Hodge Theory reading seminar: exercise sheet 2.

Exercise taken from the following books:

- Complex Geometry (D. Huybrechts);
- Hodge Theory and Complex Algebraic Geometry, I (C. Voisin)
- Period Mappings and Period Domains (J. Carlson, S. Müller-Stach, C.Peters)

Exercise 1. *Let $Y \subset X$ a smooth hypersurface in a complex manifold X of dimension n and let α be a meromorphic section of K_X with at most simple poles along Y. Locally one can write $\alpha = h \cdot \frac{dz_1}{z_1} \wedge dz_2 \wedge \ldots \wedge dz_n$, with z_1 defining Y. One sets $Res_Y(\alpha) = (h \cdot dz_2 \wedge \ldots \wedge dz_n)|_Y$.*

- *Show that $Res_Y(\alpha)$ is well-defined and that it yields an element in $H^0(Y, K_Y)$;*
- *Consider α as an element in $H^0(K_X \otimes \mathcal{O}(Y))$ and compare the definition of the residue with the adjunction formula $K_Y \cong (K_X \otimes \mathcal{O}(Y))|_Y$;*
- *Consider a smooth hypersurface $Y \subset \mathbb{P}^n$ defined by a homogeneous polynomial $f \in H^0(\mathbb{P}^n, \mathcal{O}(n+1))$, and let $\alpha := \sum (-1)^i \frac{z_i}{f} dz_0 \wedge \ldots \wedge \hat{dz_i} \wedge \ldots \wedge dz_n$ a meromorphic section of $K_{\mathbb{P}^n}$ with simple poles along Y. Show that $Res_Y(\alpha) \in H^0(Y, K_Y)$ defines a trivializing section of K_Y.*

Exercise 2. *Show that the canonical bundle K_X of a complete intersection $X = V(f_1) \cap \ldots \cap V(f_n) \subset \mathbb{P}^n$ is isomorphic to $\mathcal{O}(\sum deg(f_i) - n - 1))|_X$.*

Exercise 3. *Show that the surface $\Sigma_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ is isomorphic to the hypersurface $V(x_0^n y_1 - x_1^n y_2) \subset \mathbb{P}^1 \times \mathbb{P}^2$, where $[x_0, x_1]$ and $[y_0, y_1, y_2]$ are homogeneous coordinates of (respectively) $\mathbb{P}^1$ and $\mathbb{P}^2$.*

Exercise 4. *Are there holomorphic vector field on $\mathbb{P}^n$, i.e. global section of $\mathcal{T}_{\mathbb{P}^n}$, which vanish only in a finite number of points? If yes, how many?*

Exercise 5. *Show that $h^n(\mathbb{P}^n, \mathcal{O}(k)) = 0$ if $k > -n-1$, while if $k \leq -n-1$, $h^n(\mathbb{P}^n, \mathcal{O}(k)) = \binom{-k-1}{-n-k-1}$.*

Exercise 6. *Let X a connected, compact complex manifold of dimension n, and let L an holomorphic line bundle on X. Suppose that there exists some integer $N > 0$ such that $H^0(X, L^{\otimes N}) \neq 0$. Show that if $H^n(X, L \otimes K_X) \neq 0$, the line bundle L is trivial.*

Exercise 7. *Let L an holomorphic line bundle of degree $d > 2g(C) - 2$ on a compact curve C, where we have that the genus $g(C)$ satisfies $deg(K_C) = 2g - 2$. Show that $H^1(C, L) = 0$, and deduce that $H^1(C, K_C \otimes L) = 0$ for any line bundle L with $deg(L) > 0$.*