MA3D5 Galois Theory

Homework Assignment 2

The deadline is **2pm Thursday, week 5.** Please hand in your answers to **questions 2, 3 and 4** the MA3D5 Galois Theory box outside the Undergraduate Office.

1. Let $f \in \mathbb{Q}[x]$ be a polynomial of degree n. Show that the splitting field of f has degree $\leq n!$.

2. Let p, q be distinct primes.
   (a) Show that $\sqrt{p} \notin \mathbb{Q}(\sqrt{q})$.
   (b) Determine with proof the degree $[\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}]$.
   (c) Determine with proof the degree $[\mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{pq}) : \mathbb{Q}]$.
   (d) Let
   $$g(x) = x^4 - 2(p+q)x^2 + (p-q)^2.$$
   Show that $\sqrt{p} + \sqrt{q}$ is a root of g. Deduce that g is irreducible. (**Hint:** use the fact $\mathbb{Q}(\sqrt{p} + \sqrt{q}) = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ which you proved in Assignment 1.)

3. Let $f = x^3 + x + 3$. In Assignment 1 you showed that f is irreducible, and that it has exactly one real root.
   (a) Let θ be the real root of f. Let ϕ, ϕ' be the two other roots. Compute
   $$[\mathbb{Q}(\theta) : \mathbb{Q}] \qquad [\mathbb{Q}(\theta, \phi) : \mathbb{Q}] \qquad [\mathbb{Q}(\theta, \phi, \phi') : \mathbb{Q}].$$
   (b) Without writing down the minimal polynomial for θ^2, show that $\mathbb{Q}(\theta^2) = \mathbb{Q}(\theta)$.
   (c) Write down the minimal polynomial for θ^2.

4. Let L/K be a field extension with degree $[L : K] = p$ where p is a prime. Show that L/K is a simple extension.