How Restrictive Are Information Partitions?

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Abstract: Recently, several game theorists have questioned whether information partitions are appropriate. Bacharach (2005) has considered in particular more general information patterns which may not even correspond to a knowledge operator. Such patterns arise when agents lack perfect discrimination, as in Luce's (1956) example of intransitive indifference. Yet after extending the state space to include what the agent knows, a modified information partition can still be constructed in a straightforward manner. The required modification introduces an extra set of impossible states into the partition. This allows a natural represention of the agent's knowledge that some extended states are impossible.

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Information Partitions

1. Introduction

In game theory, von Neumann and Morgenstern themselves provided us the standard model of information that a rational agent is supposed to have. Given a state space S, an agent's possible information at any time is assumed to be represented by a partition Π of S into pairwise disjoint information sets $E \subset S$. The information sets are the equivalence classes of a (symmetric reflexive and transitive) equivalence or indistinguishability relation $\sim$ with the property that any two states $s, s' \in S$ are indistinguishable iff there exists $E \in \Pi$ such that $s, s' \in E$.

Recently, several game theorists have questioned the realism of the partition model, with its implicit requirements that agents know both what they do and what they do not know — see for example Bacharach (1985, 2005), Binmore and Brandenburger (1990), Geanakoplos (1990, 1994), Samet (1990), Shin (1993), Modica and Rustichini (1994, 1999), Morris (1996). Yet the standard may be weaker than many game theorists have realized, especially for agents who have an "epistemic state" or "fix" on each state $s \in S$ in the form of a subset $E \subset S$. In the first place, it should be understood that the standard model is not meant to apply to an arbitrary state space. Rather, the much more interesting claim is that there exists a state space, perhaps artificially constructed, on which the agent has an information partition.

In this connection, Geanakoplos (1990) points out that, whenever there is a possibility correspondence $s \mapsto\!\!\!\rightarrow P(s)$ mapping each state to a unique fix, there is already a natural partition of the state space S into classes of states that are indistinguishable because they yield the same fix. This is briefly discussed in Section 2. Then Section 3 describes knowledge operators and some of their familiar properties. One of these is called "logical omniscience". Later, Section 4 shows how, given any knowledge operator satisfying this property and the requirement that any tautology is known to be true, there is a unique associated possibility correspondence. In this sense, Geanakoplos has shown how it is easy to construct an information partition for any logically omniscient agent. The same is even true of belief operators, which differ from knowledge operators because what is believed may not always be true.

To complete the preliminaries, Section 5 describes information partitions that are modified in an important way by allowing one cell of the partition to consist of states that are known to be false. It is argued that such modified partitions arise naturally when describing how knowledge evolves over time.

Not all information or knowledge can readily be described by possibility correspondences, however, or by associated knowledge operators. Section 6 illustrates this by means of a familiar example of intransitive indifference due to Luce (1956). This kind of example can be accommodated, however, within Bacharach's (2005) more general framework of information patterns, as described in Section 7. Such patterns allow several different fixes to be associated with each state $s \in S$. Despite their significant greater generality, Section 8 shows how, on the extended space of state/fix pairs, it is still always possible to construct in a straightforward manner a modified information partition of the form described in Section 5, with some state/fix pairs known to be impossible.

Section 9 contains brief concluding remarks.

2. Possibility Correspondences and Information Partitions

A common generalization of the partition model is based on a general *possibility correspondence* $P : S \twoheadrightarrow S$ with the interpretation that, for each state $s \in S$, the set $P(s)$ describes those states that the agent cannot rule out as impossible when s is the true state. Bacharach (2005) calls $P(s)$ the agent's *fix* when the true state is s, and regards P as a mapping from S to 2^S which he calls the *fix function*. I shall use both the term "fix" and the evocative alternative *epistemic state*, because $P(s) \subset S$ is supposed to represent everything the agent knows about what states in S are possible. Note that, using the terminology of Bacharach (2005) once again, a possibility correspondence exists iff each state $s \in S$ is "epistemically sufficient for itself" (or e.s.i.).

Binmore and Brandenburger (1990) also discuss how it is common to assume that the possibility correspondence satisfies at least some of the following four axioms:

(P1) $P(s) \subset S$ for all $s \in S$

(P2) $s \in P(s)$ for all $s \in S$

(P3) for all $s, s' \in S$: $s' \in P(s) \Longrightarrow P(s') \subset P(s)$

(P4) for all $s, s' \in S$: $s' \in P(s) \Longrightarrow P(s) \subset P(s')$.

It is easy to check that, given any information partition Π on S, there is an associated possibility correspondence $P : S \twoheadrightarrow S$ satisfying (P1)–(P4) such that $s \in P(s) \in \Pi$ for all $s \in S$. Conversely, define

$$P^{-}(s) := \{ s' \in S \mid P(s') = P(s) \}$$

As Morris (1996, p. 11) argues concisely, (P3) and (P4) imply that $P(s) \subset P^{-}(s)$, whereas (P2) implies that $P^{-}(s) \subset P(s)$. Therefore $P(s) = P^{-}(s)$, or $s' \in P(s) \iff P(s') = P(s)$. It follows immediately that the binary relation $\sim$ defined on S by $s \sim s' \iff s' \in P(s)$ must be an equivalence relation. Then the corresponding partition $\Pi(\sim)$ of S into different $\sim$-equivalence classes is an information partition such that $s \in P(s) \in \Pi$ for all $s \in S$. So each possibility correspondence $P : S \twoheadrightarrow S$ satisfying (P1)–(P4) is associated with a unique information partition.

Actually, as Geanakoplos (1990) points out, the sets $P^{-}(s)$ always generate a partition, even if one or more of the axioms (P2)–(P4) are violated. This is the natural information partition that arises when the agent infers that the state can only be one of those that could produce the known fix on the state. Because $P^{-}(s) \subset P(s)$ for all $s \in S$ whenever axiom (P2) is satisfied, in this case the constructed partition refines each possibility set by using knowledge about what the fix would be in other states.

3. Logically Omniscient Knowledge Operators

An agent's *knowledge operator* K on the given state space S is a mapping $K : 2^S \to 2^S$ from the power set into itself. For each event $E \subset S$, the set $KE \subset S$ is to be interpreted as the set of all states in which the agent knows that E is true. As discussed by Binmore and Brandenburger (1990) in particular, it is common to assume that the knowledge operator satisfies at least some of the following five axioms. Apart from (K1) which is discussed below, the names are taken from Bacharach (2005):

(K0) $KS = S$ (Rule of Epistemization)

(K1) $K(\cap_{\alpha \in A} E_\alpha) = \cap_{\alpha \in A} KE_\alpha$ for every family $\{ E_\alpha \mid \alpha \in A \} \subset S$ (Logical Omniscience)

(K2) $KE \subset E$ for all $E \subset S$ (Axiom of Knowledge)

(K3) $KE \subset KKE$ for all $E \subset S$ (Axiom of Transparency)

(K4) $\neg KE \subset K(\neg KE)$ for all $E \subset S$ (Axiom of Wisdom)

These axioms have been explained and justified by many writers. Several have also been given other names. Axiom (K0) indicates that all tautologies are known to be true; in some sense, it is another very weak form of logical omniscience. Axiom (K1) earns its name because it requires the agent to be able to carry out certain logical deductions. It trivially implies that $K(E \cap E') = KE \cap KE'$, a finitistic version of (K1) which Morris (1996) calls "distributivity" and Bacharach (2005) calls the "conjunction axiom". Of course, when S is a finite set, the finitistic version of (K1) is equivalent to (K1) itself. Some other important implications of (K1) will be presented shortly.

Axiom (K2), sometimes called "non-delusion" (Geanakoplos, 1990), requires that what is known must be true. It is this axiom in particular which distinguishes knowledge from belief, since beliefs can be mistaken whereas knowledge cannot be. To the extent that the following discussion does not require (K2) to be satisfied, it is also relevant to beliefs as well as to knowledge or information.

The first three axioms are usually regarded as uncontroversial relative to the last two. Of these, axiom (K3) requires the agent to know what is known (Hintikka, 1962). Finally, axiom (K4) imposes the more demanding requirement that the agent should know his or her own epistemic limitations and recognize what is unknown. Following Fagin, Halpern and Vardi (1991), axioms (K3) and (K4) respectively are often called "positive introspection" and "negative introspection".

Axiom (K1) is logically equivalent to the combination of the following two interesting axioms:

(K1′) for all $E, E' \subset S$: $E \subset E' \Longrightarrow KE \subset KE'$

(K1″) $\cap_{\alpha \in A} KE_\alpha \subset K(\cap_{\alpha \in A} E_\alpha)$ for every family $\{ E_\alpha \mid \alpha \in A \} \subset S$

Axiom (K1′) is one form of logical omnscience requiring that any event E' must be known if it can be deduced logically from a known event E. That is, all logical implications of what is known must also be known. See Bacharach (2005) especially for further discussion. Axiom (K1″) is a second form of logical omnscience requiring that, if each event E_α ($\alpha \in A$) is known, then so is the logical conjunction $\cap_{\alpha \in A} E_\alpha$.

Of course axiom (K1) trivially implies (K1″). It also implies (K1′) because, if $E \subset E'$ and (K1) holds, then

$$KE = K(E \cap E') = KE \cap KE' \subset KE'$$

Conversely, for all $\beta \in A$, because $\cap_{\alpha \in A} E_\alpha \subset E_\beta$, axiom (K1′) evidently implies that $K(\cap_{\alpha \in A} E_\alpha) \subset KE_\beta$. So $K(\cap_{\alpha \in A} E_\alpha) \subset \cap_{\alpha \in A} KE_\alpha$. And if (K1″) holds as well, then so does (K1). Hence (K1) is logically equivalent to axioms (K1′) and (K1″) combined, which explains why I have chosen to call it "logical omnscience".

Together, as is well known, the five axioms (K0)–(K4) are equivalent to the partition model of knowledge or information. This result will be discussed in the next section.

4. Associated Possibility Correspondences and Knowledge Operators

Let $P : S \twoheadrightarrow S$ be any possibility correspondence. For event E to be known when the state is s, it is necessary and sufficient that $P(s) \subset E$. Hence it is natural to construct the associated knowledge operator K_P so that, for all $E \subset S$, one has

$$K_P E := \{\, s \in S \mid P(s) \subset E \,\}$$

Conversely, let $K : 2^S \to 2^S$ be any knowledge operator. Given any state $s \in S$, a state $s' \in S$ is possible if and only if $s' \in E$ whenever KE is true because $s \in KE$. Accordingly, construct the associated possibility correspondence P_K so that, for all $E \subset S$, one has[1]

$$P_K(s) := \{\, s' \in S \mid \forall E \subset S : s \in KE \Longrightarrow s' \in E \,\} = \bigcap \{\, E \subset S \mid s \in KE \,\}$$

The following result plays a crucial role in understanding when logically omniscient knowledge operators exist. So, even though Morris (1996, p. 5, Theorem 1) demonstrates the same result, a full proof is provided here.

Proposition 1. *Each $P : S \twoheadrightarrow S$ satisfying (P1) is equivalent to a unique $K : 2^S \to 2^S$ satisfying (K0) and (K1).*

Proof: Given any $P : S \twoheadrightarrow S$ satisfying (P1), it is obvious that the associated knowledge operator K_P satisfies (K0), and easy to prove that (K1) is satisfied as well. Also K_P can

[1] In general, this construction gives a smaller set than the alternative construction $P'_K(s) := \{\, s' \in S \mid s \notin K(S \setminus \{s'\}) \,\}$ used by Morris (1996, p. 6, eq. (2.2)). But it is easy to show that the two constructions are equivalent when (K1′) is satisfied.

be used to generate the possibility correspondence $P_{K_P} : S \twoheadrightarrow S$ with

$$P_{K_P}(s) = \{ s' \in S \mid \forall E \subset S : P(s) \subset E \Longrightarrow s' \in E \}$$

Now, considering the particular set $E = P(s)$ makes it evident that, if $s' \in P_{K_P}(s)$, then $s' \in E = P(s)$. On the other hand, for any $s' \in P(s)$ it must be true that $P(s) \subset E \Longrightarrow s' \in E$, so $s' \in P_{K_P}(s)$. Hence $P_{K_P}(s) = P(s)$.

Conversely, given any K satisfying (K0) and (K1), it is obvious that the associated possibility correspondence P_K satisfies (P1). Then P_K can be used to generate the knowledge operator K_{P_K} with

$$K_{P_K}E = \{ s \in S \mid \cap\{ E' \subset S \mid s \in KE' \} \subset E \}$$

Now, if $s \in K_{P_K}E$ then $P_K(s) \subset E$, implying that $s \in K_{P_K}E$. On the other hand, $s \in K_{P_K}E$ implies that $P_K(s) = \cap\{ E' \subset S \mid s \in KE' \} \subset E$. Hence, for each $s' \in S \setminus E$, there exists $E' \subset S$ with $s \in KE'$ such that $s' \notin E'$ and so $E' \subset S \setminus \{s'\}$. But (K1) implies (K1$'$), so for all $s' \in S \setminus E$ one has $KE' \subset K(S \setminus \{s'\})$ and so $s \in K(S \setminus \{s'\})$. Therefore, $s \in \cap_{s' \in S \setminus E} K(S \setminus \{s'\})$ and so, by (K1), $s \in K\left(\cap_{s' \in S \setminus E}(S \setminus \{s'\})\right)$. Yet de Morgan's laws imply that

$$\bigcap_{s' \in S \setminus E} (S \setminus \{s'\}) = S \setminus \bigcup_{s' \in S \setminus E} \{s'\} = S \setminus (S \setminus E) = E$$

and so $s \in KE$. ∎

The equivalence between knowledge operators satisfying (K0) and (K1) and possibility correspondences satisfying (P1) extends to other properties as well. Indeed, as Morris (1996, p. 11, Lemma 2) correctly claims:

Proposition 2. *For $n = 2, 3, 4$, any knowledge operator satisfying properties (K0), (K1) and (Kn) is associated with a unique possibility correspondence satisfying properties (P1) and (Pn), for the same n.*

Proof: Each of the three proofs makes elementary use of the appropriate definitions, and is left to the reader. ∎

In particular, Proposition 2 implies that any information partition is associated with a unique knowledge operator satisfying all five axioms (K0)–(K4).

5. Modified Partitions with States Known to be Impossible

Up to now I have followed the usual epistemic models in assuming that no state $s \in S$ can be excluded as totally impossible. Yet as the example of Section 6 below shows, some interesting epistemic models may include (extended) states which the agent's variable information structure will rule out as impossible in some eventualities. Accordingly, suppose that S is first partitioned into one set F of states that are known to be false, and the complementary set T of states that could still be true.

In this case, a reasonable requirement of the possibility correspondence $P : S \twoheadrightarrow S$ is that it should satisfy $P(s) \subset T$ whenever $s \in T$, but $P(s) = S$ whenever $s \in F$. Thus, for states not already ruled out as false, the possibility set excludes all false states. But, just as any statement, true or false, can be deduced logically from a false premiss, so the hypothesis that a false state has occurred puts no restriction at all on the set of possible states; not even states known to be false can be ruled out.

Note that such a possibility correspondence necessarily violates (P4) because, if $s \in F$ and $s' \in P(s)$, then whenever $s' \in T$ one has

$$P(s) = S \not\subset P(s') \subset T = S \setminus F \subset S \setminus \{s\}$$

However, it is still possible for P to satisfy the following weakened form of (P4):

(P4′) $s' \in P(s) \Longrightarrow P(s) \subset P(s')$ for all $s, s' \in T$

Indeed, when states $s \in F$ are known to be impossible, it is still possible for $P : S \twoheadrightarrow S$ to satisfy all four conditions (P1)–(P3) and (P4′). If it does so, there must be a restricted possibility correspondence $P_{|T} : T \twoheadrightarrow T$ satisfying (P1)–(P4) on T. This restricted possibility correspondence must be associated with an information partition $\Pi_{|T}$ on the set T; then there is also a unique partition of S consisting of $\Pi_{|T}$ together with the set F of impossible states. Thus $\Pi_{|T} \cup \{F\}$ is a modified information partition in which one particular cell F is known to be impossible.

Given such a possibility correspondence $P : S \twoheadrightarrow S$ with $P(s) \subset T$ whenever $s \in T$, but $P(s) = S$ whenever $s \in F$, the unique associated knowledge operator is given by $KE := \{\, s \in S \mid P(s) \subset E \,\}$ for all $E \subset S$, as in Section 4. This must satisfy $KT = T$ and $KS = S$, while $KE = K(E \cap T) \subset T$ for all $E \neq S$, and $KE = \emptyset$ for all definitely false events $E \subset F$. All of these are intuitively appealing properties, given the partition $T \cup F$ of S into states that could possibly be true and those that are definitely false.

Because $P : S \twoheadrightarrow S$ must violate (P4), the associated knowledge operator K must violate (K4), but could still satisfy the following weakened form of (K4):

(K4′) $\neg KE \subset K(\neg KE)$ for all $E \subset T$

In particular, K could still satisfy (K0)–(K4) on all subsets of T, even though only (K4′) holds on S. Indeed, this property must hold whenever K is associated with a modified information partition $\Pi_{|T} \cup \{F\}$, where F is the set of states known to be false.

The modified information partitions considered here appear to be novel in recognizing that some states $s \in S$ may be known to be impossible. Yet this seems entirely natural. After all, as time t progresses, an agent's information should be modelled by a family of increasingly refined information partitions Π_t. In discrete time these will form an event tree, whereas in continuous time there will be a filtration. At each time t the agent will know that some event $E \in \Pi_t$ has occurred, and that states $s \in S \setminus E$ are therefore impossible. Given that E is known to have occurred at time t, information at later times $t' > t$ will be described by a partition $\Pi_{t'|E}$ of E rather than of S. Alternatively, however, they can be described equally well by the extended partition $\Pi_{t'|E} \cup \{S \setminus E\}$ of the original state space S, even though this includes the set $S \setminus E$ of states that are known to be impossible.

6. A Classical Example: Luce's Semi-Transitive Coffee Drinker

The recent literature cited in Section 1 has presented many examples of non-partitional information structures. The following classic example of intransitive indifference due to Luce (1956, p. 179) is, however, very familiar in another context, as well as being close to what seems to be a real phenomenon:

> *. . . consider the following experiment. Find a subject who prefers a cup of coffee with one cube of sugar to one with five cubes (this should not be too difficult). Now prepare 401 cups of coffee with* $\left(1 + \frac{i}{100}\right) x$ *grams of sugar,* $i = 0, 1, \cdots, 400$*, where* x [grams] *is the weight of one cube of sugar. It is evident that he will be indifferent between cup* i *and cup* $i + 1$*, for any* i*, but by choice he is not indifferent between* $i = 0$ *and* $i = 400$*.*

The obvious reason for the indifference relation to be intransitive in this example is that the coffee drinker cannot distinguish between cup i and cup $i+1$, for any i, though he can distinguish between $i = 0$ and $i = 400$. So the coffee drinker's indistinguishability relation is intransitive, though it is naturally reflexive and symmetric. There is no corresponding information partition. Incidentally, this is so even though there need be no "processing errors" of the kind noted by Geanakoplos (1990). In this example, therefore, it is hard to see how one could successfully construct an information partition as in Section 3. For this reason, the standard epistemic model described in Sections 3 and 4 does not seem applicable to Luce's coffee drinker, even when non-partitional information is allowed.

In fact a proper model of the coffee drinker's information should presumably include a description of how sweet the coffee is perceived to be, as well as of the actual quantity of sugar. Such a model therefore consists of pairs (m, q), where $m \in M$ denotes a discrete ordinal measure of perceived sweetness or an appropriate *mental state*, and $q \in Q$ denotes the true quantity of sugar. Evidently, there will be some relationship between the mental state m and the quantity q. An obvious way of expressing this relationship mathematically is through a binary relation whose graph is a subset $R \subset M \times Q$. Then $(m, q) \in R$ will mean that mental state m is possible when the true quantity is q, so the coffee drinker should regard quantity q as possible when the perception of sweetness is m. The graph R should have the property that each m is associated with at least one q, and vice versa. Hence, there must exist two non-empty valued inverse correspondences $m \mapsto\!\!\!\rightarrow Q(m)$ and $q \mapsto\!\!\!\rightarrow M(q)$ whose common graph is the set R.

In such a model, Luce's coffee drinker can be regarded as having at least one mental state $m \in M$ associated with each cup $i = 0, 1, \cdots, 400$ containing $q_i := (1+0.01i)x$ grams of sugar. Moreover, for each pair of quantities q_i, q_{i+1} there must exist a common mental state $m \in M(q_i) \cap M(q_{i+1})$ so that the two quantities cannot be distinguished. Now, though there is no information partition on the set Q, provided that the coffee drinker is aware of mental state m, there must be a trivial partition on R corresponding to the indistinguishability equivalence relation $\sim$ defined by $(m, q) \sim (m', q') \iff m = m'$.

One defect of this newly constructed partition of $M \times Q$ is that it cannot really deal with changes in the coffee drinker's perceptions. For example, the sense of taste may become dulled by a heavy cold. Such changes alter the relation R between perceptions of sweetness and actual sugar quantities; there will be a new "dulled" relation $\bar{R}$. Pairs that were previously impossible because $(m, q) \notin R$ will now become possible because $(m, q) \in \bar{R}$; the reverse is also possible in some cases. In order to accommodate such changes in perceptions, it is natural to specify a partition on the whole Cartesian product space $M \times Q$. But some extended states $(m, q) \in M \times Q$ are known not to be possible, given the variable relationship R between mental states and quantities. Accordingly one should construct, as in Section 5, a modified information partition which represents the agent's knowledge that some extended states are impossible. This modified partition will include a variable set $F \subset M \times Q$ of extended states that are definitely known to be false, together with a variable partition of $R = (M \times Q) \setminus F$, the set of extended states that are known to be possible.

A second defect of the construction concerns the formulation in terms of abstract mental states $m \in M$, instead of relatively concrete sugar quantities $q \in Q$ and possibility sets or fixes $E \subset Q$. This is a significant and probably unhelpful departure from the epistemic model with a possibility correspondence $q \mapsto\!\!\!\rightarrow P(q)$, which considers only the natural state space Q and its subsets. As a possible remedy it seems obvious that one should try constructing the possibility correspondence $P : Q \twoheadrightarrow Q$ as the composition of the two correspondences $q \mapsto\!\!\!\rightarrow M(q)$ and $m \mapsto\!\!\!\rightarrow Q(m)$ whose common graph is the set R. Such a composition $P = Q \circ M$ is defined by $P(q) := \cup_{m \in M(q)} Q(m)$. However, any such possibility correspondence merely takes us back to the epistemic models of Sections 3 and 4, for which it is easy to construct an information partition on the set Q. This is exactly the framework which, I have argued, fails to accommodate Luce's example. In fact, a more appropriate remedy seems to be quite different, and is the topic of the next section.

7. Information Patterns

In work dating back to 1984 which still remains unpublished, Bacharach (2005) mentions the possibility of more general *information patterns.* These occur when at least one state $s \in S$ may not be e.s.i. (epistemically sufficient for itself) because more than one fix $E \subset S$ may correspond to s.[2] So an *information pattern* will be defined as a correspondence $\tilde{P} : S \twoheadrightarrow 2^S$ with graph $\tilde{T}$. Thus, each different event $E \in \tilde{P}(s)$ is a possible fix when

[2] However, Bacharach's (2005) definition of an *epistemic model* involves a knowledge operator satisfying (K0)–(K2), so that a "fix function" or possibility correspondence can be constructed as in Section 4 above. Hence each state $s \in S$ must be e.s.i. after all.

the state is s; it is no longer required that there be a single fix $P(s)$ in each state s. This implies in particular that none of the axioms (P1)–(P4) need apply to information patterns; and associated knowledge operators to which axioms (K0)–(K4) might apply may not even exist.

Of course, for each $s \in S$ there is a *union possibility set* of all states that are possible in some fix that could occur in state s. This is defined by

$$P^{\cup}(s) := \bigcup \{ E \mid E \in \tilde{P}(s) \}$$

This generates a *union possibility correspondence* $P^{\cup} : S \twoheadrightarrow S$ that satisfies (P1). There is an associated *union knowledge operator* defined by

$$K^{\cup} E := \{ s \in S \mid E' \in \tilde{P}(s) \Longrightarrow E' \subset E \}$$

for all $E \subset S$. This definition reflects the idea that E is known to be true iff it is true in any fix the agent could have in some state $s \in K^{\cup} E$. Note that, because $P^{\cup}$ satisfies (P1), the associated $K^{\cup}$ satisfies (K0) and (K1). However, in general the information pattern $\tilde{P}$ cannot be recovered from $K^{\cup}$; only $P^{\cup}$ can. So information patterns are more general than possibility correspondences or fix functions. They are also more general than knowledge operators.

Information patterns can accommodate Luce's example much more successfully than possibility correspondences can. For example, suppose that the set of possible quantities is $Q = [0, 6x] \subset \Re$, and that the set of possible mental states is $M = \{ m_k \mid k = 1, 2, \ldots, 11 \}$. Suppose too that $R \subset M \times Q$ satisfies

$$(m, q) \in R \iff q \in Q(m) \iff m \in M(q)$$

where

$$Q(m_k) := I_k := [\tfrac{1}{2}(k-1)x, \tfrac{1}{2}(k+1)x \,) \quad (k = 1, 2, \ldots, 11)$$

Now, instead of composing the two correspondences $q \mapsto\!\!\!\rightarrow M(q)$ and $m \mapsto\!\!\!\rightarrow Q(m)$ into the possibility correspondence $P = Q \circ M$, one can instead define the information pattern

$\tilde{P} : Q \twoheadrightarrow 2^Q$ by

$$\tilde{P}(q) := \{\, Q(m) \mid m \in M(q) \,\} = \begin{cases} \{I_1\} & \text{if } 0 \le q < \frac{1}{2}x \\ \{\, I_{k-1}, I_k \,\} & \text{if } \frac{1}{2}(k-1)x \le q < \frac{1}{2}kx \\ & \qquad\qquad (k = 2, 3, \ldots, 11) \\ \{I_{11}\} & \text{if } 11x/2 \le q < 6x \end{cases}$$

In other words, $\tilde{P}(q) \subset 2^Q$ is the range of all possible fixes as the mental state m varies over the set $M(q)$, whereas $P^{\cup}(q) \in 2^Q$ is the union of all those fixes. This difference allows $\tilde{P}(q)$ to describe the coffee drinker's inability to specify precisely what fix or mental state $E \subset Q$ is appropriate for each possible quantity level $q \in Q$.

Returning to general information partitions, note how the Geanakoplos construction that was briefly described in Section 3 does not work very well. Of course, one can still partition the state space S into subsets on which the entire image set $\tilde{P}(s) \subset 2^S$ of possible fixes is constant. But this requires the agent to be aware of what this entire image set is, rather than knowing just one particular fix $E \in 2^S$.

An alternative construction would be to consider the correspondence $\hat{P} : 2^S \twoheadrightarrow S$ defined by $\hat{P}(E) := \{\, s \in S \mid E \in \tilde{P}(s) \,\}$ for each $E \subset S$. Thus, $\hat{P}(E)$ is the set of states in which E is a possible fix. But this construction does not work very well either. For one thing, it does not yeld a partition or even a possibility correspondence. Also, when $E_1 \neq E_2$ but $\hat{P}(E_1) = \hat{P}(E_2)$, this construction fails to recognize the agent's ability to distinguish between the two fixes E_1 and E_2.

Nevertheless, incorporating the sets $\hat{P}(E)$ in a more extensive construction does yield a suitable information partition. This is the final topic of the paper.

8. Constructing an Information Partition

Consider now the extended space $S^* := S \times 2^S$ whose members are state/fix pairs (s, E) with $s \in S$ and $E \subset S$. Given any information pattern $\tilde{P} : S \twoheadrightarrow 2^S$, a natural modified information partition Π^* will now be defined on S^*.

The construction is almost trivial. It involves first setting

$$T^* := \text{ Graph } \tilde{P} := \{ (s, E) \in S^* \mid E \in \tilde{P}(s) \}; \quad F^* := S^* \setminus T^*$$

as the set of all states in S^* that are logically possible, and the complement which consists of all states that are definitely impossible or false. Then the new possibility correspondence $P^* : S^* \twoheadrightarrow S^*$ is defined for all $(s, E) \in S^*$ by

$$P^*(s, E) := \begin{cases} \{ (s', E') \in S^* \mid E = E' \in \tilde{P}(s') \} = \hat{P}(E) \times \{E\} & \text{if } (s, E) \in T^* \\ S^* & \text{if } (s, E) \in F^* \end{cases}$$

Evidently $P^*(s, E) \subset T^*$ iff $(s, E) \in T^*$.

The associated knowledge operator $K^* : 2^{S^*} \to 2^{S^*}$ is defined for all extended events $E^* \subset S^*$ by

$$\begin{aligned} K^* E^* :=& \{ (s, E) \in S^* \mid P^*(s, E) \subset E^* \} \\ =& \begin{cases} \{ (s, E) \in T^* \mid \hat{P}(E) \times \{E\} \subset E^* \} & \text{if } E^* \neq S^* \\ S^* & \text{if } E^* = S^* \end{cases} \end{aligned}$$

This definition implies that $K^* E^* = K^*(E^* \cap T^*) \subset T^*$ for all $E^* \neq S^*$, that $K^* T^* = T^*$, $K^* S^* = S^*$, and $K^* E^* = \emptyset$ whenever $E^* \subset F^*$. Also, whenever $E \subset S$ satisfies $\hat{P}(E) \neq \emptyset$ one has $K^*(E \times \{E\}) = \hat{P}(E) \times \{E\}$, implying that the agent knows when the fix is E, and knows also that the state s must satisfy $E \in \tilde{P}(s)$. Furthermore, $P^*(s, E) = K^*(E \times \{E\})$ whenever $(s, E) \in T^*$.

Finally, define the extended indistinguishability relation $\sim^*$ on T^* by

$$(s, E) \sim^* (s', E') \iff (s', E') \in P^*(s, E) \iff E = E'$$

where the last equivalence follows from the fact that $E \in \tilde{P}(s)$ and $E' \in \tilde{P}(s')$ for all (s, E), $(s', E') \in T^*$. The relation $\sim^*$ is an exact representation of the agent's ignorance; if $(s, E) \not\sim^* (s', E')$ when both (s, E) and $(s', E') \in T^*$, this can only be because $E \neq E'$, in which case the agent should be able to distinguish between the two fixes or epistemic states. Moreover, the relation $\sim^*$ is clearly an equivalence relation that partitions T^* into indistinguishability classes or information sets. Thus, by modelling the agent's knowledge explicitly in the simplest possible way, one is able to construct a rather obvious information partition.

The above constructions require only the existence of the correspondence $\tilde{P} : S \twoheadrightarrow 2^S$. Such unrestricted information patterns actually apply better to models of belief which, unlike models of knowledge, do not require each possible fix to include the true state. Models of knowledge, on the other hand, should satisfy the additional axiom:

($\tilde{\text{P}}$2) $s \in E$ for all $E \in \tilde{P}(s)$

In this case it is natural to redefine the extended space S^* so that it becomes

$$S^* := \{ (s, E) \in S \times 2^S \mid s \in E \}$$

Thus, the new S^* consists entirely of state/fix pairs with the property that the state is consistent with the fix. Provided that the original possibility correspondence $P : S \twoheadrightarrow S$ satisfies ($\tilde{\text{P}}$2), its graph T^* is still a subset of the new S^*, so all the previous constructions still apply.

9. Concluding Remarks

The above results should not be surprising. When the true state is $s \in S$, each possible fix $E \in \tilde{P}(s)$ is allowed by the information pattern. Any such fix is supposed to represent exactly what the agent knows. An agent who is always fully aware of this fix will always know what he or she knows and does not know. So there will be an information partition in the relevant extended space of state/fix pairs.

Recently Modica and Rustichini (1994, 1999) have discussed an "awareness" axiom which is close to (K4) or (P4). In the second paper, they also introduced "unawareness" in a model of epistemic logic. Translating this condition into set-theoretic language, they allow knowledge operators which violate the logical omniscience axiom (K1), even in its weaker finitistic version. In particular, even when both KE and $E \subset E'$, the agent may still remain totally "unaware" of event E'. Such unawareness requires that neither KE'

nor $\neg KE'$ be true. It remains to be seen what implications their model has for information patterns. However, the results of this paper suggest that, if there is to be an interesting departure from the partition model of information, even on an enriched state space, then the agent had better have no information pattern at all. There must be more vagueness about what is known than information patterns allow, and/or more significant departures from logical omniscience.

References

Bacharach, M. (1985) "Some Examples of a Claim of Aumann in an Axiomatic Model of Knowledge", *Journal of Economic Theory*, vol. 37, pp. 167–190.

Bacharach, M. (2005) "When Do We Have Information Partitions?", to appear in M.O.L. Bacharach, M.A.H. Dempster and J.L. Enos (eds.) *Mathematical Models in Economics: Oxford University Mathematical Economic Seminar 25th Anniversary Volume* (Oxford: Oxford University Press — in press).

Binmore, K. (1990) *Essays on the Foundations of Game Theory* (Cambridge, Mass. and Oxford: Basil Blackwell).

Binmore, K. and A. Brandenburger (1990) "Common Knowledge and Game Theory", in Binmore (1990) ch. 4, pp. 105–150.

Fagin, R., Halpern, J.Y., Moses, Y. and M.Y. Vardi (1995) *Reasoning about Knowledge* (Cambridge, MA: MIT Press).

Fagin, R., Halpern, J.Y. and M.Y. Vardi (1991) "A Model Theoretical Analysis of Knowledge", *Journal of the Association of Computing Machinery*, vol. 38, pp. 382–428.

Geanakoplos, J. (1989, 1990) "Game Theory without Partitions, and Applications to Speculation and Consensus", Yale University, Cowles Foundation Discussion Paper No. 914; Santa Fe Institute, Working Paper 90-018; to appear in *Journal of Economic Theory*.

GEANAKOPLOS, J. (1994) “Common Knowledge”, in R.J. Aumann and S. Hart (eds.) *Handbook of Game Theory with Economic Applications, Vol. II* (Amsterdam: North-Holland) ch. 40, pp. 1437–1496.

HINTIKKA, J. (1962) *Knowledge and Belief* (Ithaca, NY: Cornell University Press).

LISMONT, L. AND P. MONGIN (1994) “On the Logic of Common Belief and Common Knowledge”, *Theory and Decision*, vol. 37, pp. 75–106.

LUCE, R.D. (1956) “Semiorders and a Theory of Utility Discrimination”, *Econometrica*, vol. 24, pp. 178–191.

MODICA, S. AND A. RUSTICHINI (1994) “Awareness and Partitional Information Structures”, *Theory and Decision*, vol. 37, pp. 107–124.

MODICA, S. AND A. RUSTICHINI (1999) “Unawareness and Partitional Information Structures”, *Games and Economic Behavior*, vol. 27, pp. 265–298.

MORRIS, S. (1996) “The Logic of Belief and Belief Change: A Decision Theoretic Approach”, *Journal of Economic Theory*, vol. 69, pp. 1–23.

SAMET, D. (1990) “Ignoring Ignorance and Agreeing to Disagree”, *Journal of Economic Theory*, vol. 52, pp. 190–207.

SHIN, H.S. (1993) “Logical Structure of Common Knowledge”, *Journal of Economic Theory*, vol. 60, pp. 1–13.