EXERCISE SHEET 4

Let X be a compact complex manifold. We write

$$h^{i,j}(X) := h^j(X, \Omega^i_X), \quad b_i(X) := h^i(X, \mathbb{C}), \quad e(X) := \sum (-1)^i b_i(X).$$

Problem 0.1. Let X be a compact complex manifold. Show that $h^{1,0} \leq b_1/2$.

Problem 0.2. Let ω be a holomorphic $(n-1)$-form on a complex manifold X of dimension n. Show that $d\omega = 0$.

Problem 0.3. Let X be a compact complex surface. Show that

$$2h^{1,0} \leq b_1 \leq h^{0,1} + h^{1,0} \leq 2h^{0,1}.$$

Problem 0.4. Let X be a K3 surface in the sense of Exercise sheet 3. Compute $e(X)$, $h^{i,j}$ and b_i. (Without using the fact X is Kähler. *Hint*: Use the above 3 problems.)

Problem 0.5. Let $X := (\mathbb{C}^n \backslash \{0\})/\Gamma$ be the quotient, where the group $\Gamma = \mathbb{Z} = \langle \gamma \rangle$ acts on $\mathbb{C}^n$ by

$$\gamma \cdot (z_1, \dots, z_n) = (2z_1, \dots 2z_n).$$

- Show that X is a compact complex manifold diffeomorphic to $S^1 \times S^{2n-1}$.
- Compute b_1 (or all b_i). Is X Kähler?

Problem 0.6. Let X be a compact complex surface. Prove the following formulas by using the Hirzebruch–Riemann–Roch formula;

$$h^{0,0} - h^{0,1} + h^{0,2} = \frac{1}{12}(c_1(X)^2 + c_2(X)),$$
$$h^{1,0} - h^{1,1} + h^{1,2} = \frac{1}{6}(c_1(X)^2 - 5c_2(X)),$$
$$h^{2,0} - h^{2,1} + h^{2,2} = \frac{1}{12}(c_1(X)^2 + c_2(X)).$$

Problem 0.7. Let X be the example given in Problem 0.5 and assume $n = 2$. Compute $h^{i,j}(X)$ by using 0.6.

References

[1] S. Kobayashi, *Complex geometry* (in Japanese), Iwanami shoten.