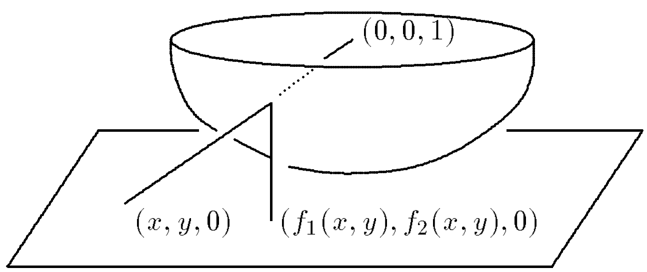
What is projective space? It's similar to, but easier to work with than, the more familiar real euclidean space, beacuse any two lines—not just non-parallel ones—meet in one point. Enjoy this tour of projective space; and a round of applause for all the pretty pictures!

Boy's will be Boy's

by Brian Sanderson

Imagine walking in a straight line off to infinity in the plane. Add a point at each end of the line and make these two points one. The line has become a circle. The universe could be like this: shine a light out to infinity and it could eventually return to hit you on the back of the head. This kind of space is called *projective space* (and can be rather hard to visualise!).

What about these added *points at infinity*? How many are there, and how do they fit together? Each straight line has been completed (to make a circle) by adding a single point. Surprisingly, more than one line can share the same added point. To get a handle on this let us 'bring infinity closer' so we can examine it better. To do this we will construct a continuous bijection (a *homeomorphism*) from the plane to the interior of the unit disc. This map, which we call f, can be defined in geometrical terms as follows. Stand a 'bowl', a hemisphere of unit radius, on the plane at the origin, as shown here:



Suppose we have a point $(x, y, 0)$ in the xy-plane and we want to know where it is mapped to under f. From $(x, y, 0)$ draw a straight line to $(0, 0, 1)$, and from the point of intersection of this line with the bowl drop straight down to the plane to get the point, say, $(f_1(x, y), f_2(x, y), 0)$. Now if we define $f(x, y) = (f_1(x, y), f_2(x, y))$, we find that

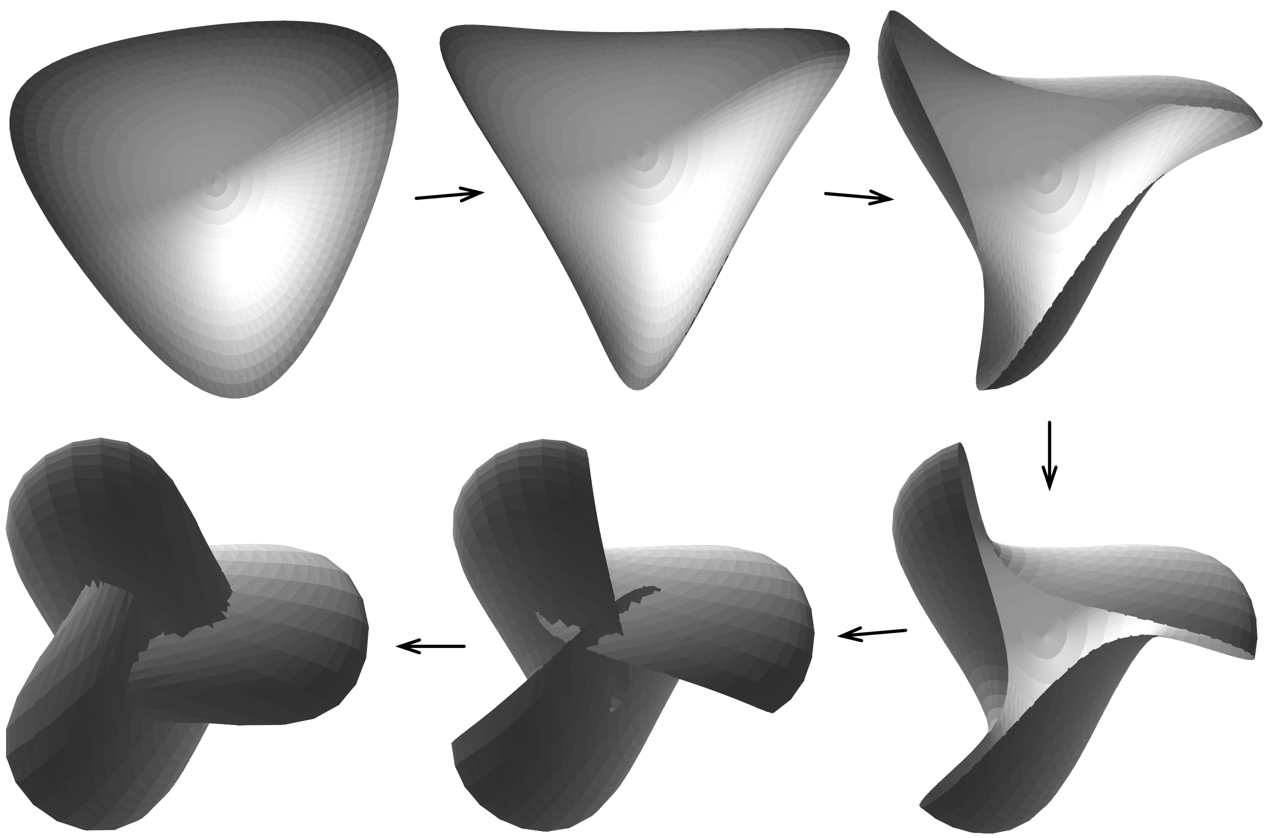
$$f(x,y) = \left(\frac{x}{\sqrt{x^2+y^2+1}}, \frac{y}{\sqrt{x^2+y^2+1}} \right)$$

is a continuous bijection from the xy-plane to the horizontal unit disc about the origin (the 'shadow' of the hemisphere), as wanted.

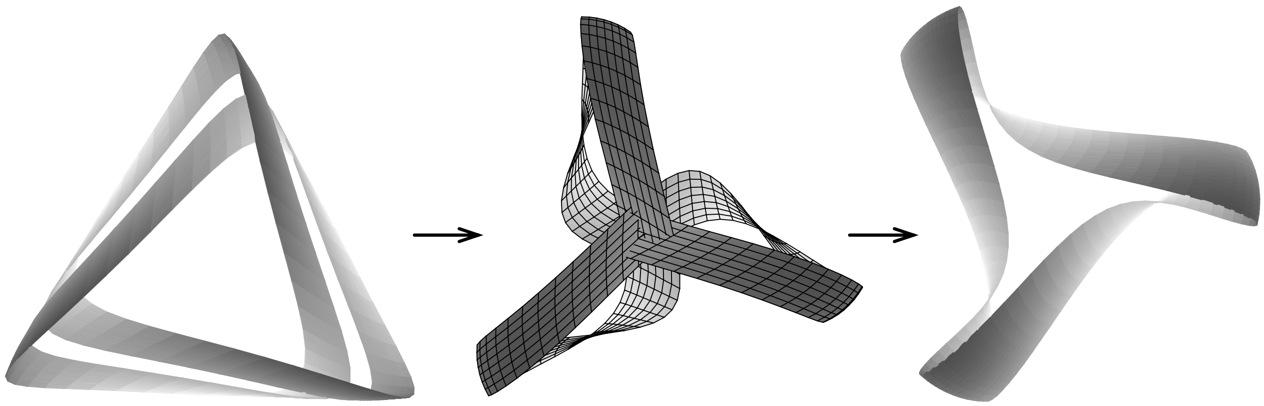
The Disc Model

We have now modelled the plane on the interior of a unit disc. Where should we put the points at infinity? The answer is obvious—put them on the unit circle, i.e., the boundary of the unit disc. But watch out: there is a trap here. Remember that points at the ends of the same line should be identified, so by considering lines through the origin in the plane we see that opposite points on the boundary of the disc are really the same point in projective space and should be identified. Suppose we take a loop of thread to represent the points on the edge of the disc. Then we can bring opposite points in contact with each other by joining two points of the thread to make a figure eight, and then putting the two loops together. The result is still a circle, the *circle at infinity*. Our construction is now complete: each original line has become a circle, and with the added circle at infinity we have made the *projective plane*.

To make a model of the projective plane we need to take a disc, made of rubber, say, and somehow join together opposite points on the edge of the disc. It is a fact that this cannot be done (in three dimensions) without self intersections; the disc must pass through itself somewhere. Here are six stages taken from an animation of a disc getting its edge together to make a projective plane:



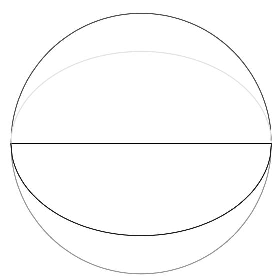
Notice that in the resulting model each little bit of the disc looks like a smoothly bent piece of surface; there are no kinks in the result. This is known as an *immersion*. We have produced a particular immersion known as *Boy's surface*. It's not clear quite what's happening in the last three pictures, so take a look at a corresponding band round the edge of the disc:



Now let's look more closely at those points at infinity. According to our construction, there is a bijection between the unit circle with opposite points identified (that's the edge of the disc) and straight lines through the origin. This raises the question: where are the points at infinity which sit on the end of straight lines which do *not* pass through the origin? Take a minute or two to think about this. Which of these points do we get to if we set out from some point away from the origin?

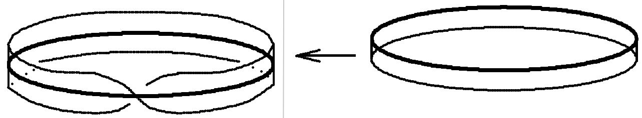
The answer is that parallel lines have the same end point at infinity. Check this by using the formula for f above. Alternatively, think about the bowl on the table: take two different parallel lines in the (x, y) plane, and for each line find the plane containing it and the point $(0, 0, 1)$. These planes intersect in a horizontal line which meets opposite points on the boundary of the bowl. When these are projected down onto the disc, the two lines are mapped to curves which end at the same points on the boundary of the disc.

The image of the three lines parallel to the x-axis and through $(0, 0)$, $(0, -1)$, and $(0, 1)$ are shown below.

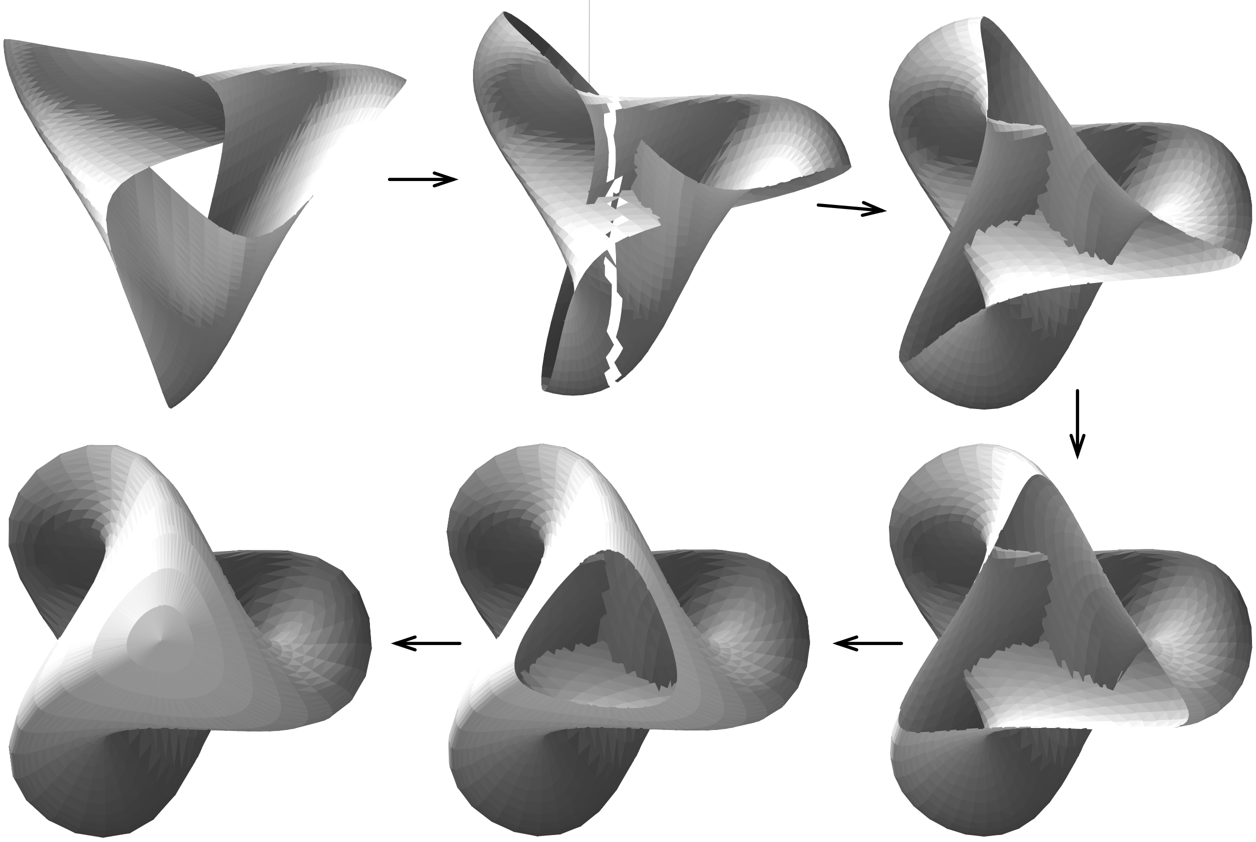


The Möbius Band Model

The disc model described above is fine, but there are other ways to look at the same immersion of the projective plane. Consider cutting out the central disc of radius a half from the unit disc. What is left? The answer is a band with two edges. Now we want to identify (glue together) all pairs of opposite points on the outer edge. If you try this with a strip of paper, you'll get the famous one-sided Möbius band. This is literally child's play. (Notice that if you start with a band of paper then scissors as well as glue will be needed.)



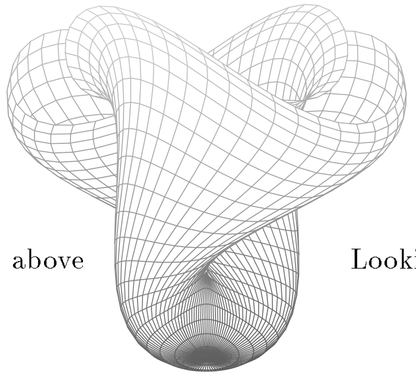
Now consider that half disc we discarded. We could equally well have discarded a very small disc and still have a Möbius band left over. In fact by shrinking the small disc to a single point we see that a projective plane can be described by: take a Möbius band and squeeze its edge to a single point! Here is a sequence of pictures, taken from an animation, showing a Möbius band doing just that.



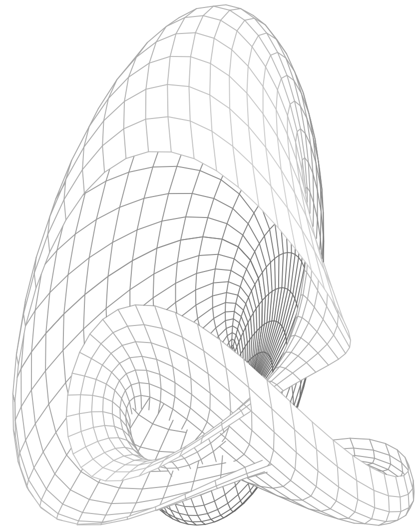
The two sequences of pictures we've seen so far are in fact complementary. The projective plane, as we have seen, can be made by taking a disc and a Möbius band and sewing their edges together. As we increase the size of the disc, the complementary Möbius band shrinks away to its central circle which is also then the edge of the disc. The first sequence shows just the disc growing; the second sequence shows the Möbius band growing during the reverse process as the disc (not shown) shrinks to its centre point. The viewpoint for the two sequences is different. If we think of looking at the last sequence as 'looking from above', the other sequence is 'looking from below'.

Consider the bottom left pictures in each case. These are views of the *Boy's surface model* of the projective plane. This has a pretty 3-fold symmetry and would make a nice three-legged stool with rather fat legs. Let's look at it from the side: suppose we grow a disc up a leg, rather than from the centre of the seat as before:

From above

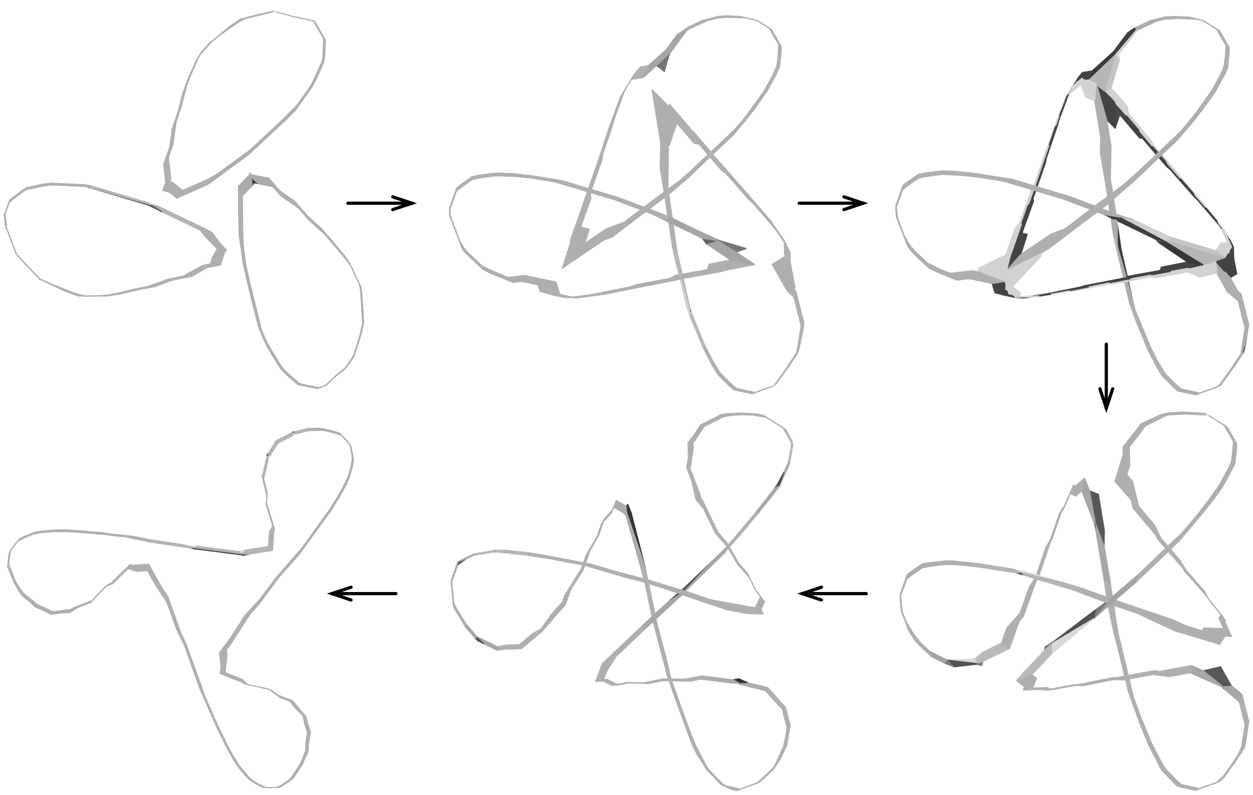


Looking in



Slicing the models

Another way to understand the model is to stand it on its three legs and take horizontal slices:



These ways of thinking of the projective plane hide its symmetry and homogeneity. For instance, the circle at infinity stands out from all the other circles. To avoid confusion (to create confusion?) the straight lines together with their added points and the circle at infinity are all called just *lines* in projective geometry. Notice that we can now say "any two distinct lines meet in exactly one point". In some ways this makes projective geometry easier than euclidean geometry, where a pair of lines may or may not meet, thus increasing the work in proofs involving lines. So can we find a model of the projective plane which treats all lines equally? Yes we can.

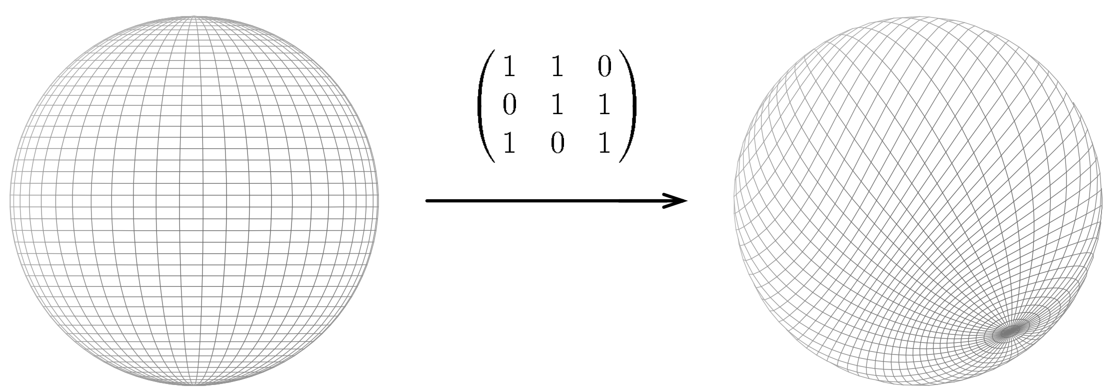
The Sphere Model

Go back to the bowl sitting on the plane. The bowl can be identified with the disc directly below it and so we can regard the projective plane as the bowl with opposite points on the rim identified. This is not yet a homogeneous view, but consider putting another upside down bowl on top of the first one thus making a sphere. Now if we identify a point in the top bowl with the opposite point in the bottom bowl (draw a line through $(0, 0, 1)$) we still have one point for each point in the projective plane. So the projective plane can now be described as: 'take a sphere and make opposite points equal'. The lines are now just the great circles on the sphere (with opposite points made equal) and no line is preferred over any other.

Projective Geometry

Projective geometry is the study of properties of projective space which remain invariant under projective transformations (which are described below). For example, lines will map to lines. An example of a projective transformation can be seen in a photograph or in a picture, provided the laws of perspective (projective geometry) have not been violated! Angles and distances are not preserved; in fact, almost any convex quadrilateral could be a valid picture of a unit square. A projective transformation in general can be described as follows.

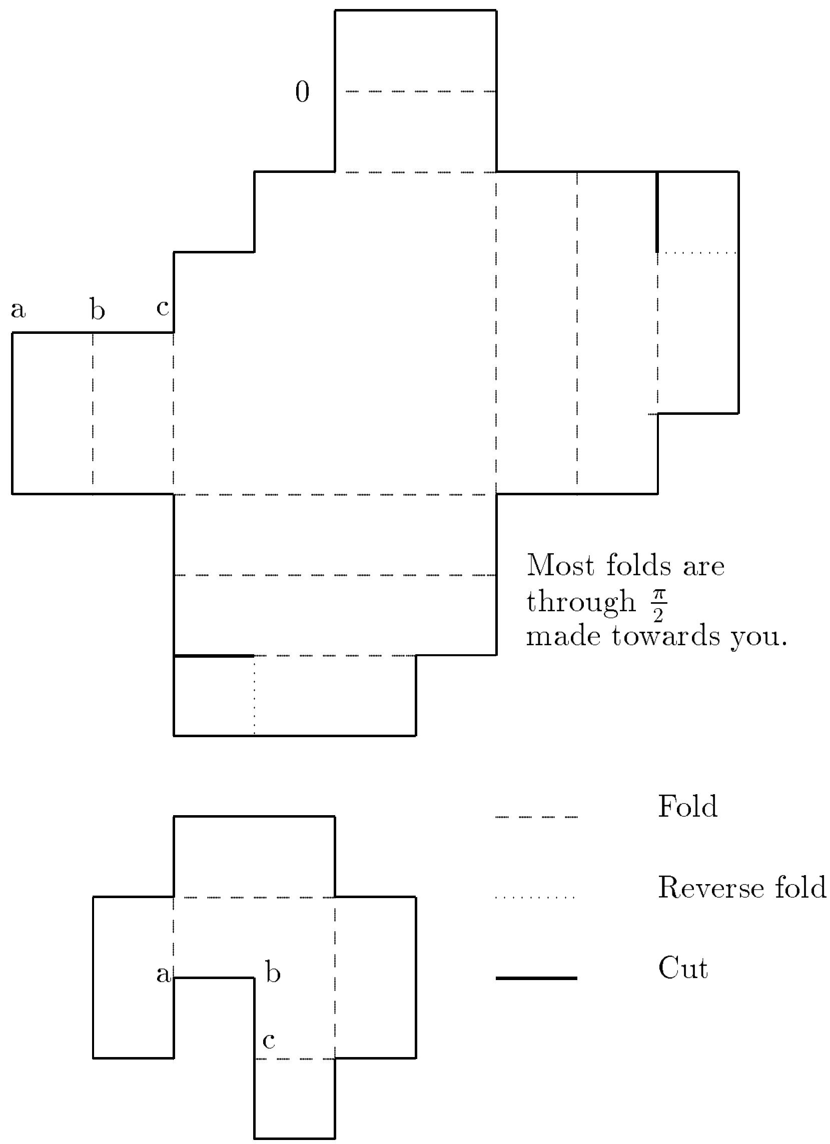
Consider our sphere model as the unit sphere in 3-space with centre the origin, and apply a linear isomorphism (given by a matrix). The result is in general a squashed sphere in 3-space. Compose with the projection to the unit 2-sphere. This induces a *projective transformation*. Below is a picture of the result of such a transformation on the 2-sphere.



$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Instructions for making a paper model of Boy's surface

Cut out and glue three copies of the following pair of surfaces. First glue the pairs as shown by the labels; then put the three pieces together by identifying the 0 labels.



References

Boy, Werner; *Über die Curvatura Integra und die Topologie der geschlossener Flächen*, Dissertation, Göttingen (1901). *Math. Ann.* **57**(1903), 151–184.

Francis, George K.; *A topological picture book*, Springer-Verlag (1987).

Reid, Miles; *How to knit a projective plane*, Manifold ???????

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