HODGE THEORY PROBLEM SHEET 1

These exercises were pulled from the following books:

(H) Huybrechts, *Complex Geometry*
(V) Voisin, *Hodge Theory and Complex Algebraic Geometry, I*

You might find more information if you consult the book itself.

Q1. (Hodge decomposition for curves) Let X be a compact connected complex curve. Consider the differential $d : \mathcal{O}_X \to \Omega_X$.

(1) Show that we have an exact sequence
$$0 \to \mathbb{C} \to \mathcal{O}_X \stackrel{d}{\to} \Omega_X \to 0$$
(2) Use Serre duality to show that $H^1(X, \Omega_X) \cong \mathbb{C}$ and Poincaré duality to show that $H^2(X, \mathbb{C}) = \mathbb{C}$.
(3) Show that we get a short exact sequence of cohomology groups
$$0 \to H^0(X, \Omega_X) \stackrel{\alpha}{\to} H^1(X, \mathbb{C}) \stackrel{\beta}{\to} H^1(X, \mathcal{O}_X) \to 0$$
(4) Show that the map $\gamma\colon H^0(X, \Omega_X) \to H^1(X, \mathcal{O}_X)$ defined by $\gamma(\omega) = \beta(\overline{\alpha(\omega)})$ is injective.
(5) Deduce from Serre duality that it is also surjective and that we have the decomposition $H^1(X, \mathbb{C}) = H^0(X, \Omega_X) \oplus \overline{H^0(X, \Omega_X)}$ where $\overline{H^0(X, \Omega_X)} \cong H^1(X, \mathcal{O}_X)$

Q2. Suppose (X, ω) is a compact Kähler manifold. The Kähler cone $\mathcal{K}_X \subset H^{1,1}(X, \mathbb{R})$ is the set of all Kähler classes, where $H^{1,1}(X, \mathbb{R}) = H^{1,1}(X) \cap H^2(X, \mathbb{R})$.

(1) Show that $\mathcal{K}_X$ is a strictly convex, open cone in $H^{1,1}(X, \mathbb{R})$
(2) Suppose $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ where β is Kähler. Show that $\alpha + t\beta \in \mathcal{K}_X$ for large t.
(3) Show that ω is harmonic.
(4) If $H^2(X, \mathcal{O}_X) = 0$ deduce that the $\mathcal{K}_X$ is actually open in $H^2(X, \mathbb{R})$.
(5) Under this assumption show that X is projective.

Q4. Let X be a Kähler manifold. Describe Poincaré duality by using the Hodge decomposition $H^k(X; \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$ and Serre duality $H^{p,q}(X) \cong H^{k-p,k-q}(X)^\star$.

Q5. Recall the Fubini-Study metric $\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\log|z|^2$ on $\mathbb{P}^n$. Prove $\int_{\mathbb{P}^n} \omega_{FS}^n = 1$. (The case $n = 1$ is done in (H).) Let $A \in GL_{n+1}(\mathbb{C})$ and consider the induced isomorphism $f_A\colon \mathbb{P}^n \to \mathbb{P}^n$. Show that $f_A^*(\omega_{FS}) = \omega_{FS}$ if and only if $A \in U_{n+1}$

Q6. Show that $h^{p,q}(\mathbb{P}^n) = 0$ for $p \neq q$ and $h^{p,p}(\mathbb{P}^n) = 1$ for $p \leq n$. Use the exponential sequence to show that $Pic(\mathbb{P}^n) \cong \mathbb{Z}$