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Voting power and voting blocs*

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Abstract. We investigate the method of power indices to study voting power of members of a legislature that has voting blocs. Our analysis is theoretical, intended to contribute to a theory of positive political science in which social actors are motivated by the pursuit of power as measured by objective power indices. Our starting points are the papers by Riker (Behavioural Science, 1959, "A test of the adequacy of the power index") and Coleman (American Sociological Review, 1973, "Loss of Power"). We argue against the Shapley–Shubik index and show that anyway the Shapley–Shubik index per head is inappropriate for voting blocs. We apply the Penrose index (the absolute Banzhaf index) to a hypothetical voting body with 100 members. We show how the power indices of individual bloc members can be used to study the implications of the formation of blocs and how voting power varies as bloc size varies. We briefly consider incentives to migrate between blocs. This technique of analysis has many real world applications to legislatures and international bodies. It can be generalised in many ways: our analysis is *a priori* (assuming formal voting and ignoring actual voting behaviour) but can be made empirical with voting data reflecting behaviour; it examines the consequences of two blocs but can easily be extended to more.

It has long been argued that voting power indices can be used as the basis for a precise political theory capable of leading to rigorous analysis. This proposal was first made by Simon (1957) who, commenting on the pioneering paper by Shapley and Shubik (1954), which defined the first *a priori* measure of voting power, observed that his intuition led him to believe that their index "agrees pretty well" with reality, but that its adequacy as a model needed to be tested. The first to attempt this was Riker (1959) who put the problem as follows: "The economists once invented the Economic Man whose aim in life was to maximise profit or a suitable generalisation of it. Game theory suggests the possibility of a theory of coalitions. Presumably, such a theory relates to the Political Man. Does the Political Man seek to maximise "power"? To determine this one must develop an index of power and then discover whether in actual cases real men attempt to maximise what it measures."

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Many later scholars have calculated power indices for particular voting bodies in empirical applications but Riker's question has not been answered. Partly this is because few have addressed it since almost all of them have had the more limited objective of attempting to quantify the *relative* voting powers of members within a *given* decision-making system.[1] They have tried to find *power distributions* in voting games with fixed characteristics and very few have tried to answer Riker's question. One result of this has been that voting power analysis is often dismissed – by those who might be its users, such as government ministers and public officials, as well as some academics – on the grounds that it is capable only of giving results for static situations, and therefore effectively useless for analysing institutional changes (which are inherently more important and interesting), or behaviour, which require a different, essentially dynamic, mode of analysis.[2]

An important exception is the work of Coleman (1970, 1971, 1973) whose approach is fundamentally dynamic, because voting is conceived as being about decisions leading to an action taken by a collectivity. In Coleman's framework a voting body may decide to take an action, or it may not, and the main questions, given the rules by which it makes decisions, are: first, how likely it is that the collectivity might take action, and second, how much control can social actors exert over it. In his well-known 1971 paper, he proposed new power measures within this framework,[3] and subsequently used them in his much less well known, but important 1973 article. His power indices were different from those used by Riker but they were used to address the same problem.

Our paper follows Coleman (1973) in trying to develop an approach that is capable of answering Riker's question by exploring the possibility of using voting power indices for dynamic analysis when the nature of the voting body changes as a result of the formation or mutation of voting blocs. Our approach is different from that of Riker in two major respects: we avoid the use of the Shapley–Shubik index (SSI), for which there are compelling grounds,[4] and we make no attempt at empirical testing here. We adopt a similar methodological approach to Coleman (1973), with the difference that our measure of power is the Penrose index (Penrose (1946)), which, in this particular context, differs only in name from that used by Coleman ('the power to prevent action') but we think there are advantages in this terminology.[5]

We begin with a short discussion of Coleman's approach and his critique of the use of game theoretic power indices. This is followed by a discussion of voting blocs, a description of the voting scenario, why the Shapley–Shubik index per head is inapplicable for analysing power in relation to voting blocs, the Penrose index, and then the results of applying this to a hypothetical legislature. Our conclusion is that this framework is applicable and capable of generating useful results in real-world contexts.

Coleman's Contribution to Voting Power Theory

Coleman (1971) argued against the use of cooperative game theory in general, and the game-theoretic SSI in particular, for the analysis of voting power. In fact in that paper he gave a fundamental theoretical critique of the SSI based, first, on its arbitrary use of orderings of members to give different weight to coalitions of different sizes and, second, its characterisation of voting as a group of rivals bargaining among themselves over a fixed payoff in a game. Coleman argued that voting was not intrinsically linked to bargaining and that in many actual voting contexts the consequences of a collective action are fixed. In Coleman's dynamic perspective, collective decisions concern *action* rather than how to divide up a given fixed payoff among the players. The consequences of any action are fixed and not subject to bargaining. This allows the relaxation of some of the analytical constraints that come from game theory, such as the requirement that the power indices of the different players should add up to a constant (often referred to as the 'efficiency axiom') and the restriction that the quota has to be at least half the total number of votes (the restriction of the analysis to 'proper games'). In this perspective a voting power index measures absolute not relative power and is therefore useful for considering how power changes as a result of members participating in coalitions, for which game theory is ill suited.

Coleman's approach shifts the main focus of the analysis from the relative powers of the members in relation to each other to the relationship between the powers of individual members and that of the collective body. This relationship is where much of the real concern lies in discussing institutions. Mathematically, within this framework, a power index is the probability of an action in some sense – usually when all voting outcomes are considered equiprobable. This is a useful property which is destroyed by normalisation, making the power indices of all the voters add up to 1. In this sense there is a fundamental difference between what we refer to here as the Banzhaf index (that is, the normalised Banzhaf index) and the Penrose index.

We do not wish to argue here against the use of cooperative game theory in general to model voting. Only that the results it leads to are of limited empirical interest. However, we do argue against the SSI on grounds both of the lack of realism of its assumptions and also its failure to produce results that are acceptable from an empirical perspective.[6]

Now we describe the analytical framework of the rest of the paper in terms of voting blocs before describing the power indices approach.

Voting Power and Voting Blocs Within a Global Voting Body

When a social actor, whether an individual or a group, relinquishes independent political power by joining a group (or a larger group than the one he or it is in) and agrees to be bound by its decisions, his (or its) power will either increase or decrease. For example, a country which, as a member of a global organisation, gives up its independence in certain matters within the organisation, in order to join a powerful bloc, may gain or lose power. The bloc will be more powerful than the country could be by itself because of its greater size, but the country has only limited power over decisions taken by the bloc's members about how it should vote in the global organisation. The country's power, as a member of the bloc, is a compound of these two factors. Another example is a parliament containing party groups whose representatives agree to a strict whipping discipline combined with majority voting within the group. Belonging to a large party group both enhances and constrains a member's power – the larger it is the more powerful but the less control any member has over its decisions.

In the following sections we present a theoretical investigation that uses power indices to find the trade-offs involved when blocs are formed in a hypothetical legislature. We assume a simple model of a legislature and use the Penrose power index to measure formal voting power when there are blocs of members who vote together in accordance with a prior agreement such as a party whip.

Formal Definitions and Notation

We assume a legislature with a large number of members; where notation is needed for this the number of members is n. The global legislature, denoted by G, H, etc, is assumed to consist of one or more blocs, denoted (for example) B, C, W, $W1$, $W2$, etc, and a number of individuals, i, j, etc. Actually it is not necessary to distinguish between individuals and blocs since any individual can be treated formally as a bloc consisting of a single member. The decision rule is represented by a number q, which denotes the quota in terms of the number of votes needed to take a decision to act. It will be convenient to denote the global body, using set notation, in terms of its membership and decision rule, as for example, $G = \{q; B, C, D, \ldots, \{i\}, \{j\}, \ldots\}$. Generally we assume q is a simple majority: if n is even, $q = n/2 + 1$ and if n is odd, $q = (n + 1)/2$. The scenario is shown schematically in Figure 1.

The Fallacy of SSI per Head as a Measure of Individual Indirect Voting Power

Before describing the Penrose power indices and the calculations, it is perhaps useful to digress briefly to consider why it is not appropriate to use the Shapley–Shubik index in the manner Riker did in his 1959 study.

Riker attempted to test the adequacy of the SSI as a measure of absolute voting power by looking at migrations between party blocs in an actual

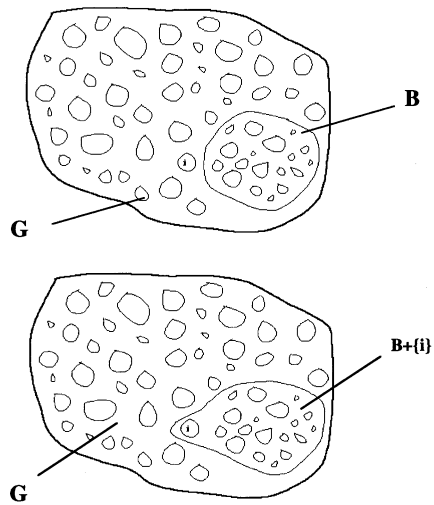

Figure 1. Schematic representation of a voting bloc.

legislature.[7] He computed the indices for all party blocs before and after every migration and sought evidence that these could have been motivated by the deputies who migrated seeking to increase their *a priori* voting power. His findings were negative. However, although it addressed a crucially important question the study was deficient in several respects and its findings should not be taken as serious evidence against power indices, but as inconclusive. Its most serious methodological flaw was its use of the SSI per head to measure the voting power of an individual member of a bloc. This was computed as the index for the bloc divided by the number of bloc members. In using this measure, Riker was assuming that the SSI could be composed in a simple way by just multiplying together the index of the bloc in the legislature and the index of the member internally within the bloc (which is just $1/m$, if m is the bloc membership). But this is quite wrong, as Owen (1995) shows. It is worth explaining this in more detail.

Owen discusses at length the derivation of power indices for a composed game, giving the appropriate modifications of both the SSI and the Banzhaf index (the latter including a derivation of the validity of the power index used in the present paper, which we describe in the next section). He also gives a method of computation for the properly defined SSI for the individual bloc members in the composed game and applies it to the US presidential election game. The indices he obtains in this way are quite different from the SSI's per head derived simply from the results for the states game with the same data, and this difference illustrates the error in Riker's method.[8]

It is perhaps worth examining this example in detail to emphasise the point. Table 1 is compiled from Owen's results given in chapter XII, Tables 4.1 and

Table 1. SSI and SSI per head: US Presidential Electoral College

State	Electoral votes w	Population m	SSI $\varphi(\nu)$	SSI/Head* $\varphi(\nu)/m$	Individual SSI* $\varphi_i(u)$
California	45	19,953,134	0.08831	4.4259	7.8476
Florida	17	6,789,433	0.03147	4.6351	4.7326
Alabama	9	3,444,165	0.01641	4.7646	3.4849
Alaska	3	302,173	0.005412	17.910	3.9253

Source. Owen (1995), Tables XII 4.1 and 4.2. In Owen's notation, ν represents the states game, so $\varphi(\nu)$ is the SSI of a state, u represents the composed game, so $\varphi_i(u)$ represents the appropriate SSI for a citizen, i, of the state being considered.
*($\times 10^{-9}$).

4.2, which give the SSIs for the states and for individual citizens of the states respectively. In the table we have reported the indices for four states only since this is sufficient to our purpose. We have computed the SSI per head using the indices from Table 4.1. There is clearly a large difference and we conclude that the SSI per head is not the right measure of indirect power for members of a bloc.

The fact that the SSI does not compose in the simple way assumed by Riker, which would allow the use of SSI per head, is not in itself a sufficient argument against using an approach based on that index to measure power in voting blocs. As Owen shows, a suitable modification of it can be defined, which can be calculated with the right algorithm. However we consider the theoretical arguments and empirical evidence against its use, described above, as decisive and therefore do not use it.[9] We now return to the description of the power indices used in this paper.

Voting Powers of Blocs and Individual Members

We make the basic assumptions that all decisions are made by simple majority and all members vote. Every member has the independent right to vote 'aye' or 'no' in any ballot or roll-call. We model the formal power of an actor, whether an individual or a bloc, who is a member of this body, as a probability. The power of an actor (whether an individual or bloc), over decisions taken internally within the body in which it votes, is the probability that it swings the vote, a power index. The power of actor a in voting body V, written P_a^V, is defined generally as,

$$P_a^V = \Pr\,[\text{Actor } a \text{ swings the vote in } V] \tag{1}$$

Expression (1) is the probability that the combined votes of all the other members of V are just short of a majority, such that adding the vote(s) of a to them will produce a majority. This obviously depends on the particular data for the voting body consisting of the sizes of all the blocs, their number, the number of votes cast by actor a, the decision rule and the model of probabilistic voting. Thus, the power of an individual i internally over decisions taken within a bloc B is then written P_i^B while the power of the bloc B within the global body G is denoted by P_B^G.

The power indices for all the actors are found using the general definition in equation (1) applied to the voting model assumed and the data. This definition also requires a model of probabilistic voting. This can be either a description of actual behaviour, taking into account relationships between members and party blocs, or a stylised model in which all actors vote for or against an action with equal probability and independently. The power indices from the former approach would measure *behavioural* power, while the latter would be an *a priori* power index measuring power deriving from the bloc structure and the voting rules in a *constitutional* sense. The former requires data on actual voting behaviour; Coleman[10] showed how an estimate of the variance of the size of the 'aye' vote could be used for this purpose. The latter, which is the approach followed here, requires only a stylised model of probabilistic voting to compute the *a priori* power indices.[11]

In this study the power indices are found in two general ways.

(i) If a voting body consists only of individuals, and does not contain any blocs, the power index for any member is given by the binomial distribution. Thus, the power of an actor who is an individual member within bloc B, which has m members is simply the binomial probability that the number of other members who vote 'aye' is exactly one vote less than the number required for a decision. That is $m/2$, or $(m-1)/2$, depending on whether m is even or odd.

(ii) To find the power of an actor which is a particular bloc within a legislature which also contains other blocs, that are in general of different sizes, is more difficult computationally, and requires the use of a computer program that implements an appropriate power indices algorithm. In this study we use the algorithm known as the method of generating functions to compute the power indices for bodies that have blocs (Brams & Affuso, 1976; Leech & Leech, 2004).

Each of these calculations gives us the (absolute) voting power of a certain actor within a given voting body. Our main interest however is in the power of individuals in relation to voting blocs, for which we need further notation. It is unnecessary for this purpose now to label the individual so we can drop the actor subscript from the power index.[12] It is however necessary to label the bloc structure. Thus we denote the power of an individual acting as a member of bloc B in global body G as $P(B, G)$, and the power of an individual acting independently (that is, formally, a bloc with one member) in the same body as $P(\{i\}, G)$.[13]

Thus we can write the voting power of an individual member of bloc B[14] as,

$$P(B, G) = P_i^B P_B^G. \tag{2}$$

The (indirect) power of a member of bloc B is the product of his or her power over decisions of the bloc and the power of the bloc over the decisions of the global legislature. This can be compared with $P(\{i\}, G) = P_{\{i\}}^G$, the power of an independent member, in order to determine if there is a net power gain or loss when i joins B.[15]

Power Index Calculations for a Hypothetical Legislature

We now report the calculations for the power indices for a hypothetical legislature assuming one and then two blocs. The one-bloc case is described first in order to demonstrate the power of blocs and to show the trade-off faced by individuals, described above, and also the optimum bloc size. Then we generalise it and show that the two-bloc situation gives rise to a rich variety of cases including monopolar and bipolar power structures. We then discuss the incentives that individual members have to migrate that the differences in voting power create.

Power with One Bloc

We assume there is one bloc, W, whose number of members is w. Then we can write, for the global legislature, $G = \{q; W, \{i\}, \{j\}, \dots\}$, the indirect power of a bloc member:

$$P(W, G) = P_i^W \cdot P_W^G. \tag{3}$$

The two components of (3) are evaluated separately. The value of P_i^W is found analytically as a binomial probability. This depends on the parity of w, and we must use different formulae for odd and even bloc sizes:

$$P_i^W = \binom{w-1}{\frac{w-1}{2}} 0.5^{w-1}, \quad \text{if } w \text{ is odd;} \quad \binom{w-1}{\frac{w}{2}} 0.5^{w-1} \text{ if } w \text{ is even.} \tag{4}$$

The value of P_W^G can also be found analytically in this case, but it is better, as a general strategy for these calculations, where we wish to allow for a general bloc structure, to evaluate it numerically.

If w is large enough, then (4) can be replaced by the approximation,[16]

$$P_i^W = \sqrt{\frac{2}{\pi w}} = \frac{0.79788}{\sqrt{w}} \tag{5}$$

Expression (5) is Penrose's square root rule which states that the power of a member of a large voting body is approximately inversely proportional to the square root of size of the body (Penrose, 1946). Since in this paper our interest is in relatively small voting blocs, including very small ones, we will use (4) only. However (5) is useful when the voting blocs contain very many members, for example, where they are constituencies with thousands of electors or countries with millions.

Power with two blocs

When there are two blocs, labelled $W1$ and $W2$, with w_1 and w_2 members, the global legislature can be written, $G = \{q; W1, W2, \{i\}, \{j\}, \dots\}$. The power indices we are interested in are written:

$$P(W1, G) = P_i^{W1} \cdot P_{W1}^G, \quad P(W2, G) = P_i^{W2} \cdot P_{W2}^G, \quad P(\{i\}, G) = P_{\{i\}}^G.$$

We find P_i^{W1} and P_i^{W2} as binomial probabilities, and P_{W1}^G, P_{W2}^G and $P_{\{i\}}^G$ numerically as before.

Voting Power and Voting Blocs: An Example

Here we report the results for a legislature with $n = 100$ members.[17] The assumptions throughout are that the legislature makes its decisions by a simple majority of 51 votes, that is $q = 51$, and that each bloc uses a simple majority rule internally to determine how it votes.

We first consider the one-bloc case. Figure 2 illustrates the trade-off between the power indices for the bloc as a whole, P_W^G, and of a bloc member within it, P_i^W, as the bloc size, w, increases, for all values of w from 2 to 50. As the size of the bloc increases its power increases, eventually approaching 1 when it has an absolute majority, $w = 51$. Its power index gets very close to 1 long before it has an absolute majority, however, illustrating how very powerful even minority blocs can be. On the other hand, the power of one of its members to control the bloc in an internal vote falls continuously to about 0.08, in the limiting case when $w = 100$.

Figure 3 shows the trade-off between these two power indices. The saw-tooth appearance of the line shows the sensitivity of the power index for an individual member within the bloc to the parity of the bloc size in small blocs. This comes about because, for example, a member of a bloc with 4 members

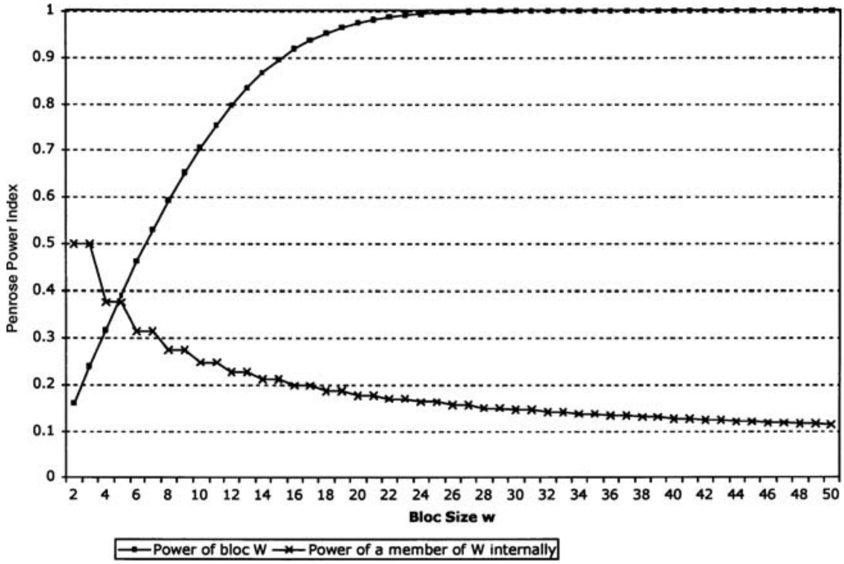

Figure 2. Bloc power and internal power of a bloc member.

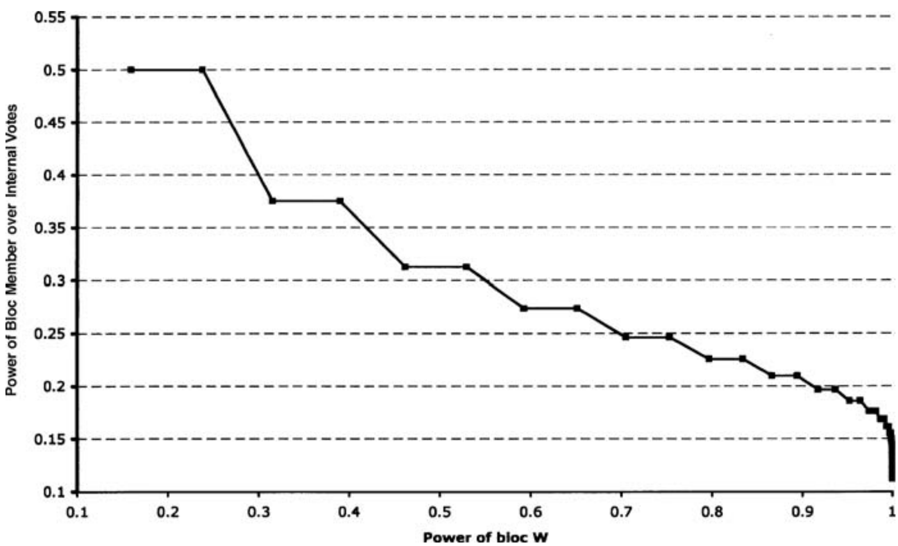

Figure 3. Tradeoff between bloc power and bloc member's internal power.

has the same internal voting power within the bloc as he or she would have if the bloc had 5 members, so both have the same value of P_i^W.[18] However the bloc with 5 members has more power in the legislature and a greater value of P_W^G.

Figure 4 shows the relationship between the indirect power of a bloc member, $P(W, G)$, defined in equation (3), and bloc size. However the (indirect)

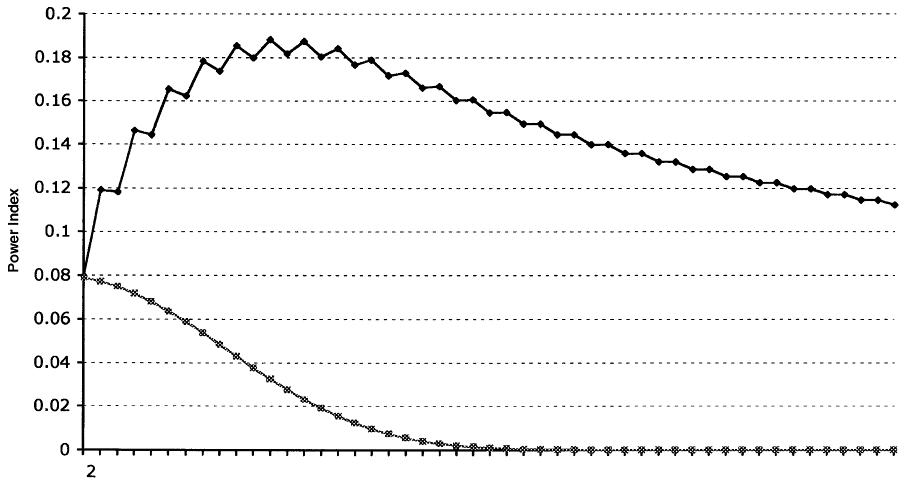

Figure 4. Power of a bloc member and a non-member.

power of a bloc member does not grow continuously; it grows to a point and then declines. The bloc size that maximises the member's power is $w = 13$. Up to this point the bloc is powerful in the legislature but because the number of members is small, each individual member is influential internally; beyond that point the power of the bloc increases at a diminishing rate while the addition of new members dilutes the internal power of individual members. As the bloc grows in size and the number of independent members declines, it becomes rapidly more powerful. At the same time the power of each independent member falls rapidly and continuously, becoming virtually zero once the bloc has more than about 20 members, $w > 20$. On the other hand, however much this dilution proceeds, the power of a bloc member still far exceeds that of a non-member, $P(\{i\}, G)$. The saw-tooth effect for small bloc sizes is also reflected in this diagram.

Figure 5 extends this analysis to the case where there are two blocs, $W1$ and $W2$. The chart shows the power of a member of the bloc $W1$ in G as w_1 varies, and $W2$ is of a fixed size, for different values of w_2, that is, $P(W1, G)$ where $G = \{q; W1, W2, \{i\}, \{j\}, \dots\}$. The power of a member of bloc $W1$ is less the greater is w_2. Table 2 shows the relation between the optimum value of w_1, for which power is maximised, and w_2.

Figure 6 shows the powers of members of $W1$, $W2$ and non-members, i, in terms of the size of the bloc $W1$ for the four cases: $w_2 = 10, 20, 30, 40$. It is noticeable how in all four diagrams a major effect is that the two large blocs reduce each other's power substantially when they are of comparable size while one of them is very dominant when their sizes differ. In some cases this is to the advantage of individuals who are not bloc members who become more powerful than bloc members.

Table 2. Optimum w_1

w_2	Optimum w_1	Power of member of $W1$
0	13	0.1883
5	17	0.1789
10	21	0.1596
15	27	0.1459
20	31	0.1359
30	41	0.1227
40	49	0.1142

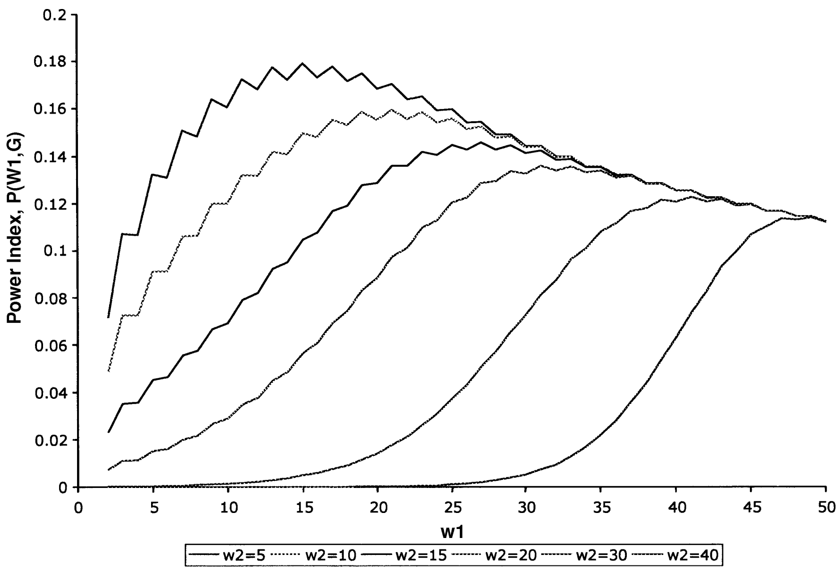

Figure 5. Power of a bloc member, two blocs.

Figure 6(a) is the case where $w_2 = 10$. When w_1 is small $P(\{i\}, G)$ is equal to $P\{W1, G\}$ and the bloc is too small to matter. As $W1$ increases in size and becomes more powerful, $W2$ loses power, as does, after a while, the independent member i. The optimum size of $W1$ is $w_1 = 21$ when its members' power is at its maximum.

Figure 6(b) shows the case where $w_2 = 20$. Now it is advantageous to belong to either $W2$ or $W1$ until $w_1 = 26$ when members of bloc $W2$ have less power than independent members. For values of $w_1 > 26$ independent members have an incentive to join W1 but not $W2$; members of $W2$ have an incentive to leave and become independent or join $W1$.

Figure 6(c) shows the situation when $w_2 = 30$. Now $W2$ is very powerful when w_1 is small, and members of $W1$ have less power than independent members until $w_1 = 28$. In this range, there are strong incentives to join W2 and weak incentives for members of $W1$ to leave and become independent. Between $w_1 = 28$ and $w_1 = 32$ there is an intermediate range where the power

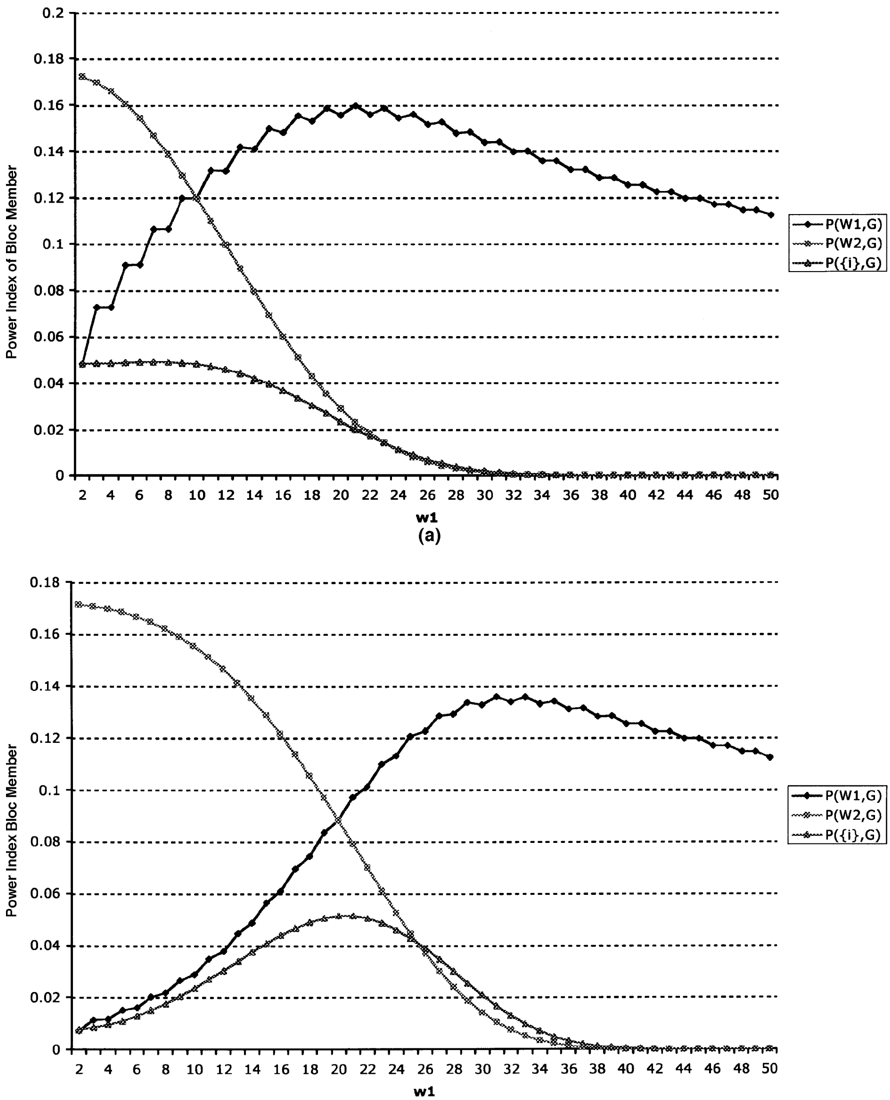

Figure 6. (a) Two blocs, $w_2 = 10$, $n = 100$. (b) Two blocs, $w_2 = 20$, $n = 100$. (c) Two blocs, $w_2 = 30$, $n = 100$. (d) Two blocs, $w_2 = 40$, $n = 100$.

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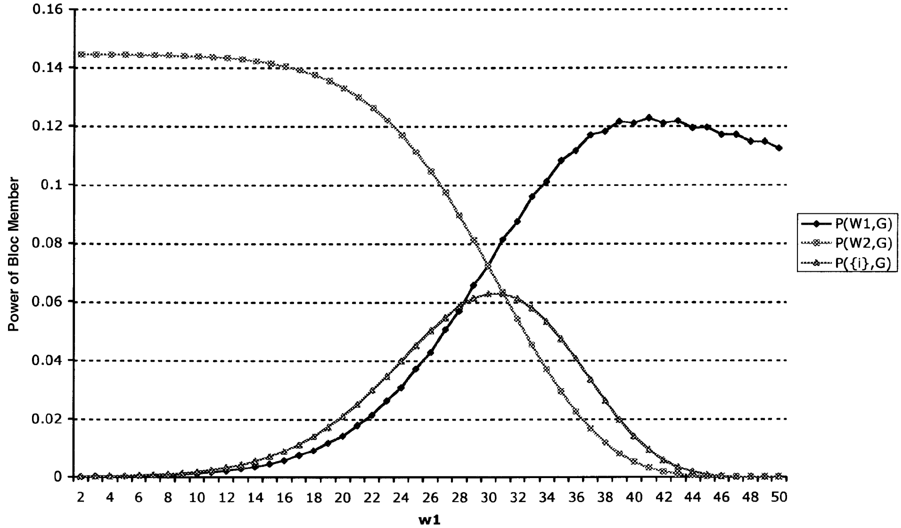

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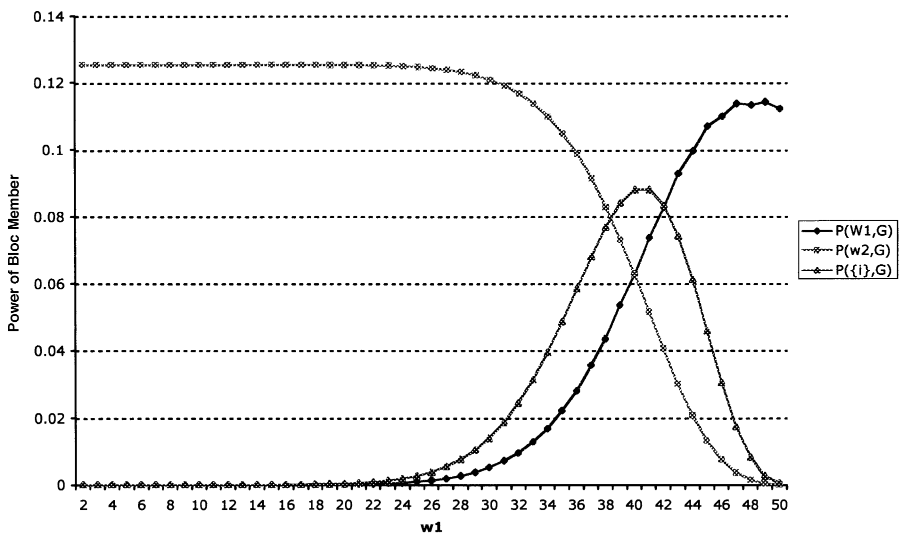

(d)

Figure 6. (*Continued*)

of the independent member is at its peak but still below that of a member of either bloc. Above $w_1 = 32$ an independent has greater power than a member of bloc $W2$ (even though that bloc controls 30 percent of the votes), such is the power of $W1$. In this bipolar situation, the power of $W1$, even though it is the dominant bloc, is much less than that of $W2$ was when $W1$ was small.

Figure 6(d) shows the case where $W2$ is just short of an overall majority, $w_2 = 40$. Now, when $W1$ becomes big enough to rival W2, the power of

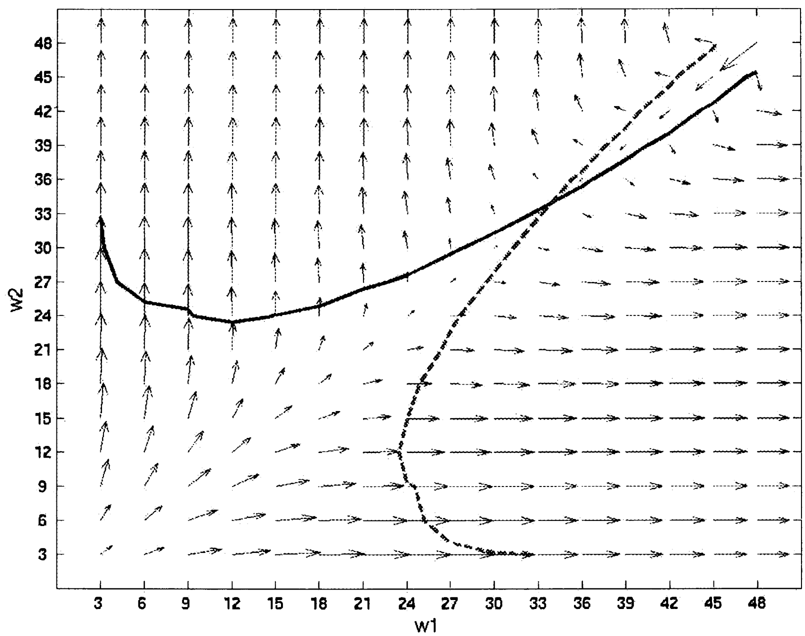

Figure 7. Incentives to migrate between two blocs.

an independent is greater than any member of either bloc. This is a truly bipolar situation in which there are two powerful blocs opposing one another which are each so large that their members' power over internal decisions is dissipated.

Figure 7 shows the incentives facing individual members to migrate between blocs when there are two blocs. The diagram shows the range of values of w_1 and w_2, where neither bloc has an absolute majority. The incentives to migrate are measured by the differences in power indices for an individual who is a bloc member and a non-member. The diagram is constructed from the vector $[P(W1, G) - P(\{i\}, G), P(W2, G) - P(\{i\}, G)]$ for every pair of values of (w_1, w_2). The arrows indicate the direction and strength of the resultant as an indication of the strength of the incentive to migrate and the consequent direction of change of the bloc sizes. The lines are the zero contours where there is no incentive that would lead one of the blocs to change: the power of a non-member of a bloc is equal to that of a bloc member. The diagram says nothing about possible migrations beyond incentives. It particular, it takes no account of incentives of bloc members to accept migrants.

In Figure 7, along the 45-degree line, when both blocs are equal, there is an incentive for them both to change unless $w_1(=w_2) = 33$. Below this value, the incentive is for both blocs to grow, above it to shrink. The set of points where $w_1 = w_2$ has a knife-edge property, since when $w_1 \neq w_2$, the incentive is for the larger bloc to grow and the smaller one to decline. The point $w_1 = w_2 = 33$ has a saddle point property where it is stable in one dimension and unstable in another.

Generalisations

The numerical analysis described above is for a special case of a simple legislature with 100 voting members and at most two blocs. But the methodology of which it is illustrative does not depend on these simplifications. It can be applied straightforwardly to cases in which there are more blocs, as in most real legislatures containing a number of party groupings and voting bodies that use weighted voting, such as intergovernmental organisations like the EU Council. However in order to be empirically valid such analyses involve much more complexity which must be handled in a coherent way.

We must consider the effects of relaxing the limitation to 100 voters, in particular to allow larger voting bodies.[19] When the number of voters increases, while holding the bloc sizes constant, we can show that the substantive results for the powers of the blocs do not change. Under our assumptions the global voting body, G, can be closely approximated by an "oceanic game" for which we have analytical results from Dubey and Shapley (1979). An oceanic game is a limiting case of a legislature in which the number of voters is considered to increase without limit, while each voter has a progressively smaller weight, in the limit infinitesimal, such that the bloc sizes remain fixed. Dubey and Shapley showed that, in our notation, for an oceanic game in which w_1 and w_2 are fixed percentages, the limiting power indices of the blocs $W1$ and $W2$, P^G_{W1} and P^G_{W2}, tend to the values they would have in a body comprising only the two blocs, say $H = \{q; W1, W2\}$, in which the decision rule is amended to $q = 50 - (100 - w_1 - w_2)/2$. In this case there are two possibilities: either one bloc has all the power and the other has none, or they are both equally powerful, $P^G_{W1} = P^G_{W2} = 0.5$. The powers of the members of the blocs, P_i^{W1} and P_i^{W2} and the power of a voter who is not a member of a bloc, $P^G_{\{i\}}$ will tend to zero in the limit. They can be evaluated exactly (or by normal approximation to the binomial distribution) where n is large but finite.

Conclusion

This paper has proposed the use of Penrose power indices to study the power of actors in a voting body with blocs. We have looked at the simple case of a legislature with 100 members where there are one or two blocs, such as party groups, in which the whip is applied on the basis of simple majority voting among its members.

We have shown that the power of an individual bloc member can be modelled in terms of two contrasting components: the power of the bloc within the legislature deriving from the internal discipline that creates the power of combined forces, that increases with bloc size; and the power of the individual member over bloc decisions, which declines with bloc size. Analysing this trade-off leads to useful insights for voting situations involving more than one voting body or multiple layers of decision making, or for changes in voting systems or bloc structures, for example following elections.

The model and the general approach described here can be extended in many ways. First, the analysis here is entirely *a priori* in the sense that no account is taken of preferences or actual voting behaviour. This analysis is especially useful for an understanding of the power implications of voting rules when considered as formal constitutions. However, the approach is more general since the basic definition of a power index (in equation (1)) can be adapted to allow for actual or empirically observed voting behaviour if the appropriate data on voting patterns is available. Second, we have considered a stylised legislature with only two blocs. This can be generalised easily to take account of more voting blocs, as for example parties in a real legislature or where weighted voting is used, such as intergovernmental international organisations. Thirdly, the analysis and results hold for larger legislatures.

Notes

1. Surveys of the literature on the measurement of voting power are given by Straffin (1994) and Felsenthal and Machover (1998). See also Holler (1982).
2. Another common criticism is that power indices studies of relative voting power do not reflect the importance of the decisions to be taken by the particular voting body of interest. A voting body is taken as a given and the results obtained are not dependent on whether for example it is a major international organisation or a minor organ of local government.
3. The power (of the collective body as a whole) to act, and the power (of a member) to prevent action and to initiate action.
4. See Coleman (1971), Felsenthal and Machover (1998), Leech (2002). Riker did a lot of other work on voting power measurement but its success was limited by his reliance on the Shapley–Shubik index.
5. Actually the more commonly used name for this index in the literature is the absolute Banzhaf index. We prefer to use the term Penrose index (after its original inventor) and reserve the name Banzhaf index for its normalised version that is used as a measure of relative voting power. We make this distinction to emphasise the importance we attach to the non-normalised index as an analytical tool for answering a different set of questions- in particular Riker's question about power seeking behaviour – than computing power shares.
6. The study by Leech (2002) found that when the SSI was applied to real-world shareholder-voting games – where there is strong empirical knowledge about the power of large blocs of shares – its values were implausible. By contrast the Banzhaf indices were not.
7. The French National Assembly over two-year period 1953 and 1954.
8. Owen (1995), Chapter XII. It is beyond the scope of the present paper to describe the mathematics of the derivation. We refer the interested reader to Owen.
9. A second major criticism of Riker's study is that his data set was not good for empirically testing the adequacy of the power index since very few of the migrations he observed involved members of the large and powerful party blocs, and the period he took was very short. This suggests a need for more empirical testing of power indices using better data.

10. Coleman (1973).
11. A recent application where *a priori* power indices are appropriate for the study of the fairness of voting rules, is Leech and Leech (2006)
12. All individuals within a given bloc have the same power.
13. In this notation, when we consider variation in the first argument of $P(B, G)$, B, with G held constant, it is understood that the bloc B changes but the other blocs do not change. Changes in the size of the bloc B occur by way of changes in the number of individuals who do not belong to the other blocs, all of which are assumed constant.
14. It is sometimes appropriate to refer to this as the indirect voting power to emphasise that the member is working through the group.
15. This comparison assumes that when individual i joins the bloc the characteristics of the global voting body do not change. This is strictly false but has been ignored for ease of exposition. Write $G = \{q; B, C, D, E, \ldots\}$. Then the relevant comparison should be between $P(B, G)$ and $P(\{i\}, H)$ where $H = \{q; B - \{i\}, \{i\}, C, D, E, \ldots\}$. This point must be allowed for in empirical applications.
16. This approximation is based on Stirling's formula. See Feller (1950, p. 180). See also Penrose (1946), Coleman (1973).
17. The spreadsheets containing the calculations are available from the authors.
18. If w is an even number, then the internal powers of a member of bloc W and of another bloc bigger by one member, say $W + \{j\}$, can easily be shown to be equal, that
$$P_i^W = \binom{w-1}{\frac{w}{2}} 0.5^{w-1} = P_i^{W+\{j\}} = \binom{w}{\frac{w}{2}} 0.5^w \quad .$$
19. Our assumption that $n = 100$ was made for illustrative purposes, and not for any reasons of computational limitation. The computer algorithm we used, *ipgenf*, from Leech and Leech (2004), which uses the method of generating functions, can compute power indices for much larger bodies (and the specific implementation allows up to $n = 200$). However the method is not applicable to all voting bodies and an approximation method may be required. See Leech and Leech (2004) for computational details and alternative software.

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