrm{Ind}(G \setminus v)$ is null-homotopic then

$$\mathrm{Ind}(G) \simeq \mathrm{Ind}(G \setminus v) \vee \Sigma\,\mathrm{Ind}(G \setminus N[v])$$

Consequences: $\mathrm{Ind}(G)$ vs. $\mathrm{Ind}(G \setminus v)$

$$\mathrm{Ind}(G \setminus N[v]) \hookrightarrow \mathrm{Ind}(G \setminus v) \hookrightarrow \mathrm{Ind}(G) \to \Sigma\,\mathrm{Ind}(G \setminus N[v]) \to \cdots$$

- If $\mathrm{Ind}(G \setminus N[v]) \simeq *$ then

$$\mathrm{Ind}(G) \simeq \mathrm{Ind}(G \setminus v)$$

e.g. when $N(w) \subset N(v)$ for some w (fold lemma)

- If $\mathrm{Ind}(G \setminus N[v]) \hookrightarrow \mathrm{Ind}(G \setminus v)$ is null-homotopic then

$$\mathrm{Ind}(G) \simeq \mathrm{Ind}(G \setminus v) \vee \Sigma\,\mathrm{Ind}(G \setminus N[v])$$

e.g. when $N[w] \subset N[v]$ for some w

Even better consequences: $\mathrm{Ind}(G)$ vs. $\mathrm{Ind}(G-e)$

$$\Sigma\,\mathrm{Ind}(G\setminus N[e]) \hookrightarrow \mathrm{Ind}(G) \hookrightarrow \mathrm{Ind}(G-e) \to \Sigma^2\,\mathrm{Ind}(G\setminus N[e]) \to \cdots.$$

Even better consequences: $\mathrm{Ind}(G)$ vs. $\mathrm{Ind}(G-e)$

$$\Sigma\,\mathrm{Ind}(G\setminus N[e]) \hookrightarrow \mathrm{Ind}(G) \hookrightarrow \mathrm{Ind}(G-e) \to \Sigma^2\,\mathrm{Ind}(G\setminus N[e]) \to \cdots.$$

- If $\mathrm{Ind}(G \setminus N[e]) \simeq *$ then

$$\mathrm{Ind}(G) \simeq \mathrm{Ind}(G-e)$$

Even better consequences: $\mathrm{Ind}(G)$ vs. $\mathrm{Ind}(G-e)$

$$\Sigma\,\mathrm{Ind}(G\setminus N[e]) \hookrightarrow \mathrm{Ind}(G) \hookrightarrow \mathrm{Ind}(G-e) \to \Sigma^2\,\mathrm{Ind}(G\setminus N[e]) \to \cdots.$$

- If $\mathrm{Ind}(G\setminus N[e]) \simeq *$ then

$$\mathrm{Ind}(G) \simeq \mathrm{Ind}(G-e)$$

e.g. when $G\setminus N[e]$ has an isolated point

Even better consequences: $\mathrm{Ind}(G)$ vs. $\mathrm{Ind}(G-e)$

$$\Sigma\,\mathrm{Ind}(G\setminus N[e]) \hookrightarrow \mathrm{Ind}(G) \hookrightarrow \mathrm{Ind}(G-e) \to \Sigma^2\,\mathrm{Ind}(G\setminus N[e]) \to \cdots.$$

- If $\mathrm{Ind}(G \setminus N[e]) \simeq *$ then

$$\mathrm{Ind}(G) \simeq \mathrm{Ind}(G-e)$$

e.g. when $G \setminus N[e]$ has an isolated point

if the residue graph of e is contractible, then the removal of e does not affect the homotopy type

Even better consequences: $\mathrm{Ind}(G)$ vs. $\mathrm{Ind}(G-e)$

$$\Sigma\,\mathrm{Ind}(G\backslash N[e]) \hookrightarrow \mathrm{Ind}(G) \hookrightarrow \mathrm{Ind}(G-e) \to \Sigma^2\,\mathrm{Ind}(G\backslash N[e]) \to \cdots.$$

- If $\mathrm{Ind}(G \setminus N[e]) \simeq *$ then

$$\mathrm{Ind}(G) \simeq \mathrm{Ind}(G-e)$$

e.g. when $G \setminus N[e]$ *has an isolated point*

if the residue graph of e is contractible, then the removal of e does not affect the homotopy type

- If $\mathrm{Ind}(e \sqcup (G \setminus N[e])) \hookrightarrow \mathrm{Ind}(G)$ is null-homotopic then

$$\mathrm{Ind}(G-e) \simeq \mathrm{Ind}(G) \vee \Sigma^2\,\mathrm{Ind}(G \setminus N[e])$$

Even better consequences: $\mathrm{Ind}(G)$ vs. $\mathrm{Ind}(G-e)$

$$\Sigma\,\mathrm{Ind}(G\setminus N[e]) \hookrightarrow \mathrm{Ind}(G) \hookrightarrow \mathrm{Ind}(G-e) \to \Sigma^2\,\mathrm{Ind}(G\setminus N[e]) \to \cdots.$$

- If $\mathrm{Ind}(G \setminus N[e]) \simeq *$ then

$$\mathrm{Ind}(G) \simeq \mathrm{Ind}(G-e)$$

e.g. when $G \setminus N[e]$ *has an isolated point*

if the residue graph of e is contractible, then the removal of e does not affect the homotopy type

- If $\mathrm{Ind}(e \sqcup (G \setminus N[e])) \hookrightarrow \mathrm{Ind}(G)$ is null-homotopic then

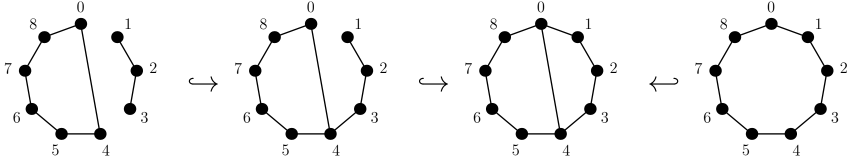
$$\mathrm{Ind}(G-e) \simeq \mathrm{Ind}(G) \vee \Sigma^2\,\mathrm{Ind}(G \setminus N[e])$$

e.g. when e belongs to an induced P_4 'away from' $G \setminus N[e]$.

Example 1 [Kozlov]

A 'natural' proof of $\mathrm{Ind}(C_n) \simeq \Sigma\,\mathrm{Ind}(C_{n-3})$.

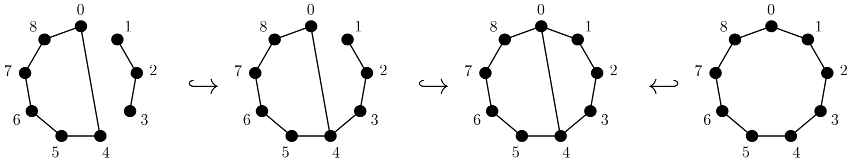
A ‘natural’ proof of $\mathrm{Ind}(C_n) \simeq \Sigma\,\mathrm{Ind}(C_{n-3})$.



$$\mathrm{Ind}(C_n''') \hookleftarrow \mathrm{Ind}(C_n'') \hookleftarrow \mathrm{Ind}(C_n') \hookrightarrow \mathrm{Ind}(C_n)$$

Example 1 [Kozlov]

A ‘natural’ proof of $\mathrm{Ind}(C_n) \simeq \Sigma\,\mathrm{Ind}(C_{n-3})$.

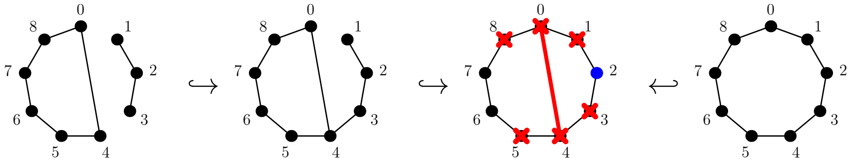


$$\mathrm{Ind}(C_n''') \hookleftarrow \mathrm{Ind}(C_n'') \hookleftarrow \mathrm{Ind}(C_n') \hookrightarrow \mathrm{Ind}(C_n)$$

- each induced map is a homotopy equivalence

Example 1 [Kozlov]

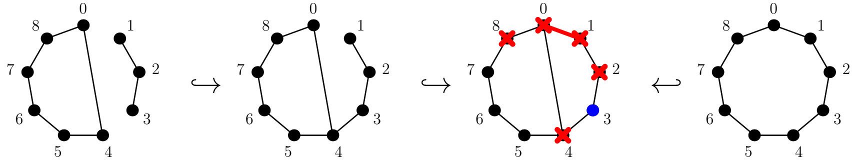
A 'natural' proof of $\mathrm{Ind}(C_n) \simeq \Sigma\,\mathrm{Ind}(C_{n-3})$.



$$\mathrm{Ind}(C_n''') \hookleftarrow \mathrm{Ind}(C_n'') \hookleftarrow \mathrm{Ind}(C_n') \hookrightarrow \mathrm{Ind}(C_n)$$

- each induced map is a homotopy equivalence because $\mathrm{Ind}(G \setminus N[e])$ has an isolated vertex

A 'natural' proof of $\mathrm{Ind}(C_n) \simeq \Sigma\, \mathrm{Ind}(C_{n-3})$.

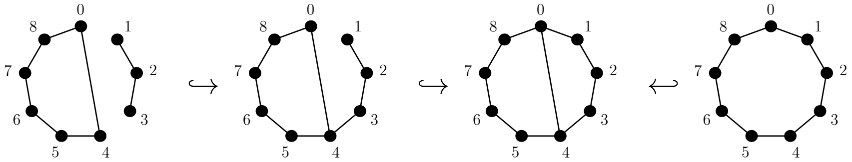


$$\mathrm{Ind}(C_n''') \hookleftarrow \mathrm{Ind}(C_n'') \hookleftarrow \mathrm{Ind}(C_n') \hookrightarrow \mathrm{Ind}(C_n)$$

- each induced map is a homotopy equivalence because $\mathrm{Ind}(G \setminus N[e])$ has an isolated vertex

Example 1 [Kozlov]

A 'natural' proof of $\mathrm{Ind}(C_n) \simeq \Sigma\,\mathrm{Ind}(C_{n-3})$.

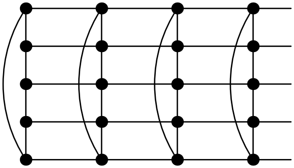


$$\mathrm{Ind}(C_n''') \hookleftarrow \mathrm{Ind}(C_n'') \hookleftarrow \mathrm{Ind}(C_n') \hookrightarrow \mathrm{Ind}(C_n)$$

- each induced map is a homotopy equivalence because $\mathrm{Ind}(G \setminus N[e])$ has an isolated vertex
- $\mathrm{Ind}(C_n''') \simeq \mathrm{Ind}(C_{n-3}) * S^0 = \Sigma\,\mathrm{Ind}(C_{n-3})$.

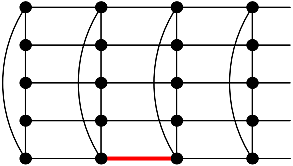
Example 2 [Thapper,Jonsson]

$$\mathrm{Ind}(P_m \times C_5)$$



Example 2 [Thapper,Jonsson]

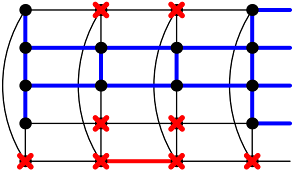
$$\mathrm{Ind}(P_m \times C_5)$$



The edge e can be removed

Example 2 [Thapper,Jonsson]

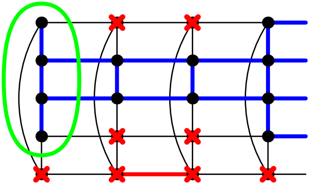
$$\mathrm{Ind}(P_m \times C_5)$$



The edge e can be removed, because $\mathrm{Ind}(G \setminus N[e])$ is contractible.

Example 2 [Thapper,Jonsson]

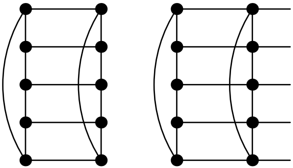
$\mathrm{Ind}(P_m \times C_5)$



The edge e can be removed, because $\mathrm{Ind}(G \setminus N[e])$ is contractible.

Example 2 [Thapper,Jonsson]

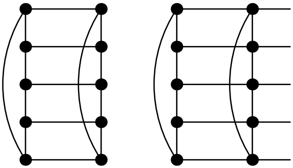
$$\mathrm{Ind}(P_m \times C_5)$$



$$\simeq S^1 * \mathrm{Ind}(P_{m-2} \times C_5)$$

Example 2 [Thapper,Jonsson]

$$\mathrm{Ind}(P_m \times C_5)$$



$$\simeq S^1 * \mathrm{Ind}(P_{m-2} \times C_5)$$

Theorem (A.)

$$\mathrm{Ind}(P_m \times C_7) \simeq S^5 * \mathrm{Ind}(P_{m-4} \times C_7)$$

Powers of cycles

Theorem (A.)

$$\mathrm{Ind}(C_n^r) \simeq \Sigma^2 \,\mathrm{Ind}(C_{n-(3r+3)}^r) \vee X_{n,r}$$

Theorem (A.)

$$\mathrm{Ind}(C_n^r) \simeq \Sigma^2\,\mathrm{Ind}(C_{n-(3r+3)}^r) \vee X_{n,r}$$

where

$$X_{n,r} \simeq \Sigma^3 \bigvee_{i=4r+6}^{6r+3} \bigvee^{k_i} \mathrm{Ind}(P_{n-i}^r)$$

and

$$k_i = \begin{cases} \frac{1}{2}(i-4r-5)(i-2r-2) & \text{for } i \leq 5r+4, \\ \frac{1}{2}(6r+4-i)(i-2r-1) & \text{for } i \geq 5r+5. \end{cases}$$

Powers of cycles

Theorem (A.)

$$\operatorname{Ind}(C_n^r) \simeq \Sigma^2 \operatorname{Ind}(C_{n-(3r+3)}^r) \vee X_{n,r}$$

where

$$X_{n,r} \simeq \Sigma^3 \bigvee_{i=4r+6}^{6r+3} \bigvee^{k_i} \operatorname{Ind}(P_{n-i}^r)$$

and

$$k_i = \begin{cases} \frac{1}{2}(i-4r-5)(i-2r-2) & \text{for } i \leq 5r+4, \\ \frac{1}{2}(6r+4-i)(i-2r-1) & \text{for } i \geq 5r+5. \end{cases}$$

For example

$$\operatorname{Ind}(C_n) \simeq \Sigma^2 \operatorname{Ind}(C_{n-6})$$

$$\operatorname{Ind}(C_n^2) \simeq \Sigma^2 \operatorname{Ind}(C_{n-9}^2) \vee \bigvee^4 \Sigma^3 \operatorname{Ind}(P_{n-14}^2) \vee \bigvee^5 \Sigma^3 \operatorname{Ind}(P_{n-15}^2)$$

Plan

We will extend C_n^r to a graph with more edges

$$C_n^r \hookrightarrow (C_n^r \cup \{e_1, \ldots, e_k\})$$

This leads to an inclusion

Plan

We will extend C_n^r to a graph with more edges

$$C_n^r \hookrightarrow (C_n^r \cup \{e_1, \ldots, e_k\})$$

This leads to an inclusion

$$\mathrm{Ind}(C_n^r \cup \{e_1, \ldots, e_k\}) \hookrightarrow \mathrm{Ind}(C_n^r)$$

Plan

We will extend C_n^r to a graph with more edges

$$C_n^r \hookrightarrow (C_n^r \cup \{e_1, \ldots, e_k\})$$

This leads to an inclusion

$$\begin{array}{ccc} \mathrm{Ind}(C_n^r \cup \{e_1, \ldots, e_k\}) & \hookrightarrow & \mathrm{Ind}(C_n^r) \\ \downarrow \simeq & & \\ \Sigma^2 \,\mathrm{Ind}(C_{n-(3r+3)}^r) & & \end{array}$$

Plan

We will extend C_n^r to a graph with more edges

$$C_n^r \hookrightarrow (C_n^r \cup \{e_1, \ldots, e_k\})$$

This leads to an inclusion

$$\begin{array}{ccc} \mathrm{Ind}(C_n^r \cup \{e_1, \ldots, e_k\}) & \hookrightarrow & \mathrm{Ind}(C_n^r) \\ \downarrow \simeq & & \\ \Sigma^2 \, \mathrm{Ind}(C_{n-(3r+3)}^r) & & \end{array}$$

$+$ every single edge addition splits

Plan

We will extend C_n^r to a graph with more edges

$$C_n^r \hookrightarrow (C_n^r \cup \{e_1, \dots, e_k\})$$

This leads to an inclusion

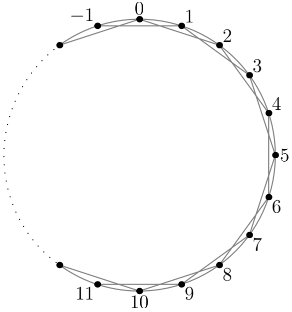
$$\mathrm{Ind}(C_n^r \cup \{e_1, \dots, e_k\}) \hookrightarrow \mathrm{Ind}(C_n^r)$$

$$\mathrm{Ind}(C_n^r \cup \{e_1, \dots, e_k\}) \xrightarrow{\simeq} \Sigma^2 \,\mathrm{Ind}(C^r_{n-(3r+3)})$$

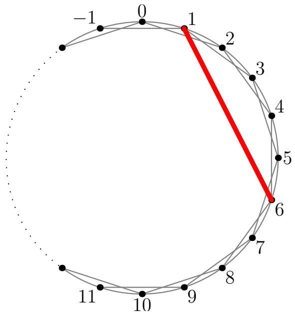
$+$ every single edge addition splits . Then

$$\begin{aligned} \mathrm{Ind}(C_n^r) &\simeq \mathrm{Ind}(C_n^r \cup \{e_1\}) \vee X_1 \\ &\simeq \mathrm{Ind}(C_n^r \cup \{e_1, e_2\}) \vee X_1 \vee X_2 \\ &\quad \dots \\ &\simeq \Sigma^2 \,\mathrm{Ind}(C^r_{n-(3r+3)}) \vee X_{n,r} \end{aligned}$$

Pictorial proof when $r = 2$ — splittings



Pictorial proof when $r = 2$ — splittings



$$\mathrm{Ind}(e \sqcup (G \setminus N[e])) \hookrightarrow \mathrm{Ind}(G \cup e) \hookrightarrow \mathrm{Ind}(G) \to \Sigma^2\,\mathrm{Ind}(G \setminus N[e]) \to \cdots.$$

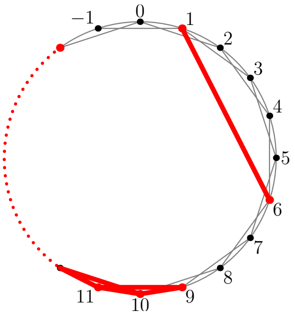
If we show

$$\mathrm{Ind}(e \sqcup (G \setminus N[e])) \hookrightarrow \mathrm{Ind}(G \cup e)$$

is null-homotopic, then we get

$$\mathrm{Ind}(G) \simeq \mathrm{Ind}(G \cup e) \vee \Sigma^2\,\mathrm{Ind}(G \setminus N[e])$$

Pictorial proof when $r = 2$ — splittings



$$\mathrm{Ind}(e \sqcup (G \setminus N[e])) \hookrightarrow \mathrm{Ind}(G \cup e) \hookrightarrow \mathrm{Ind}(G) \to \Sigma^2\, \mathrm{Ind}(G \setminus N[e]) \to \cdots .$$

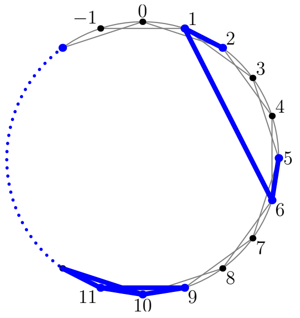
If we show

$$\mathrm{Ind}(e \sqcup (G \setminus N[e])) \hookrightarrow \mathrm{Ind}(G \cup e)$$

is null-homotopic, then we get

$$\mathrm{Ind}(G) \simeq \mathrm{Ind}(G \cup e) \vee \Sigma^2\, \mathrm{Ind}(G \setminus N[e])$$

Pictorial proof when $r = 2$ — splittings



$$\mathrm{Ind}(e \sqcup (G \setminus N[e])) \hookrightarrow \mathrm{Ind}(G \cup e) \hookrightarrow \mathrm{Ind}(G) \to \Sigma^2\,\mathrm{Ind}(G \setminus N[e]) \to \cdots.$$

But there is a factorization

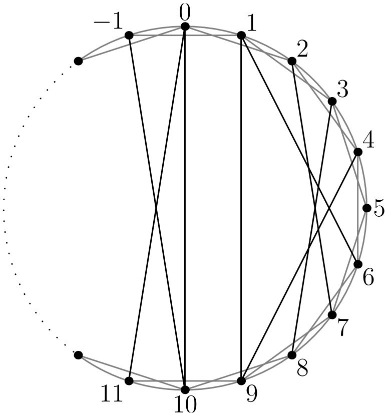
$$\mathrm{Ind}(e \sqcup (G \setminus N[e])) \hookrightarrow \mathrm{Ind}(P_4 \sqcup (G \setminus N[e])) \hookrightarrow \mathrm{Ind}(G \cup e)$$

through a contractible space, so

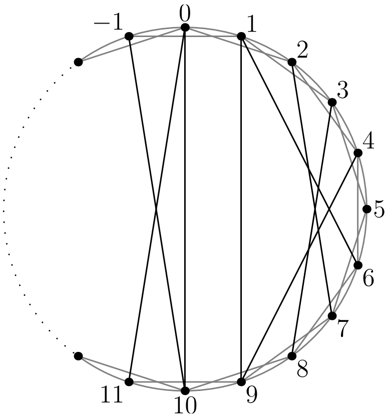
$$\mathrm{Ind}(G) \simeq \mathrm{Ind}(G \cup e) \vee \Sigma^2\,\mathrm{Ind}(G \setminus N[e])$$

Pictorial proof when $r = 2$ — splittings

Similar arguments are used to add edges until we get the graph:

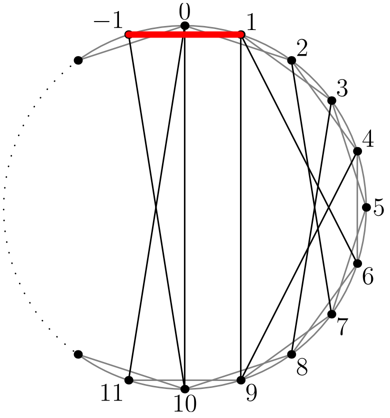


We just need to find $\operatorname{Ind}()$ of the graph



Pictorial proof when $r = 2$ — modeling $\Sigma^2\,\mathrm{Ind}(C^2_{n-9})$

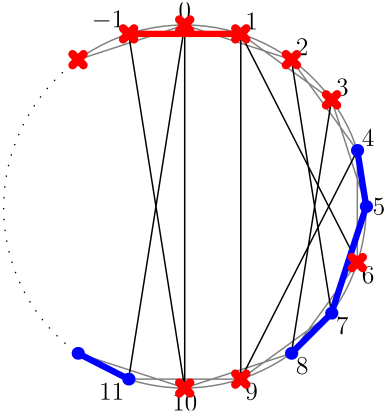
We just need to find $\mathrm{Ind}()$ of the graph



The edge e can be removed

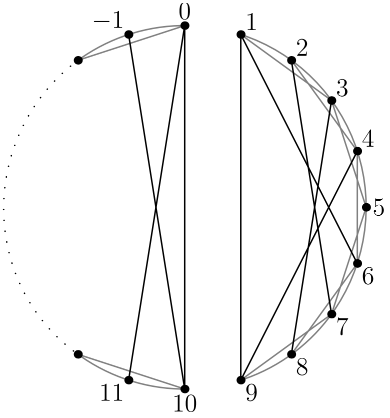
Pictorial proof when $r = 2$ — modeling $\Sigma^2 \operatorname{Ind}(C_{n-9}^2)$

We just need to find $\operatorname{Ind}()$ of the graph

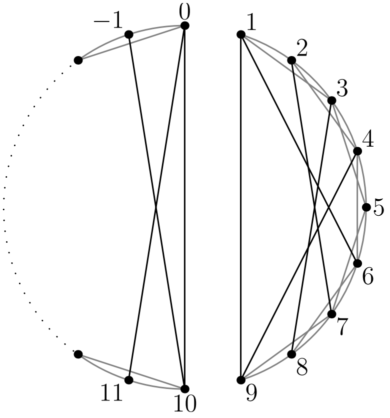


The edge e can be removed, because $\operatorname{Ind}(G \setminus N[e]) \simeq *$.

Repeating such removals we are left with

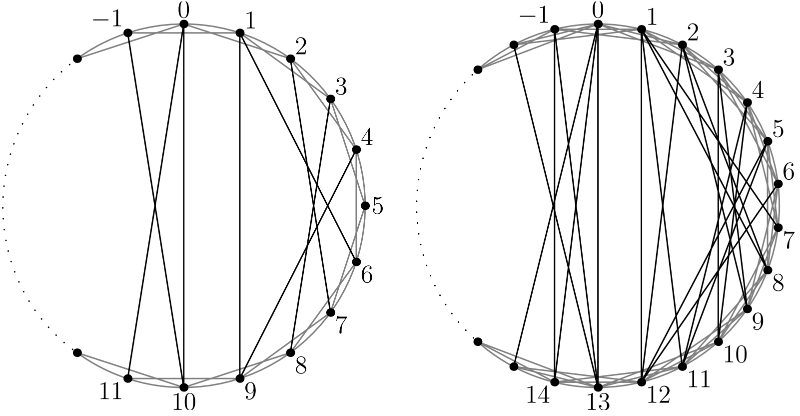


Repeating such removals we are left with



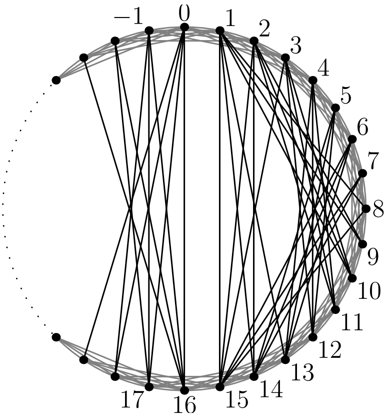
$$\mathrm{Ind}(C^2_{n-9}) \quad * \quad S^1.$$

Models for $\Sigma^2\,\mathrm{Ind}(C^r_{n-(3r+3)})$

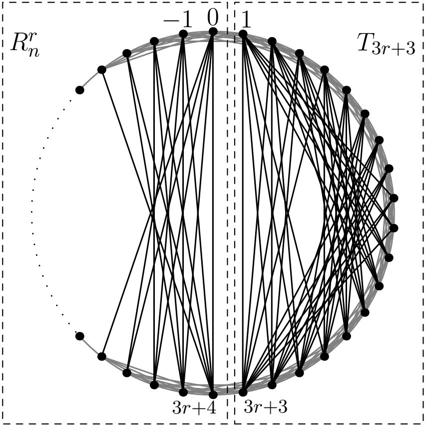


$r = 2$ $\qquad$ $r = 3$

Models for $\Sigma^2 \operatorname{Ind}(C^r_{n-(3r+3)})$



$r = 4$



$r = 5$

The last slide

Other methods of calculating the homotopy type of $\mathrm{Ind}(C_n^r)$ were proposed by

- Damiano Testa, *private comm.*
- Duško Jojić, *Shellability of complexes of directed trees*, `arxiv/1109.4475`

and this one is

- M.A., *Splittings of independence complexes and the powers of cycles*, `arxiv/1106.6250`

The last slide

Other methods of calculating the homotopy type of $\mathrm{Ind}(C_n^r)$ were proposed by

- Damiano Testa, *private comm.*
- Duško Jojić, *Shellability of complexes of directed trees*, `arxiv/1109.4475`

and this one is

- M.A., *Splittings of independence complexes and the powers of cycles*, `arxiv/1106.6250`

Thank you!