Lecture Outline

Determinants
- Determinants of Order 2
- Determinants of Order 3
- Characterizing the Determinant Function
- Rules for Determinants
- Expansion by Alien Cofactors and the Adjugate Matrix
- Minor Determinants

The Inverse Matrix
- Definition and Existence
- Orthogonal Matrices
- Partitioned Matrices
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Determinants of Order 2: Definition

Consider again the pair of linear equations

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 &= b_1 \\
    a_{21}x_1 + a_{12}x_2 &= b_2
\end{align*}
\]

with its associated coefficient matrix

\[
A = \begin{pmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{pmatrix}
\]

Let us define \( D := a_{11}a_{22} - a_{21}a_{12} \).

Provided that \( D \neq 0 \), there is a unique solution given by

\[
\begin{align*}
    x_1 &= \frac{1}{D}(b_1a_{22} - b_2a_{12}), \\
    x_2 &= \frac{1}{D}(b_2a_{11} - b_1a_{21})
\end{align*}
\]

The number \( D \) is called the determinant of the matrix \( A \),
and denoted by either \( \det(A) \) or more concisely, \( |A| \).
Determinants of Order 2: Simple Rule

Thus, for any $2 \times 2$ matrix $A$, its determinant $D$ is

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

For this special case of order 2 determinants, a simple rule is:

1. multiply the diagonal elements together;
2. multiply the off-diagonal elements together;
3. subtract the product of the off-diagonal elements from the product of the diagonal elements.

Note that

$$|A| = a_{11}a_{22} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + a_{21}a_{12} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$
Cramer’s Rule in the $2 \times 2$ Case

Using determinant notation, the solution to the equations

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 &= b_1 \\
    a_{21}x_1 + a_{12}x_2 &= b_2
\end{align*}
\]

can be written in the alternative form

\[
\begin{align*}
    x_1 &= \frac{1}{D} \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}, \\
    x_2 &= \frac{1}{D} \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}
\end{align*}
\]

This accords with Cramer’s rule for the solution to $Ax = b$, which is the vector $x = (x_i)_{i=1}^n$ each of whose components $x_i$ is the fraction with:

1. denominator equal to the determinant $D$ of the coefficient matrix $A$ (provided, of course, that $D \neq 0$);
2. numerator equal to the determinant of the matrix $(A_{-i}, b)$ formed from $A$ by replacing its $i$th column with the $b$ vector of right-hand side elements.
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Determinants of Order 3: Definition

Determinants of order 3 can be calculated from those of order 2 according to the formula

\[
\begin{vmatrix} a_{11} & a_{22} & a_{23} \\ a_{32} & a_{23} & a_{33} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{21} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{21} & a_{22} \\ a_{31} & a_{22} & a_{33} \end{vmatrix} = \sum_{j=1}^{3} (-1)^{1+j} a_{1j} |\mathbf{C}_{1j}|\]

where, for \( j = 1, 2, 3 \), the \( 2 \times 2 \) matrix \( \mathbf{C}_{1j} \) is the \((1,j)\)-cofactor obtained by removing both row 1 and column \( j \) from \( \mathbf{A} \).

The result is the following sum

\[
\begin{vmatrix} a_{11} & a_{22} & a_{33} \\ a_{11} & a_{23} & a_{32} + a_{12}a_{23}a_{31} \\ - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \end{vmatrix}
\]

of \( 3! = 6 \) terms, each the product of 3 elements chosen so that each row and each column is represented just once.
Determinants of Order 3: Cofactor Expansion

The determinant expansion

\[
|A| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} \\
- a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}
\]

is very symmetric, suggesting (correctly) that the cofactor expansion along the first row \((a_{11}, a_{12}, a_{13})\)

\[
|A| = \sum_{j=1}^{3} (-1)^{1+j} a_{1j}|C_{1j}|
\]

gives the same answer as the two cofactor expansions

\[
|A| = \sum_{j=1}^{3} (-1)^{r+j} a_{rj}|C_{rj}| = \sum_{i=1}^{3} (-1)^{i+s} a_{is}|C_{is}|
\]

along, respectively:

- the \(r\)th row \((a_{r1}, a_{r2}, a_{r3})\)
- the \(s\)th column \((a_{1s}, a_{2s}, a_{3s})\)
Determinants of Order 3: Alternative Expressions

The same result

\[ |A| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \]

can be obtained as either of the two expansions

\[ |A| = \sum_{j_1=1}^{3} \sum_{j_2=1}^{3} \sum_{j_3=1}^{3} \epsilon_{j_1j_2j_3} a_{1j_1} a_{2j_2} a_{3j_3} = \sum_{\pi \in \Pi} \text{sgn}(\pi) \prod_{i=1}^{3} a_{i\pi(i)} \]

Here \( \epsilon_j = \epsilon_{j_1j_2j_3} \in \{-1, 0, 1\} \) denotes the Levi-Civita symbol associated with the mapping \( i \mapsto j_i \) from \( \{1, 2, 3\} \) into itself.

Also, \( \Pi \) denotes the set of all \( 3! = 6 \) possible permutations on \( \{1, 2, 3\} \), with typical member \( \pi \), whose sign is denoted by \( \text{sgn}(\pi) \).
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Characterizing the Determinant Function

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The Determinant Function

When $n = 1, 2, 3$, the determinant mapping $A \mapsto |A| \in \mathbb{R}$ specifies the determinant $|A|$ of each $n \times n$ matrix $A$ as a function of its $n$ row vectors $(a_i)_{i=1}^n$.

For a general natural number $n \in \mathbb{N}$, consider any mapping

$$D_n \ni A \mapsto D(A) = D((a_i)_{i=1}^n) \in \mathbb{R}$$

defined on the domain $D_n$ of $n \times n$ matrices.

**Notation:** Let $D(A/b_r)$ denote the new value $D(a_1, \ldots, a_{r-1}, b_r, a_{r+1}, \ldots, a_n)$ of the function $D$ after the $r$th row $a_r$ of the matrix $A$ has been replaced by the new row vector $b_r$. 
Row Multilinearity

Definition
The function $D_n \ni A \mapsto D(A)$ of A's $n$ rows $(a_i)_{i=1}^n$ is (row) multilinear just in case,
for each row number $i \in \{1, 2, \ldots, n\}$,
each pair $b_i, c_i \in \mathbb{R}^n$ of new versions of row $i$,
and each pair of scalars $\lambda, \mu \in \mathbb{R}$, one has

$$D(A/\lambda b_i + \mu c_i) = \lambda D(A/b_i) + \mu D(A/c_i)$$

Formally, the mapping $\mathbb{R}^n \ni a_i \mapsto D(A/a_i) \in \mathbb{R}$ should be linear,
for fixed each row $i \in \mathbb{N}_n$.

That is, $D$ is a linear function of the $i$th row vector $a_i$ on its own,
when all the other rows $a_h \ (h \neq i)$ are fixed.
The Three Characterizing Properties

Definition
The function $D_n \ni A \mapsto D(A)$ is alternating just in case for every transposition matrix $T$, one has $D(TA) = -D(A)$ — i.e., interchanging any two rows reverses its sign.

Definition
The mapping $D_n \ni A \mapsto D(A)$ is of the determinant type just in case:

1. $D$ is multilinear in its rows;
2. $D$ is alternating;
3. $D(I_n) = 1$ for the identity matrix $I_n$.

Exercise
Show that the mapping $D_n \ni A \mapsto |A| \in \mathbb{R}$ is of the determinant type provided that $n \leq 3$. 
Lemma

Suppose that $D_n \ni A \mapsto D(A)$ is multilinear in its rows. For any fixed $B \in D_n$, the value of $D(AB)$ can be expressed as the linear combination

$$D(AB) = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_n=1}^{n} a_{1j_1} a_{2j_2} \cdots a_{nj_n} D(L_{j_1j_2\ldots j_n}B)$$

of its values at all possible matrices

$$L_jB = L_{j_1j_2\ldots j_n}B := (b_{jr})_{r=1}^{n}$$

whose $r$th row, for each $r = 1, 2, \ldots, n$, equals the $j_r$th row $b_{jr}$ of the matrix $B$. 

First Implication of Multilinearity in the $n \times n$ Case
Characterizing $2 \times 2$ Determinants

1. In the case of $2 \times 2$ matrices, the lemma tells us that multilinearity implies

$$D(AB) = a_{11}a_{21}D(b_1, b_1) + a_{11}a_{22}D(b_1, b_2)$$
$$+ a_{12}a_{21}D(b_2, b_1) + a_{12}a_{22}D(b_2, b_2)$$

where $b_1 = (b_{11}, b_{21})$ and $b_2 = (b_{12}, b_{22})$ are the rows of $B$.

2. If $D$ is also alternating, then $D(b_1, b_1) = D(b_2, b_2) = 0$ and $D(B) = D(b_1, b_2) = -D(b_2, b_1)$, implying that

$$D(AB) = a_{11}a_{22}D(b_1, b_2) + a_{12}a_{21}D(b_2, b_1)$$
$$= (a_{11}a_{22} - a_{12}a_{21})D(B)$$

3. Imposing the additional restriction $D(B) = 1$ when $B = I_2$, we obtain the ordinary determinant $D(A) = a_{11}a_{22} - a_{12}a_{21}$.

4. Then, too, one derives the product rule $D(AB) = D(A)D(B)$. 
First Implication of multilinearity: Proof

Each element of the product $C = AB$ satisfies $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$.

Hence each row $c_i = (c_{ik})_{k=1}^{n}$ of $C$ can be expressed as the linear combination $c_i = \sum_{j=1}^{n} a_{ij} b_j$ of $B$'s rows.

For each $r = 1, 2, \ldots, n$ and arbitrary selection $b_{j_1}, \ldots, b_{j_{r-1}}$ of $r - 1$ rows from $B$, multilinearity therefore implies that

$$D(b_{j_1}, \ldots, b_{j_{r-1}}, c_r, c_{r+1}, \ldots, c_n) = \sum_{j_r=1}^{n} a_{ij_r} D(b_{j_1}, \ldots, b_{j_{r-1}}, b_{j_r}, c_{r+1}, \ldots, c_n)$$

This equation can be used to show, by induction on $k$, that

$$D(C) = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_k=1}^{n} a_{1j_1} a_{2j_2} \cdots a_{kj_k} D(b_{j_1}, \ldots, b_{j_k}, c_{k+1}, \ldots, c_n)$$

for $k = 1, 2, \ldots, n$, including for $k = n$ as the lemma claims. □
Additional Implications of Alternation

Lemma

Suppose $D_n \ni A \mapsto D(A)$ is both row multilinear and alternating.

Then for all possible $n \times n$ matrices $A, B,$ and for all possible permutation matrices $P^\pi$, one has:

1. $D(AB) = \sum_{\pi \in \Pi} \prod_{i=1}^{n} a_{i\pi(i)} D(P^\pi B)$

2. $D(P^\pi B) = \text{sgn}(\pi) D(B)$.

3. Under the additional assumption that $D(I_n) = 1$, one has:

   determinant formula: $D(A) = \sum_{\pi \in \Pi} \text{sgn}(\pi) \prod_{i=1}^{n} a_{i\pi(i)}$;

   product rule: $D(AB) = D(A)D(B)$
First Additional Implication of Alternation: Proof

Because $D$ is alternating, one has $D(B) = 0$ whenever two rows of $B$ are equal.

It follows that for any matrix $(b_{ji})_{i=1}^n = L_jB$ whose $n$ rows are all rows of the matrix $B$, one has $D((b_{ji})_{i=1}^n) = 0$ unless these rows are all different.

But if all the $n$ rows of $(b_{ji})_{i=1}^n = L_jB$ are different, there exists a permutation $\pi \in \Pi$ such that $L_jB = P^{\pi}B$.

Hence, after eliminating terms that are zero, the sum

$$D(AB) = \sum_{j_1=1}^n \sum_{j_2=1}^n \cdots \sum_{j_n=1}^n a_{j_1} a_{j_2} \cdots a_{j_n} D((b_{jr})_{r=1}^n)$$

$$= \sum_{j_1=1}^n \sum_{j_2=1}^n \cdots \sum_{j_n=1}^n a_{j_1} a_{j_2} \cdots a_{j_n} D(L_{j_1j_2\ldots j_n}B)$$

as stated in part 1 of the Lemma.
Because $D$ is alternating, one has $D(T\pi B) = -D(P\pi B)$ whenever $T$ is a transposition matrix.

Suppose that $\pi = \tau^1 \circ \cdots \circ \tau^q$ is one possible “factorization” of the permutation $\pi$ as the composition of transpositions.

But $\text{sgn}(\tau) = -1$ for any transposition $\tau$.

So $\text{sgn}(\pi) = (-1)^q$ by the product rule for signs of permutations.

Note that $P\pi = T^1 T^2 \cdots T^q$ where $T^p$ denotes the permutation matrix corresponding to the transposition $\tau^p$, for each $p = 1, \ldots, q$

It follows that

$$D(P\pi B) = D(T^1 T^2 \cdots T^q B) = (-1)^q D(B) = \text{sgn}(\pi) D(B)$$

as required.
Third Additional Implication: Proof

In case $D(I_n) = 1$, applying parts 1 and 2 of the Lemma (which we have already proved) with $B = I_n$ gives immediately

$$D(A) = \sum_{\pi \in \Pi} \prod_{i=1}^{n} a_{i\pi(i)} D(P^\pi) = \sum_{\pi \in \Pi} \text{sgn}(\pi) \prod_{i=1}^{n} a_{i\pi(i)}$$

But then, applying parts 1 and 2 of the Lemma for a general matrix $B$ gives

$$D(AB) = \sum_{\pi \in \Pi} \prod_{i=1}^{n} a_{i\pi(i)} D(P^\pi B)$$

$$= \sum_{\pi \in \Pi} \text{sgn}(\pi) \prod_{i=1}^{n} a_{i\pi(i)} D(B) = D(A)D(B)$$

as an implication of the first equality on this slide.

This completes the proof of all three parts.

\[\square\]
Formal Definition and Cofactor Expansion

Definition

The determinant $|A|$ of any $n \times n$ matrix $A$ is defined so that $\mathcal{D}_n \ni A \mapsto |A|$ is the unique (row) multilinear and alternating mapping that satisfies $|I_n| = 1$.

Definition

For any $n \times n$ determinant $|A|$, its $rs$-cofactor $|C_{rs}|$ is the $(n - 1) \times (n - 1)$ determinant of the matrix $C_{rs}$ obtained by omitting row $r$ and column $s$ from $A$.

The cofactor expansion of $|A|$ along any row $r$ or column $s$ is

$$|A| = \sum_{j=1}^{n} (-1)^{r+j} a_{rj} |C_{rj}| = \sum_{i=1}^{n} (-1)^{i+s} a_{is} |C_{is}|$$

Exercise

Prove that these cofactor expansions are valid, using the formula

$$|A| = \sum_{\pi \in \Pi} \prod_{i=1}^{n} \text{sgn}(\pi) a_{i\pi(i)}$$
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Eight Basic Rules (Rules A–H of EMEA, Section 16.4)

Let $|A|$ denote the determinant of any $n \times n$ matrix $A$.

1. $|A| = 0$ if all the elements in a row (or column) of $A$ are 0.
2. $|A^\top| = |A|$, where $A^\top$ is the transpose of $A$.
3. If all the elements in a single row (or column) of $A$ are multiplied by a scalar $\alpha$, so is its determinant.
4. If two rows (or two columns) of $A$ are interchanged, the determinant changes sign, but not its absolute value.
5. If two of the rows (or columns) of $A$ are proportional, then $|A| = 0$.
6. The value of the determinant of $A$ is unchanged if any multiple of one row (or one column) is added to a different row (or column) of $A$.
7. The determinant of the product $|AB|$ of two $n \times n$ matrices equals the product $|A| \cdot |B|$ of their determinants.
8. If $\alpha$ is any scalar, then $|\alpha A| = \alpha^n |A|$.
The Transpose Rule 2: Verification

The transpose rule 2 is key: for any statement about how $|A|$ depends on the rows of $A$, there is an equivalent statement about how $|A|$ depends on the columns of $A$.

Exercise

Verify Rule 2 directly for $2 \times 2$ and then for $3 \times 3$ matrices.

Proof of Rule 2

The expansion formula implies that

$$|A| = \sum_{\pi \in \Pi} \text{sgn}(\pi) \prod_{i=1}^{n} a_{i\pi(i)} = \sum_{\pi \in \Pi} \text{sgn}(\pi) \prod_{j=1}^{n} a_{\pi^{-1}(j)j}$$

But the product rule for signs of permutations implies that $\text{sgn}(\pi) \text{sgn}(\pi^{-1}) = \text{sgn}(\iota) = 1$, with $\text{sgn}(\pi) = \pm 1$. Hence $\text{sgn}(\pi^{-1}) = 1 / \text{sgn}(\pi) = \text{sgn}(\pi)$.

So, because $\pi \leftrightarrow \pi^{-1}$ is a bijection,

$$|A| = \sum_{\pi^{-1} \in \Pi} \text{sgn}(\pi^{-1}) \prod_{j=1}^{n} a_{\pi^{-1}(j)j} = |A^\top|$$

after using the expansion formula with $\pi$ replaced by $\pi^{-1}$. \qed
Verification of Rule 6

Exercise
Verify Rule 6 directly for $2 \times 2$ and then for $3 \times 3$ matrices.

Proof of Rule 6 Recall the notation $E_{r+\alpha q}$ for the matrix resulting from adding the multiple of $\alpha$ times row $q$ of $I$ to its $r$th row.

Recall too that $E_{r+\alpha q}A$ is the matrix that results from applying the same row operation to the matrix $A$.

Finally, recall the formula $|A| = \sum_{j=1}^{n} a_{rj}|C_{rj}|$ for the cofactor expansion of $|A|$ along the $r$th row.

The corresponding cofactor expansion of $E_{r+\alpha q}A$ is then

$$|E_{r+\alpha q}A| = \sum_{j=1}^{n} (a_{rj} + \alpha a_{qj})|C_{rj}| = |A| + \alpha|B|$$

where $B$ is derived from $A$ by replacing row $r$ with row $q$.

Unless $q = r$, the matrix $B$ will have its $q$th row repeated, implying that $|B| = 0$ because the determinant is alternating.

So $q \neq r$ implies $|E_{r+\alpha q}A| = |A|$ for all $\alpha$, which is Rule 6. 

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Verification of the Other Rules

Apart from Rules 2 and 6, note that we have already proved the product Rule 7, whereas the interchange Rule 4 just restates alternation.

Now that we have proved Rule 2, note that Rules 1 and 3 follow from multilinearity, applied in the special case when one row of the matrix is multiplied by a scalar.

Also, the proportionality Rule 5 follows from combining Rule 4 with multilinearity.

Finally, Rule 8, concerning the effect of multiplying all elements of a matrix by the same scalar, is easily checked because the expansion of $|\mathbf{A}|$ is the sum of many terms, each of which involves the product of exactly $n$ elements of $\mathbf{A}$. 
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Expansion by Alien Cofactors

Expanding along either row \( r \) or column \( s \) gives

\[
|A| = \sum_{j=1}^{n} a_{rj} |C_{rj}| = \sum_{i=1}^{n} a_{is} |C_{is}|
\]

when one uses matching cofactors.

Expanding by alien cofactors, however, from either the wrong row \( i \neq r \) or the wrong column \( j \neq s \), gives

\[
0 = \sum_{j=1}^{n} a_{rj} |C_{ij}| = \sum_{i=1}^{n} a_{is} |C_{ij}|
\]

This is because the answer will be the determinant of an alternative matrix in which:

- either row \( i \) has been duplicated and put in row \( r \);
- or column \( j \) has been duplicated and put in column \( s \).
The Adjugate Matrix

Definition
The adjugate (or “(classical) adjoint”) $\text{adj} \ A$ of an order $n$ square matrix $A$ has elements given by $(\text{adj} \ A)_{ij} = |C_{ji}|$.

It is therefore the transpose of the cofactor matrix $C^+$ whose elements are the respective cofactors of $A$. 
Main Property of the Adjugate Matrix

Theorem
\((\text{adj } A)A = A(\text{adj } A) = |A|I_n\) for every \(n \times n\) square matrix \(A\). 

Proof.
The \((i, j)\) elements of the two product matrices are

\[
[(\text{adj } A)A]_{ij} = \sum_{k=1}^{n} |C_{ki}| a_{kj} \quad \text{and} \quad [A(\text{adj } A)]_{ij} = \sum_{k=1}^{n} a_{ik} |C_{jk}|
\]

These are expansions by:
- alien cofactors in case \(i \neq j\), implying that they equal 0;
- matching cofactors in case \(i = j\), implying that they equal \(|A|\).

Hence \([(\text{adj } A)A]_{ij} = [A(\text{adj } A)]_{ij} = |A|(I_n)_{ij}\) for each pair \((i, j)\).
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Minor Determinants: Definition

Definition
Given any $m \times n$ matrix $A$, a minor determinant of order $k$ is the determinant $|A_{i_1i_2...i_kj_1j_2...j_k}|$ of a $k \times k$ submatrix $(a_{ij})$, with $1 \leq i_1 < i_2 < \ldots < i_k \leq m$ and $1 \leq j_1 < j_2 < \ldots < j_k \leq n$, that is formed by selecting all the elements that lie both:

- in one of the chosen rows $i_r$ ($r = 1, 2, \ldots, k$);
- in one of the chosen columns $j_s$ ($s = 1, 2, \ldots, k$).

Example

1. In case $A$ is an $n \times n$ matrix:
   - the whole determinant $|A|$ is the only minor of order $n$;
   - each of the $n^2$ cofactors $C_{ij}$ is a minor of order $n - 1$;

2. In case $A$ is an $m \times n$ matrix:
   - each element of the $mn$ elements of the matrix is a minor of order $1$;
   - there are $\binom{m}{k} \cdot \binom{n}{k}$ minors of order $k$. 
Principal and Leading Principal Minors

Exercise

Verify that the set of elements that make up the minor $|A_{i_1i_2...i_kj_1j_2...j_k}|$ of order $k$ is completely determined by its $k$ diagonal elements $a_{i_hj_h}$ ($h = 1, 2, \ldots, k$).

Definition

If $A$ is an $n \times n$ matrix, the minor $|A_{i_1i_2...i_kj_1j_2...j_k}|$ of order $k$ is:

- a **principal minor** if all its diagonal elements are diagonal elements of $A$;
- a **leading principal minor** if its diagonal elements are $a_{hh}$ ($h = 1, 2, \ldots, k$).

Exercise

*Explain why an $n \times n$ determinant has $2^n - 1$ principal minors.*
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Definition of Inverse Matrix

Exercise

Suppose that \( A \) is any "invertible" \( n \times n \) matrix for which there exist \( n \times n \) matrices \( B \) and \( C \) such that \( AB = CA = I \).

1. By writing \( CAB \) in two different ways, prove that \( B = C \).

2. Use this result to show that the equal matrices \( B = C \), if they exist, must be unique.

Definition

The \( n \times n \) matrix \( X \) is the unique inverse of the invertible \( n \times n \) matrix \( A \) provided that \( AX =XA = I_n \).

In this case we write \( X = A^{-1} \), so \( A^{-1} \) denotes the unique inverse.

Big question: does the inverse exist?
Existence Conditions

Theorem
An $n \times n$ matrix $A$ has an inverse if and only if $|A| \neq 0$, which holds if and only if at least one of the equations $AX = I_n$ and $XA = I_n$ has a solution.

Proof.
Provided $|A| \neq 0$, the identity $(\text{adj } A)A = A(\text{adj } A) = |A|I_n$ shows that the matrix $X := (1/|A|) \text{adj } A$ is well defined and satisfies $XA = AX = I_n$, so $X$ is the inverse $A^{-1}$.

Conversely, if either $XA = I_n$ or $AX = I_n$ has a solution, then the product rule for determinants implies that $1 = |I_n| = |AX| = |XA| = |A||X|$, and so $|A| \neq 0$. The rest follows from the paragraph above.
Singularity

So $A^{-1}$ exists if and only if $|A| \neq 0$.

**Definition**

1. In case $|A| = 0$, the matrix $A$ is said to be **singular**;
2. In case $|A| \neq 0$, the matrix $A$ is said to be **non-singular** or **invertible**.
Example and Application to Simultaneous Equations

Exercise

Verify that

\[
A = \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix} \implies A^{-1} = C := \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2}
\end{pmatrix}
\]

by using direct multiplication to show that \(AC = CA = I_2\).

Example

Suppose that a system of \(n\) simultaneous equations in \(n\) unknowns is expressed in matrix notation as \(Ax = b\).

Of course, \(A\) must be an \(n \times n\) matrix.

Suppose \(A\) has an inverse \(A^{-1}\).

Premultiplying both sides of the equation \(Ax = b\) by this inverse gives \(A^{-1}Ax = A^{-1}b\), which simplifies to \(Ix = A^{-1}b\).

Hence the unique solution of the equation is \(x = A^{-1}b\).
Cramer’s Rule: Statement

Notation
Given any $m \times n$ matrix $A$,
let $[A_{-j}, \mathbf{b}]$ denote the new $m \times n$ matrix
in which column $j$ has been replaced by the column vector $\mathbf{b}$.

Evidently $[A_{-j}, \mathbf{a}_j] = A$.

Theorem
Provided that the $n \times n$ matrix $A$ is invertible,
the simultaneous equation system $Ax = b$
has a unique solution $x = A^{-1}b$ whose $i$th component
is given by the ratio of determinants $x_i = |[A_{-i}, \mathbf{b}]|/|A|$.
This result is known as Cramer’s rule.
Cramer’s Rule: Proof

Proof.
Given the equation system $AX = b$, each cofactor $|C_{ji}|$ of the coefficient matrix $A$ is also the $(j, i)$ cofactor of the matrix $|[A_{-i}, b]|$.

Expanding the determinant $|[A_{-i}, b]|$ by cofactors along column $i$ therefore gives $\sum_{j=1}^{n} b_j |C_{ji}| = \sum_{j=1}^{n} (\text{adj } A)_{ij} b_j$, by definition of the adjugate matrix.

Hence the unique solution to the equation system has components

$$x_i = (A^{-1}b)_i = \frac{1}{|A|} \sum_{j=1}^{n} (\text{adj } A)_{ij} b_j = \frac{1}{|A|} |[A_{-i}, b]|$$

for $i = 1, 2, \ldots, n$. 

$\square$
Rule for Inverting Products

**Theorem**

Suppose that $A$ and $B$ are two invertible $n \times n$ matrices.

Then the inverse of the matrix product $AB$ exists, and is the reverse product $B^{-1}A^{-1}$ of the inverses.

**Proof.**

Using the associative law for matrix multiplication repeatedly gives:

\[
(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I)B = B^{-1}(IB) = B^{-1}B = I
\]

and

\[
(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = (AI)A^{-1} = AA^{-1} = I.
\]

These equations confirm that $X := B^{-1}A^{-1}$ is the unique matrix satisfying the double equality $(AB)X = X(AB) = I$. \qed
Rule for Inverting Chain Products

Exercise

Prove that, if $A$, $B$ and $C$ are three invertible $n \times n$ matrices, then $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

Then use mathematical induction to extend this result in order to find the inverse of the product $A_1A_2 \cdots A_k$ of any finite chain of invertible $n \times n$ matrices.
Matrices for Elementary Row Operations

Example
Consider the following two out of the three possible kinds of elementary row operation:

1. of multiplying the $r$th row by $\alpha \in \mathbb{R}$, represented by the matrix $S_r(\alpha)$;
2. of multiplying the $q$th row by $\alpha \in \mathbb{R}$, then adding the result to row $r$, represented by the matrix $E_{r+\alpha q}$.

Exercise
*Find the determinants and, when they exist, the inverses of the matrices $S_r(\alpha)$ and $E_{r+\alpha q}$.*
Outline

Determinants
- Determinants of Order 2
- Determinants of Order 3
- Characterizing the Determinant Function
- Rules for Determinants
- Expansion by Alien Cofactors and the Adjugate Matrix
- Minor Determinants

The Inverse Matrix
- Definition and Existence
- Orthogonal Matrices
- Partitioned Matrices
Inverting Orthogonal Matrices

An $n$-dimensional square matrix $Q$ is said to be **orthogonal** just in case its columns form an orthonormal set — i.e., they must be pairwise orthogonal unit vectors.

**Theorem**

A square matrix $Q$ is orthogonal if and only if it satisfies $Q^\top Q = I$.

**Proof.**

The elements of the matrix product $Q^\top Q$ satisfy

$$(Q^\top Q)_{ij} = \sum_{k=1}^{n} q_{ik}q_{kj} = \sum_{k=1}^{n} q_{ki}q_{kj} = q_i \cdot q_j$$

where $q_i$ (resp. $q_j$) denotes the $i$th (resp. $j$th) column vector of $Q$.

But the columns of $Q$ are orthonormal iff $q_i \cdot q_j = \delta_{ij}$ for all $i, j = 1, 2, \ldots, n$, and so iff $Q^\top Q = I$. 


Exercises on Orthogonal Matrices

Exercise

Show that if the matrix $Q$ is orthogonal, then so is $Q^\top$.

Use this result to show that a matrix is orthogonal if and only if its row vectors also form an orthonormal set.

Exercise

Show that any permutation matrix is orthogonal.
Rotations in $\mathbb{R}^2$

Example

In $\mathbb{R}^2$, consider the anti-clockwise rotation through an angle $\theta$ of the unit circle $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$.

It maps:

1. the first unit vector $(1, 0)$ of the canonical basis to the column vector $(\cos \theta, \sin \theta)^\top$;

2. the second unit vector $(0, 1)$ of the canonical basis to the column vector $(-\sin \theta, \cos \theta)^\top$.

So the rotation can be represented by the rotation matrix

$$R_\theta := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

with these vectors as its columns.
Rotations in $\mathbb{R}^2$ Are Orthogonal Matrices

Because $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = -\cos(\theta)$, the transpose of $R_\theta$ satisfies $R_\theta^\top = R_{-\theta}$, and so is the clockwise rotation through an angle $\theta$ of the unit circle $S^1$.

Since clockwise and anti-clockwise rotations are inverse operations, it is no surprise that $R_\theta^\top R_\theta = I$.

We verify this algebraically by using matrix multiplication

$$R_\theta^\top R_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

because $\cos^2 \theta + \sin^2 \theta = 1$, thus verifying orthogonality.

Similarly

$$R_\theta R_\theta^\top = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$
Outline

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The Inverse Matrix
- Definition and Existence
- Orthogonal Matrices
- Partitioned Matrices
Partitioned Matrices: Definition

A partitioned matrix is a rectangular array of different matrices.

Example
Consider the \((m + \ell) \times (n + k)\) matrix

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]

where the four submatrices \(A, B, C, D\)
are of dimension \(m \times n, m \times k, \ell \times n\) and \(\ell \times k\) respectively.

For any scalar \(\alpha \in \mathbb{R}\),
the scalar multiple of a partitioned matrix is

\[
\alpha \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
\alpha A & \alpha B \\
\alpha C & \alpha D
\end{pmatrix}
\]
Partitioned Matrices: Addition

Suppose the two partitioned matrices

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} E & F \\ G & H \end{pmatrix}
\]

have the property that the following four pairs of corresponding matrices have equal dimensions:
(i) $A$ and $E$; (ii) $B$ and $F$; (iii) $C$ and $G$; (iv) $D$ and $H$.

Then the sum of the two matrices is

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} A + E & B + F \\ C + G & D + H \end{pmatrix}
\]
Partitioned Matrices: Multiplication

Provided that the two partitioned matrices

\[
\begin{pmatrix}
A & B \\
C & D \\
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
E & F \\
G & H \\
\end{pmatrix}
\]

along with their sub-matrices are all compatible for multiplication, the product is defined as

\[
\begin{pmatrix}
A & B \\
C & D \\
\end{pmatrix}
\begin{pmatrix}
E & F \\
G & H \\
\end{pmatrix} =
\begin{pmatrix}
AE + BG & AF + BH \\
CE + DG & CF + DH \\
\end{pmatrix}
\]

This adheres to the usual rule for multiplying rows by columns.
The rule for transposing a partitioned matrix is

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}^\top = 
\begin{pmatrix}
A^\top & C^\top \\
B^\top & D^\top
\end{pmatrix}
\]

So the original matrix is symmetric iff \( A = A^\top, D = D^\top, B = C^\top, \) and \( C = B^\top. \)

It is diagonal iff \( A, D \) are both diagonal, while \( B = 0 \) and \( C = 0. \)

The identity matrix is diagonal with \( A = I, D = I, \) possibly identity matrices of different dimensions.
Partitioned Matrices: Inverses, I

For an \((m + n) \times (m + n)\) partitioned matrix to have an inverse, the equation

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix} = \begin{pmatrix}
AE + BG & AF + BH \\
CE + DG & CF + DH
\end{pmatrix} = \begin{pmatrix}
I_m & 0_{n \times m} \\
0_{m \times n} & I_n
\end{pmatrix}
\]

should have a solution for the matrices \(E, F, G, H\), given \(A, B, C, D\).

Assuming that \(A\) has an inverse, we can:

1. construct new first \(m\) equations by premultiplying the old ones by \(A^{-1}\);
2. construct new second \(n\) equations by:
   - premultiplying the new first \(m\) equations by \(C\);
   - then subtracting this product from the old second \(n\) equations.

The result is

\[
\begin{pmatrix}
I_m & A^{-1}B \\
0 & D - CA^{-1}B
\end{pmatrix}
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix} = \begin{pmatrix}
A^{-1} & 0_{n \times m} \\
-CA^{-1} & I_n
\end{pmatrix}
\]
Partitioned Matrices: Inverses, II

To take the next step, assume the matrix $X := D - CA^{-1}B$ also has an inverse $X^{-1} = (D - CA^{-1}B)^{-1}$.

Given \[
\begin{pmatrix}
I_m & A^{-1}B \\
0 & D - CA^{-1}B
\end{pmatrix}
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix}
= \begin{pmatrix}
A^{-1} & 0_{n \times m} \\
-C A^{-1} & I_n
\end{pmatrix},
\]
we can then premultiply the second $n$ equations by $X^{-1}$, then subtract $A^{-1}B$ times the new second $n$ equations from the old first $m$ equations to obtain

\[
\begin{pmatrix}
I_m & 0_{n \times m} \\
0_{m \times n} & I_n
\end{pmatrix}
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix}
= \begin{pmatrix}
E & F \\
G & H
\end{pmatrix}
= Z
\]

where $Z := \begin{pmatrix}
A^{-1} + A^{-1}BX^{-1}CA^{-1} & -A^{-1}BX^{-1} \\
-C A^{-1} & X^{-1}
\end{pmatrix}$

**Exercise**

*Use direct multiplication twice in order to verify that*

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
Z = Z
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
= \begin{pmatrix}
I_m & 0_{n \times m} \\
0_{m \times n} & I_n
\end{pmatrix}
\]
Partitioned Matrices: Extension

Exercise

Suppose that the two partitioned matrices

$$A = (A_{ij})^{k \times \ell} \quad \text{and} \quad B = (B_{ij})^{k \times \ell}$$

are both $k \times \ell$ arrays of respective $m_i \times n_j$ matrices $A_{ij}, B_{ij}$.

1. Under what conditions can the product $AB$ be defined as a $k \times \ell$ array of matrices?

2. Under what conditions can the product $BA$ be defined as a $k \times \ell$ array of matrices?

3. When either $AB$ or $BA$ can be so defined, give a formula for its product, using summation notation.

4. Express $A^\top$ as a partitioned matrix.

5. Under what conditions is the matrix $A$ symmetric?