MA3H1 Topics in Number Theory
Example Sheet 1

You should attempt all the questions on this sheet. but questions Q1(a), (c), (d), Q4, Q5, Q7 will marked for credit, and must be handed in to TA Homero Gallegos Ruiz by **3pm Friday, week 3.**

Number theory is orthogonal to the mathematics you have studied so far. Most importantly, *it demands that you think discretely.* Moreover, there is emphasis on explicit problems and problem solving.

(1) This question is random fun to get you in the right mood for the subject. Solve the following equations positive integers
  (a) $x^2 + y^2 = 100$.
  (b) $x^2 - y^2 = 8$.
  (c) $a! + b! = c!$.
  (d) $a! + b! = 25 \cdot c!$.

(2) Let $\mathbb{N}'$ be the set of all positive integers $\equiv 1 \pmod 4$. Call an element of $\mathbb{N}' \backslash \{1\}$ *irreducible* if it can't be factorized into strictly smaller elements of $\mathbb{N}'$. Is every element of $\mathbb{N}'$ a product of irreducibles? Is the expression of an element of $\mathbb{N}'$ as a product of irreducibles unique up to order?

(3) Let f be a monic polynomial with integer coefficients, and α be root of f. Show that if α is rational then α must be an integer.

(4) *Mersenne Numbers and Perfect Numbers.*
  (a) The n-th Mersenne number is $M_n = 2^n - 1$. Show that if M_n is prime then n must be prime.
  (b) A positive integer N is called *perfect* if the sum of its proper divisors is equal to itself. For example, the proper divisors of 28 are 1, 2, 4, 7, 14 and their sum is 28, so 28 is perfect. Show that if M_n is prime then $2^{n-1}M_n$ is perfect.
  (c) Don't try this one; it's the oldest unsolved problem in mathematics. Are there any odd perfect numbers?

(5) Let m, n be non-zero integers. Show that $\gcd(m,n) = \prod_{p \in \mathbb{P}} p^{\min(\mathrm{ord}_p(m), \mathrm{ord}_p(n))}$.

(6) Show that if a, b, c, n are positive integers with $\gcd(a,b) = 1$ and $ab = c^n$ then there exist positive integers x, y such that $a = x^n$, $b = y^n$. **(Hint: show that $n \mid \mathrm{ord}_p(x)$ and $n \mid \mathrm{ord}_p(y)$ for all primes p.)**

(7) Let a, b be positive integers with $\gcd(a,b) = 1$. Suppose $x^a = y^b$ where x, y are positive integers. Show that $x = n^b$, $y = n^a$ for some integer n. **(Hint: use the properties of ord_p.)**